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ON FRACTIONAL DERIVATIVES

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In this paper we shall prove a theorem of Bernstein's type for fractional derivatives in the sense of Weyl, extending some results of Montel [1], p. 175.

1. Let f be a 2π-periodic function integrable in $(0, 2\pi)$ and such that

$$\int_0^{2\pi} f(x)\,dx = 0.$$

Consider $0 < a < 1$, and write

$$(*) \qquad F(x) = \frac{1}{\Gamma(1-a)} \int_{-\infty}^{x} (x-z)^{-a} f(z)\,dz$$

for all these x at which the integral exists. $F(x)$ is defined almost everywhere (see [5], p. 134-135).

If $F(x)$ has a derivative $F'(x)$ a. e., then F' is called the *derivative of order a of the function f in the sense of Weyl* and is denoted by $f^{(a)}$.

It is known (see [5], p. 135) that

$$F(x) = \frac{1}{\Gamma(1-a)} \int_0^x f(z)\,(x-z)^{-a}\,dz + \frac{1}{2\pi} \int_0^{2\pi} f(z) r_{1-a}(x-z)\,dz$$

for $x \in (0, 2\pi)$, where

$$r_a(t) = \frac{2\pi}{\Gamma(a)} \lim_{k\to\infty} \left\{ \sum_{\nu=1}^{k} (t+2\pi\nu)^{a-1} - (2\pi)^{a-1} \frac{k^a}{a} \right\} \quad \text{for } t \in (-2\pi, 2\pi\rangle.$$

We shall assume that the functions f under consideration belong to the space L^p in the interval $(0, 2\pi)$, where $1 < p \leqslant \infty$, with the norm

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\,dx \right)^{1/p} \quad \text{if } 1 < p < \infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0<x<2\pi} |f(x)| = \lim_{p\to\infty} \|f\|_p.$$

First, we prove the following auxiliary result concerning derivatives of order a of a sequence of functions from L^p:

THEOREM 1. *Let f and f_n $(n = 1, 2, \ldots)$ be 2π-periodic functions belonging to L^p, $1 < p \leqslant \infty$, and let*

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} f_n(x)\,dx = 0.$$

If the functions f_n have derivatives $f_n^{(a)}$ of order a in the sense of Weyl for an a satisfying the inequalities $0 < a < 1-1/p$, and if $\|f_n - f\|_p \to 0$, $\|f_n^{(a)} - G\|_p \to 0$ as $n \to \infty$ for a function $G \in L^p$, then the function F defined by (*) is equivalent to an absolutely continuous function whose derivative is equal to G almost everywhere, i. e. $f^{(a)}(x) = G(x)$ for almost every x, and $f^{(a)} \in L^p$.

The proof will be carried out in the case $1 < p < \infty$; the case $p = \infty$ is obtained immediately, passing to limit as $p \to \infty$.

Denote by $F_n(x)$ the right-hand side of (*) in which $f(z)$ is replaced by $f_n(z)$. Then

$$F(x) - F_n(x) = \frac{1}{\Gamma(1-a)} \int_0^x (x-z)^{-a} [f(z) - f_n(z)]\,dz + \\ + \frac{1}{2\pi} \int_0^{2\pi} [f(z) - f_n(z)]\, r_{1-a}(x-z)\,dz.$$

By Minkowski's inequality,

$$\Big(\int_0^{2\pi} |F(x) - F_n(x)|^p dx\Big)^{1/p} \leqslant \frac{1}{\Gamma(1-a)} \Big\{\int_0^{2\pi} \Big|\int_0^x (x-z)^{-a} [f(z) - f_n(z)]\,dz\Big|^p dx\Big\}^{1/p} + \\ + \frac{1}{2\pi} \Big\{\int_0^{2\pi} \Big|\int_0^{2\pi} [f(z) - f_n(z)]\, r_{1-a}(x-z)^{-a}\,dz\Big|^p dx\Big\}^{1/p} = \frac{1}{\Gamma(1-a)} A + \frac{1}{2\pi} B.$$

Writing $q = p/(p-1)$ and applying the Hölder inequality, we estimate A as follows:

$$A \leqslant \frac{1}{(1-aq)^{1-1/p}} \Big\{\int_0^{2\pi} \Big[\int_z^{2\pi} x^{(1-aq)(p-1)} |f(z) - f_n(z)|^p dx\Big]\, dz\Big\}^{1/p} \\ \leqslant \frac{1}{(1-aq)^{(p-1)/p}}\, \frac{(2\pi)^{1-a}}{p^{1/p}(1-a)^{1/p}}\, \|f - f_n\|_p.$$

In order to estimate B, let us remark that

$$r_{1-a}(x-z) = \frac{2\pi}{\Gamma(1-a)}\left[(x-z+2\pi)^{-a}+\varrho_{1-a}(x-z)\right],$$

where $\varrho_{1-a}(x-z)$ is continuous and thus bounded in the rectangle $[0, 2\pi;\ 0, 2\pi]$. Hence

$$B \leqslant \frac{2\pi}{\Gamma(1-a)}\left\{\left[\int_0^{2\pi}\left|\int_0^{2\pi}\big(f(z)-f_n(z)\big)(x-z+2\pi)^{-a}dz\right|^p dx\right]^{1/p}+\right.$$

$$\left.+\left[\int_0^{2\pi}\left|\int_0^{2\pi}\big(f(z)-f_n(z)\big)\varrho_{1-a}(x-z)\,dz\right|^p dx\right]^{1/p}\right\}.$$

However, by the Hölder inequality,

$$\left\{\int_0^{2\pi}\left|\int_0^{2\pi}\big(f(z)-f_n(z)\big)(x-z+2\pi)^{-a}dz\right|^p dx\right\}^{1/p} \leqslant \frac{(4\pi)^{1/q-a}(2\pi)^{1/p}}{(1-aq)^{1/q}}\|f-f_n\|_p.$$

Analogously, if $\varrho_{1-a}(x-z)\leqslant M$, where M is a constant, then we have

$$\left\{\int_0^{2\pi}\left|\int_0^{2\pi}\big(f(z)-f_n(z)\big)\varrho_{1-a}(x-z)\,dz\right|^p dx\right\}^{1/p} \leqslant 2\pi M\|f-f_n\|_p.$$

Therefore,

$$\|F-F_n\|_p \leqslant C_{ap}\|f-f_n\|_p,$$

where C_{ap} is a constant, and $C_{ap}\leqslant C_a$ for $1<p<\infty$, where C_a does not depend on p. Thus $\|F-F_n\|_p\to 0$. Consequently, $F_n(x)\to F(x)$ in measure. Hence, there exists a subsequence $F_{m_n}(x)$ such that $F_{m_n}(x)\to F(x)$ a. e. Let $\bar{x}\in(0,2\pi)$ be such a point that $F_{m_n}(\bar{x})\to F(\bar{x})$. By assumption, $\|F_n'-G\|_p\to 0$.

Let us write

$$R(x) = \int_{\bar{x}}^{x} G(t)\,dt+F(\bar{x}).$$

We show that $R(x) = F(x)$ a. e. Indeed, we have

$$\int_0^{2\pi}|F_{m_n}'(x)-G(x)|\,dx \leqslant (2\pi)^{1/q}\|F_{m_n}'-G\|_p\to 0.$$

Hence

$$\int_{\bar{x}}^{x} F_{m_n}'(x)\,dx \to \int_{\bar{x}}^{x} G(x)\,dx = R(x)-F(\bar{x}).$$

But $F_{m_n}(x)$ are absolutely continuous, by the assumption. Hence

$$\int_{\bar{x}}^{x} F'_{m_n}(t)\,dt = F_{m_n}(x) - F_{m_n}(\bar{x}).$$

Thus $F_{m_n}(x) - F_{m_n}(\bar{x}) \to R(x) - F(\bar{x})$ a. e. Hence, we conclude that $R(x) = F(x)$ a. e. Consequently,

$$F(x) = \int_{\bar{x}}^{x} G(t)\,dt + F(\bar{x}) \qquad \text{a. e.,}$$

and so $F(x)$ is equivalent to an absolutely continuous function whose derivative is equal to $G(x)$ a. e. Thus, Theorem 1 is proved (cf. [3], p. 533).

2. Consider a 2π-periodic real function f Lebesgue-integrable over $\langle 0, 2\pi\rangle$, with complex Fourier coefficients c_k ($c_0 = 0$). Given a number $a > 0$, we set

$$I_a(x; f) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{(ik)^a},$$

where $i^a = e^{ia\pi/2}$ for all these x at which the series converges, and

$$f^{(a)}(x) = \frac{d^\nu}{dx^\nu} I_{\nu-a}(x; f), \qquad \text{where } \nu = [a]+1.$$

The function $f^{(a)}$ is called the *fractional derivative* of f if $a \neq 1, 2, \ldots$ In case $0 < a < 1$, $I_{1-a}(x; f)$ coincides with $F(x)$ defined by (*) (see [5], p. 134-135).

If f is a trigonometric polynomial such that

$$\int_0^{2\pi} f(x)\,dx = 0,$$

then the derivative of order a $(0 < a < 1)$ of f in the sense of Weyl is equal to the derivative in the sense of [2].

Now, we prove the following theorem of Bernstein's type, mentioned above:

THEOREM 2. *Let*

$$f \in L^p, \qquad \int_0^{2\pi} f(x)\,dx = 0, \qquad 0 < a < 1,\ 1 < p \leqslant \infty.$$

If there exists a sequence of trigonometric polynomials $\{T_n\}$, T_n *of degree not greater than* n, *such that*

$$\|f - T_n\|_p \leqslant \frac{C}{n^a}$$

for a constant $C > 0$ and $n = 1, 2, \ldots$, then f has almost everywhere a derivative $f^{(\beta)}$ of an arbitrary order β satisfying the conditions

$$0 < \beta < \min(a, 1-1/p) \quad \textit{and} \quad f^{(\beta)} \in L^p.$$

Let us remark that if we denote by $E_n(f)_p$ the best approximation of the function $f \in L^p$ by trigonometric polynomials of degrees not exceeding n in metric L^p, then the assumption of Theorem 2 can be written in the form $E_n(f)_p = O(n^{-a})$.

Proof of Theorem 2. First, we show that if $0 < \beta < a$, then the sequence $\{T_n^{(\beta)}\}$ of derivatives of order β of trigonometric polynomials T_n satisfies the Cauchy condition in L^p. Let $k < n$ and let numbers r and m be chosen in such a manner that $2^{r-1} \leqslant k < 2^r$, $2^{m-1} \leqslant n < 2^m$. Then

$$\|T_n^{(\beta)} - T_k^{(\beta)}\|_p \leqslant \|T_{2^r}^{(\beta)} - T_k^{(\beta)}\|_p + \sum_{j=r}^{\infty} \|T_{2^{j+1}}^{(\beta)} - T_{2^j}^{(\beta)}\|_p + \|T_m^{(\beta)} - T_{2^m}^{(\beta)}\|_p.$$

By Theorem 3 of [2],

$$\|T_{2^r}^{(\beta)} - T_k^{(\beta)}\|_p \leqslant \frac{7}{\beta} 2^{r\beta} \|T_{2^r} - T_k\|_p \leqslant \frac{7}{\beta} 2^{r\beta} \frac{2C}{k^a} \leqslant \frac{7}{\beta} \frac{2^{a+1}C}{2^{r(a-\beta)}},$$

and, similarly,

$$\|T_n^{(\beta)} - T_{2^m}^{(\beta)}\|_p \leqslant \frac{7}{\beta} \frac{2^{1+\beta}C}{2^{r(a-\beta)}},$$

$$\|T_{2^{j+1}}^{(\beta)} - T_{2^j}^{(\beta)}\|_p \leqslant \frac{7}{\beta} \frac{2^{1+\beta}C}{2^{j(a-\beta)}}.$$

Hence,

$$\sum_{j=r}^{\infty} \|T_{2^{j+1}}^{(\beta)} - T_{2^j}^{(\beta)}\|_p \leqslant \frac{7}{\beta} C 2^{1+\beta} \frac{2^{(\beta-a)r}}{1-2^{\beta-a}}.$$

Thus we have, for $k < n$,

$$\|T_n^{(\beta)} - T_k^{(\beta)}\|_p \leqslant \frac{C^*}{2^{(a-\beta)r}} \leqslant \frac{C^*}{k^{a-\beta}},$$

where C^* is a constant. Consequently, the sequence $\{T_n^{(\beta)}\}$ satisfies the Cauchy condition in L^p. Since L^p is complete, there exists a function $G \in L^p$ such that $\|T_n^{(\beta)} - G\|_p \to 0$. On the other hand, we have $\|T_n - f\|_p \to 0$.

First, let us suppose that

$$\int_0^{2\pi} T_n(x)\,dx = 0 \quad \text{for } n = 1, 2, \ldots$$

Then, by Theorem 1, the function f has derivative $f^{(\beta)}$ of order β such that

$$0 < \beta < \min(a, 1-1/p).$$

Now, we remove the assumption

$$\int_0^{2\pi} T_n(x)\,dx = 0.$$

Let

$$\int_0^{2\pi} T_n(x)\,dx = 2\pi a_0^{(n)},$$

and let

$$\tilde{T}_n(x) = T_n(x) - a_0^{(n)}.$$

Then

$$\int_0^{2\pi} \tilde{T}_n(x)\,dx = 0.$$

But

$$|2\pi a_0^{(n)}| = \left|\int_0^{2\pi} \big(f(x) - T_n(x)\big)\,dx\right| \leqslant (2\pi)^{1/q}\|f - T_n\|_p.$$

Hence

$$\|f - \tilde{T}_n\|_p \leqslant \|f - T_n\|_p + (2\pi)^{1/p}|a_0^{(n)}| \leqslant 2\|f - T_n\|_p \leqslant \frac{2C}{n^a}.$$

Thus, the function f and trigonometric polynomials $\tilde{T}_n$ satisfy the assumptions of the first part of the proof. Hence, $f^{(\beta)}$ exists almost everywhere, and $f^{(\beta)} \in L^p$.

Theorem 2 can be extended as follows:

THEOREM 3. *Let*

$$f \in L^p, \quad \int_0^{2\pi} f(x)\,dx = 0, \quad 0 < a < 1, \quad 1 < p \leqslant \infty,$$

and let r be a positive integer. If there exists a sequence $\{T_n\}$ of trigonometric polynomials, T_n of degree not greater than n, such that

$$\|f - T_n\|_p \leqslant \frac{C}{n^{r+a}}$$

for a constant C and $n = 1, 2, \ldots$, then f has almost everywhere the derivative $f^{(r+\beta)}$ of an arbitrary order $r+\beta$ for β such that

$$0 < \beta < \min(a, 1 - 1/p) \quad \textit{and} \quad f^{(r+\beta)} \in L^p.$$

Proof. By virtue of the well-known theorem of Bernstein's type (see [4], p. 350), the function f is equivalent to a function φ having absolutely continuous derivative $\varphi^{(r-1)}$ and the derivative $\varphi^{(r)}$ of class $\mathrm{Lip}(a, p)$, $\varphi^{(r)} \in L^p$. In view of the Jackson theorem, in the space L^p (see [4], p. 274-275), $E_n(\varphi^{(r)})_p = O(n^{-a})$ as $n \to \infty$.

Next, Theorem 2 ensures that $\varphi^{(r)}$ has the derivative $(\varphi^{(r)})^{(\beta)}$ a. e., whenever $0 < \beta < \min(a, 1-1/p)$. Moreover, $(\varphi^{(r)})^{(\beta)} \in L^p$.

Observing that

$$\big(\varphi^{(r)}(x)\big)^{(\beta)} = \varphi^{(r+\varphi)}(x) = f^{(r+\beta)}(x) \qquad \text{a. e.,}$$

we get at once the desired assertion.

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