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ON EXTERIOR FORMS AND EXTERIOR DIFFERENTIAL ON A DIFFERENTIAL SPACE OF FINITE DIMENSION

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The purpose of this paper is to discuss the concept of exterior form and exterior differential of smooth exterior forms on m-dimensional differential space (M, C) (see [1] and [4]). A smooth exterior differential form is meant as a smooth section of the bundle $\bigwedge T^*(M, C)$ which is defined in this paper.

1. The bundles $\bigwedge T(M, C)$ and $\bigwedge T^*(M, C)$

Let (M, C) be a differential space with the Hausdorff topology τ_C (see [4]). In paper [2] it is defined the tangent bundle $T(M, C)$. Any mapping X which assigns to each point $p \in M$ the vector $X(p)$ of $T_p(M, C)$ is called a *vector field* on (M, C). The vector field X is said to be *smooth* if for any function $\alpha \in C$ the function $\partial_X \alpha$ defined by the formula:

$$(\partial_X \alpha)(p) = X(p)(\alpha) \quad \text{for} \quad p \in M,$$

belongs to C. The vector field X is smooth iff we have the smooth mapping $X\colon (M, C) \to T(M, C)$. The set of all smooth vector fields on (M, C) we denote by $\mathscr{X}(M, C)$. The set $\mathscr{X}(M, C)$ may be regarded as C-module with addition of elements of $\mathscr{X}(M, C)$ and multiplication by elements of C defined by formulae:

$$(X+Y)(p) = X(p)+Y(p) \quad \text{and} \quad (\alpha \cdot X)(p) = \alpha(p)X(p)$$

for $p \in M$, $X, Y \in \mathscr{X}(M, C)$, and $\alpha \in C$. The module just defined will be denoted by the same symbol. Any element of the vector space $(T_p(M, C))^*$, dual to $T_p(M, C)$, is called a *covector of* (M, C) *at* p. A tangent covector of (M, C) at any point $p \in M$ is called a *covector of* (M, C). The set of all covectors of (M, C) is denoted by $\operatorname{set} T^*(M, C)$. We have well defined mapping $\overset{*}{\pi}$ of $\operatorname{set} T^*(M, C)$ onto M by the condition: w is an element of $(T^*_{\pi(w)}(M, C))^*$ for $w \in \operatorname{set} T^*(M, C)$. For any $X \in \mathscr{X}(M, C)$ we have defined the real function $\tilde{X}$ defined by formula: $\tilde{X}(w) = w\big(X(\pi(w))\big)$ for $w \in \operatorname{set} T^*(M, C)$. We consider the smallest differential structure on $\operatorname{set} T^*(M, C)$ containing the set

$$\{\alpha \circ \overset{*}{\pi};\ \alpha \in C\} \cup \{\tilde{X};\ X \in \mathscr{X}(M, C)\}.$$

The differential space being set $T^*(M, C)$ together with this differential structure is denoted by $T^*(M, C)$ and called the *cotangent bundle* of (M, C).

Let $V_1, \ldots, V_m \in \mathscr{X}(M, C)$ and $U \in \tau_C$. We say that $(V_1, \ldots, V_m)$ is a *local vector basis of* (M, C) *on* U, if

(a) $(V_1(p), \ldots, V_m(p))$ is a basis of $T_p(M, C)$ for $p \in U$;

(b) for any $V \in \mathscr{X}(M, C)$ there exist $\alpha_1, \ldots, \alpha_m \in C$ such that $V(p) = \sum_i \alpha_i(p) V_i(p)$ for $p \in U$.

(M, C) is said to be (see [6]) *of finite dimension m* if for any point $p \in M$ there exist $U \in \tau_C$ and a local vector basis of (M, C) on U.

For any $p \in M$ we have the exterior algebras $\bigwedge T_p(M, C)$ and $\bigwedge (T_p(M, C))^*$ of the vector spaces $T_p(M, C)$ and $(T_p(M, C))^*$, respectively. Assuming that (M, C) is of finite dimension m we have the m-dimensional spaces $T_p(M, C)$ and $(T_p(M, C))^*$, and the pairing

$$\langle \cdot | \cdot \rangle\colon \bigwedge T_p(M, C) \times \bigwedge (T_p(M, C))^* \to \boldsymbol{R},$$

defined by the formulae: $\langle t|w\rangle = 0$ for t in $\bigwedge^k T_p(M, C)$, w in $\bigwedge^l (T_p(M, C))^*$, $k \neq l$,

$$\langle v_1 \wedge \ldots \wedge v_k | w_1 \wedge \ldots \wedge w_k\rangle = \det[w_j(v_i);\ i, j \leqslant m]$$

for $v_1, \ldots, v_k$ in $T_p(M, C)$ and $w_1, \ldots, w_k$ in $(T_p(M, C))^*$.

We may assume that the sets of all elements of algebras $\bigwedge T_p(M, C)$ and $\bigwedge T_q(M, C)$ are disjoint for $p \neq q$. The set of all elements of algebras $\bigwedge T_p(M, C)$, for all $p \in M$, we denote by set $\bigwedge T(M, C)$. Assigning to every $t \in$ set $\bigwedge T(M, C)$ the point $\pi^\wedge(t) \in M$ in such a way that t is an element of $\bigwedge T_{\pi^\wedge(t)}(M, C)$, we have the mapping

$$\pi^\wedge\colon \text{set} \bigwedge T(M, C) \to M.$$

We define the differential structure $C^\wedge$ on set $\bigwedge T(M, C)$ as the smallest of all differential structures containing the set

$$(1.1) \qquad \{\alpha \circ \pi^\wedge;\ \alpha \in C\} \cup \bigcup_{k=1}^{m} \{|d\alpha_1 \wedge \ldots \wedge d\alpha_k\rangle;\ \alpha_1, \ldots, \alpha_k \in C\},$$

where for any k

$$|d\alpha_1 \wedge \ldots \wedge d\alpha_k\rangle(t) = \langle t|(d\alpha_1)_p \wedge \ldots \wedge (d\alpha_k)_p\rangle, \quad p = \pi^\wedge(t)$$

for $t \in$ set $\bigwedge T(M, C)$. Here $(d\alpha)_p(v) = v(\alpha)$ for $\alpha \in C(p)$ and v in $T_p(M, C)$. Then, we get the differential space $\bigwedge T(M, C) := (\text{set} \bigwedge T(M, C), C^\wedge)$ (see [3]). Of course, we have the smooth mapping

$$\pi^\wedge\colon \bigwedge T(M, C) \to (M, C)$$

called the *projection of the Grassmann bundle* $\bigwedge T(M, C)$.

Analogously, we define set $\bigwedge T^*(M, C)$ as the set of all covectors of (M, C) and the mapping

$$\overset{*}{\pi}{}^{\wedge}\colon \text{set} \bigwedge T^*(M, C) \to M$$

in such a way that w is in $\bigwedge(T^*_{\overset{*}{\pi}{}^{\wedge}(w)}(M, C))$ for w in set $\bigwedge T^*(M, C)$. Taking the smallest differential structure, $\overset{*}{C}{}^{\wedge}$, on set $\bigwedge T^*(M, C)$ containing the set

$$\{\alpha \circ \overset{*}{\pi}{}^{\wedge};\ \alpha \in C\} \cup \bigcup_{k=1}^{m} \{\langle X_1 \wedge \dots \wedge X_k|;\ X_1, \dots, X_k \in \mathscr{X}(M, C)\}, \tag{1.2}$$

where for any k

$$\langle X_1 \wedge \dots \wedge X_k|(w) = \langle X_1(p) \wedge \dots \wedge X_k(p)|w\rangle, \quad p = \overset{*}{\pi}{}^{\wedge}(w), \tag{1.3}$$

we define $\bigwedge T^*(M, C)$ as the differential space $(\text{set} \bigwedge T^*(M, C), \overset{*}{C}{}^{\wedge})$. We have then the smooth mapping

$$\overset{*}{\pi}{}^{\wedge}\colon \bigwedge T^*(M, C) \to (M, C)$$

called the *Grassmann bundle* $\bigwedge T^*(M, C)$.

A section ω of the Grassmann bundle $\bigwedge T^*(M, C)$, i.e., the smooth mapping

$$\omega\colon (M, C) \to \bigwedge T^*(M, C) \tag{1.4}$$

such that

$$\overset{*}{\pi}{}^{\wedge} \circ \omega = \mathrm{id}_M \tag{1.5}$$

is said to be a *smooth exterior form on* (M, C). If k is such that $\omega(p)$ is in $\bigwedge^k(T_p(M, C))^*$ for $p \in M$, then ω is said to be a *form on* (M, C) *of degree* k.

PROPOSITION 1.1. *For a differential space (M, C) of finite dimension any form ω on (M, C) of degree k is a smooth exterior form on (M, C) iff for any $X_1, \dots, X_k \in \mathscr{X}(M, C)$ the function*

$$M \ni p \mapsto \langle X_1(p) \wedge \dots \wedge X_k(p)|\omega(p)\rangle \tag{1.6}$$

belongs to C.

Proof. Let $X_1, \dots, X_k \in \mathscr{X}(M, C)$. Assume that (1.6) belongs to C. Then, by (1.4) and (1.3),

$$\langle X_1 \wedge \dots \wedge X_k|\big(\omega(p)\big) = \langle X_1(p) \wedge \dots \wedge X_k(p)|\omega(p)\rangle \quad \text{for} \quad p \in M. \tag{1.7}$$

Thus, the function $\langle X_1 \wedge \dots \wedge X_k| \circ \omega$ belongs to set (1.2). For any $\alpha \in C$ we have $(\alpha \circ \overset{*}{\pi}{}^{\wedge}) \circ \omega = \alpha \in C$. Therefore, we have the smooth mapping (1.4).

Conversely, for the smooth mapping (1.4) fulfilling (1.5), and for any $X_1, \dots, X_k \in \mathscr{X}(M, C)$, by the definition of $\overset{*}{C}{}^{\wedge}$, the function $\langle X_1 \wedge \dots \wedge X_k| \circ \omega$ belongs to C. According to (1.7) function (1.6) belongs to C. This ends the proof.

Let us remark that, denoting the set of all elements of $\bigwedge^k T_p(M, C)$ $(\bigwedge^k(T_p(M, C))^*)$ by set $\bigwedge^k T_p(M, C)$ (set $\bigwedge^k(T_p(M, C))^*$) and setting

$$\operatorname{set} \bigwedge^k T(M, C) = \bigcup_{p\in M} \operatorname{set} \bigwedge^k T_p(M, C)$$

and

$$\operatorname{set} \bigwedge^k T^*(M, C) = \bigcup_{p\in M} \operatorname{set} \bigwedge^k (T_p(M, C))^*,$$

we have

$$\operatorname{set} \bigwedge T(M, C) = \bigcup_{k=0}^{m} \operatorname{set} \bigwedge^k T(M, C)$$

and

$$\operatorname{set} \bigwedge T^*(M, C) = \bigcup_{k=0}^{m} \operatorname{set} \bigwedge^k T^*(M, C).$$

PROPOSITION 1.2. *For a differential space (M, C) of finite dimension the differential structure induced* (see [8]) *from $\bigwedge T(M, C)$ ($\bigwedge T^*(M, C)$) to* $\operatorname{set} \bigwedge^k T(M, C)$ ($\operatorname{set} \bigwedge^k T^*(M, C)$) *coincides with the smallest differential structure containing the set obtained from*

$$\{\alpha \circ \pi^{\wedge};\ \alpha \in C\} \cup \{|d\alpha_1 \wedge \ldots \wedge d\alpha_k\rangle;\ \alpha_1, \ldots, \alpha_k \in C\}, \tag{1.8}$$

$$\{\alpha \circ \overset{*}{\pi}{}^{\wedge};\ \alpha \in C\} \cup \{\langle X_1 \wedge \ldots \wedge X_k|;\ X_1, \ldots, X_k \in \mathscr{X}(M, C)\} \tag{1.9}$$

by the restriction of each function to $\operatorname{set} \bigwedge^k T(M, C)$ ($\operatorname{set} \bigwedge^k T^*(M, C)$).

Proof. The fact that the set of all functions obtained from (1.8) by the restriction to $\operatorname{set} \bigwedge^k T(M, C)$ is contained in the differential structure induced from (1.1) to $\operatorname{set} \bigwedge^k T(M, C)$ is obvious. To prove the inverse inclusion let us take any $\alpha_1, \ldots, \alpha_l \in C$, where $l \neq k$, and $t \in \operatorname{set} \bigwedge^k T(M, C)$. Then, by definition of the function $|d\alpha_1 \wedge \ldots \wedge d\alpha_k\rangle$ we have

$$|d\alpha_1 \wedge \ldots \wedge d\alpha_l\rangle(t) = \langle t|(d\alpha_1)_p \wedge \ldots \wedge (d\alpha_l)_p\rangle = 0, \quad p = \pi^{\wedge}(t).$$

Thus, the restriction of the function $|d\alpha_1 \wedge \ldots \wedge d\alpha_l\rangle$ to $\operatorname{set} \bigwedge^k T(M, C)$ vanishes. So, it belongs to the smallest differential structure containing the set of all restrictions of functions of set (1.8) to $\operatorname{set} \bigwedge^k T(M, C)$. The proof of the dual part of Proposition 1.2 is similar.

2. Two points of view on smooth k-forms

Proposition 1.2 allows us to consider $\operatorname{set} \bigwedge^k T(M, C)$ ($\operatorname{set} \bigwedge^k T^*(M, C)$) together with the smallest differential structure containing the set of all restrictions to $\operatorname{set} \bigwedge^k T(M, C)$ (to $\operatorname{set} \bigwedge^k T^*(M, C)$) of functions belonging to set (1.8) (to set (1.9)) as a differential subspace of $\bigwedge T(M, C)$ ($\bigwedge T^*(M, C)$). This subspace will be denoted by $\bigwedge^k T(M, C)$ ($\bigwedge^k T^*(M, C)$). Denoting by π^k (by $\overset{*}{\pi}{}^k$) the restriction of $\pi^{\wedge}$ (of $\overset{*}{\pi}{}^{\wedge}$) to $\operatorname{set} \bigwedge^k T(M, C)$ (to $\operatorname{set} \bigwedge^k T^*(M, C)$) we have the smooth mappings

$$\pi^k\colon \bigwedge^k T(M, C) \to (M, C)$$

and

$$\overset{*}{\pi}{}^k\colon \textstyle\bigwedge^k T^*(M, C) \to (M, C).$$

These mappings will be called the *projections of bundles* $\bigwedge^k T(M, C)$ and $\bigwedge^k T^*(M, C)$, respectively.

Now, each of the smooth exterior forms of degree k may be considered as a smooth section of the bundle $\bigwedge^k T^*(M, C)$, i.e., as a smooth mapping

$$\omega\colon (M, C) \to \textstyle\bigwedge^k T^*(M, C) \tag{2.1}$$

such that $\overset{*}{\pi}{}^k \circ \omega = \mathrm{id}_M$.

Let us remark that we may consider skew symmetric k-C-linear mappings

$$\theta\colon (\mathscr{X}(M, C))^k \to C,$$

where $(\mathscr{X}(M, C))^k$ is kth Cartesian power of the set $\mathscr{X}(M, C)$. The set of all such mappings θ with addition and multiplication by functions belonging to C is the C-module. Denote this C-module by k-sklin(M, C). On the other hand, under the assumption of finite dimension of (M, C) we have the C-module of all smooth exterior forms (2.1) with addition and multiplication by elements of C defined in the natural way. Denote this C-module by $\Gamma(\bigwedge^k T^*(M, C))$. Assigning to every ω in $\Gamma(\bigwedge^k T^*(M, C))$ the mapping $\bar{\omega}$ defined by the formula

$$\bar{\omega}(X_1, \ldots, X_k)(p) = \langle X_1(p) \wedge \ldots \wedge X_k(p) | \omega(p)\rangle \tag{2.2}$$

for $p \in M$, $X_1, \ldots, X_k \in \mathscr{X}(M, C)$ we obtain the C-linear mapping

$$\omega \mapsto \bar{\omega}\colon \Gamma(\textstyle\bigwedge^k T^*(M, C)) \to k\text{-sklin}(M, C). \tag{2.3}$$

PROPOSITION 2.1. *For any differential space* (M, C) *of finite dimension the* C-*linear mapping* (2.3) *is an isomorphism of* C-*modules.*

Proof. To prove that (2.3) is one-one, take ω from $\Gamma(\bigwedge^k T^*(M, C))$ and suppose that $\bar{\omega} = 0$. Let $p \in M$. Then there exist a neighbourhood U of p and a local basis $(V_1, \ldots, V_k)$ of (M, C) on U. Then for any $i_1, \ldots, i_k \in \{1, \ldots, m\}$ we have

$$0 = \bar{\omega}(V_{i_1}, \ldots, V_{i_k})(p) = \langle V_{i_1}(p) \wedge \ldots \wedge V_{i_k}(p) | \omega(p)\rangle.$$

From (a) of Section 1 it follows that $\omega(p) = 0$.

Let now θ be any element of k-sklin(M, C). From the hypothesis that (M, C) is of finite dimension it follows that $\mathscr{X}(M, C)$ is an R. Sikorski's differential module (see [6]). Then, for any $p \in M$ and any $v_1, \ldots, v_k$ in $T_p(M, C)$ we get well defined real number $\theta_p(v_1, \ldots, v_k)$ such that for any $X_1, \ldots, X_k \in \mathscr{X}(M, C)$ satisfying the equalities $X_i(p) = v_i$, $i = 1, \ldots, k$,

$$\theta_p(v_1, \ldots, v_k) = \theta(X_1, \ldots, X_k)(p). \tag{2.4}$$

From k-C-linearity and skew-symmetry of θ it follows that we have got the k-$\boldsymbol{R}$-linear and skew-symmetric mapping

$$\theta_p\colon (T_p(M, C))^k \to \boldsymbol{R}.$$

This yields the existence of the linear mapping

$$\tilde{\theta}_p \colon \bigwedge^k T_p(M, C) \to \boldsymbol{R}$$

such that

$$\tilde{\theta}_p(v_1 \wedge \ldots \wedge v_k) = \theta_p(v_1, \ldots, v_k) \tag{2.5}$$

for any $v_1, \ldots, v_k$ in $T_p(M, C)$. Then by duality there exists exactly one $\omega(p)$ in $\bigwedge^k (T_p(M, C))^*$ such that

$$\langle v_1 \wedge \ldots \wedge v_k | \omega(p) \rangle = \tilde{\theta}_p(v_1 \wedge \ldots \wedge v_k) \quad \text{for} \quad v_1, \ldots, v_k \quad \text{in } T_p(M, C).$$

This, (2.4) and (2.5) yields

$$\langle X_1(p) \wedge \ldots \wedge X_k(p) | \omega(p) \rangle = \theta(X_1, \ldots, X_k)(p).$$

We get by (2.2) that $\bar{\omega}(X_1, \ldots, X_k) = \theta(X_1, \ldots, X_k)$. From Proposition 1.1 it follows that ω is a smooth form of degree k on (M, C) and we have $\bar{\omega} = \theta$. This ends the proof.

3. Exterior differential

R. Sikorski in the paper [7] has considered the exterior differential of skew-symmetric forms on abstract differential module. In particular, we have the exterior differential of forms treated as elements of k-skewlin(M, C). The exterior differential of such forms is defined by the standard formula:

$$d\theta(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \partial_{X_i} \theta(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{k+1}) + \\ + \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k+1}), \tag{3.1}$$

where $[X_i, X_j]$ is the Lie bracket of X_i and X_j, for $X_1, \ldots, X_n$ in $\mathscr{X}(M, C)$. Correctness of definition of the operation d by formula (3.1) does not require the hypothesis that (M, C) is of finite dimension, while we assume this hypothesis to correctly define the exterior differential of forms treated as an element of $\Gamma(\bigwedge^k T^*(M, C))$. The exterior differential of forms, which are the elements of $\Gamma(\bigwedge^k T^*(M, C))$, we shall denote by the same symbol d. $\bigwedge^0 T^*(M, C)$ may be identified with C.

PROPOSITION 3.1. *If (M, C) is of finite dimension, then there exists exactly one operation which assigns to every ω in $\Gamma(\bigwedge^k T^*(M, C))$ the element $d\omega$ of $\Gamma(\bigwedge^{k+1} T^*(M, C))$,*

$$\langle X_1(p) \wedge \ldots \wedge X_{k+1}(p) | d\omega(p) \rangle = d\bar{\omega}(X_1, \ldots, X_{k+1})(p), \quad p \in M, \tag{3.2}$$

for $X_1, \ldots, X_{k+1}$ in $\mathscr{X}(M, C)$, $k = 0, 1, \ldots, m$. The operation d satisfies the following conditions:

(i) *$d\alpha(p)(v) = v(\alpha)$ for v in $T_p(M, C)$ and $\alpha \in C$,*

(ii) *$d(\omega + \omega_1) = d\omega + d\omega_1$ for ω, ω_1 in $\Gamma(\bigwedge^k T^*(M, C))$,*

(iii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ *for* ω *in* $\Gamma(\bigwedge^k T^*(M, C))$ *and* η *in* $\Gamma(\bigwedge^l T^*(M, C))$,

(iv) $d \circ d = 0$.

Proof. It follows from Proposition 2.1 and results of [7] by an easy verification.

The properties of exterior differential allow us to define the de Rham cohomology groups for differential spaces. There is evident way of introducing the singular homology groups for differential spaces. To get the natural duality between de Rham cohomology groups and singular homology groups the Stokes's formula on differential space would be useful.

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