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Some relations between starlike and convex functions

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Abstract. Let S be the class of regular and univalent functions in K_1 ($K_r = \{z: |z| \leqslant r\}$) such that $f(0) = 0$, $f'(0) = 1$ and let S^*, S^c be the subclasses of functions $f \in S$ starshaped w.r.t. the origin and convex, resp. Let S_R denote the class of all functions $f \in S$ such that

$$K_R \subset f(K_1).$$

The intersections $S^* \cap S_R$, $S^c \cap S_R$ will be denoted S_R^* and S_R^c resp.

It is well known that if a function $g \in S^*$, then

$$f(z) = \int_0^z \frac{g(t)}{t}\, dt \tag{1}$$

belongs to the class S^c.

This note deals with the following problem: given $g \in S_R^*$, find "the best possible" ϱ (depend on R only) such that the function f of form (1) belongs to S_ϱ^c.

1. Introduction. Let S^* denote the class of functions

$$z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in $|z| < 1$ and such that each $g \in S^*$ maps $|z| < 1$ one-to-one onto a domain starshaped with respect to the origin. This is equivalent to the analytic condition: $\operatorname{re} \dfrac{zg'(z)}{g(z)} > 0$ in $|z| < 1$.

An analytic function f of form (1.1) is said to be *convex* ($f \in S^c$) if it maps $|z| < 1$ one-to-one onto a convex domain. Analytically this is equivalent to that $\operatorname{re} \left(1 + \dfrac{zf''(z)}{f'(z)}\right) > 0$ in $|z| < 1$.

It is very easy to check that the integral

$$\int_0^z \frac{g(\xi)}{\xi}\, d\xi \tag{1.2}$$

transforms S^* onto the class S^c of all convex functions of form (1.1).

Suppose that $K_r = \{z: |z| < r\}$. Let $S_R^* (\frac{1}{4} \leqslant R \leqslant 1)$, $S_R^* \subset S^*$, denote the class of starlike functions g satisfying

$$K_R \subset g(K_1). \tag{1.3}$$

Moreover, let $S_\varrho^c (\frac{1}{2} \leqslant \varrho \leqslant 1)$ be the class of convex functions f satisfying

$$K_\varrho \subset f(K_1). \tag{1.4}$$

We recall that $S_{1/4}^* = S^*$, $S_{1/2}^c = S^c$, $S_1^* = S_1^c = \{z\}$ (cf. [3], p. 3, 13, 80).

This means that the integral (1.2) transforms the classes $S_{1/4}^*$, S_1^* onto $S_{1/2}^c$, S_1^c resp.

In this communication I shall present the connection between the class S_R^* and S_ϱ^c under transformation (1.2). In the further considerations we shall use the following lemmas.

LEMMA 1 [2]. *Suppose* $g \in S_R^* (\frac{1}{4} < R < 1)$. *Then*

$$-G(-|z|) \leqslant |g(z)| \leqslant G(|z|), \tag{1.5}$$

where

$$G(z) = \frac{4}{(2+a)^a} \frac{[a(1+z)+2\lambda(z)]^a}{[1+z+\lambda(z)]^2} \frac{z}{(1-z)^a} \tag{1.6}$$

with

$$\lambda(z) = [1+(a^2-2)z+z^2]^{1/2}, \qquad 0 < a < 2.$$

The connection between a and R is given by

$$R = 4[(2-a)^{2-a}(2+a)^{2+a}]^{-1/2}. \tag{1.7}$$

The function G maps the unit disk K_1 onto starshaped domain being the sum of the K_R and the angle $\{w: |\mathrm{Arg} w| < \pi a/2\}$ (see [6]).

Let us put

$$F(z) = \int_0^z \frac{G(\xi)}{\xi} d\xi, \tag{1.8}$$

with G defined by (1.6). Examining the behaviour of the boundary of $G(K_1)$ under transformation (1.2) we find that $F(K_1)$ is the convex circular domain symmetric w.r.t. the real axis whose boundary consists of an arc situated on the boundary of the disk K_ϱ and two half straight-lines (or segments) starting from the end points that arc and tangent to ∂K_ϱ (see [7]). Of course, ϱ depend on R. That dependence is given by the formula

$$\varrho = \int_{-1}^0 \frac{G(t)}{t} dt. \tag{1.9}$$

After some calculations we obtain that

$$(1.10)\qquad \varrho = \varrho(a) = 16a\int_q^1 t^a \frac{(2+a)t^2+2-a}{[(2+a)^2t^2-(2-a)^2]^2}\,dt,$$

where

$$q = \sqrt{\frac{2-a}{2+a}}.$$

LEMMA 2 [7]. *Suppose that* $f\in S_\varrho^c$ $(\frac{1}{2}<\varrho<1)$. *Then*

$$(1.11)\qquad -F(-|z|)\leqslant |f(z)|\leqslant F(|z|),$$

where F *is given by* (1.8).

The method of the proofs of Lemmas 1, 2 is based on the fact that the problem of determining the extremal values of $|f(z)|$ is equivalent to the extremal problem for the Green's function in a class of domains which satisfy some additional conditions (cf. [1], [4]).

2. Main results.

THEOREM 1. *Suppose that*

$$g\in S_R^*,\qquad f(z) = \int_0^z \frac{g(\xi)}{\xi}\,d\xi.$$

Then $f\in S_\varrho^c$, *where* ϱ *is defined by* (1.9) *or* (1.10).

Proof. Let $z\in K_1$. The length of the segment $[0, f(z)]$ is equal to $|f(z)|$. On the other hand

$$|f(z)| = \int_L |f'(z)|\,|dz|,\qquad \text{where}\qquad L = \{z\colon z = f^{-1}(\xi),\ \xi\in[0, f(z)]\}.$$

Changing a parametrization of the curve L we obtain:

$$|f(z)| = \int_0^{|z|} |f'(re^{i\theta})|\,dr = \int_0^{|z|}\left|\frac{g(er^{i\theta})}{r}\right|dr.$$

Now, from (1.5) we have:

$$|f(z)| = \int_0^{|z|}\left|\frac{g(re^{i\theta})}{r}\right|dr \geqslant \int_0^{|z|}\frac{-G(-r)}{r}\,dr = \int_{-|z|}^0 \frac{G(t)}{t}\,dt.$$

Hence, letting $|z|\to 1$ gives $\min_{|z|=1}|f(z)|\geqslant\varrho$. Consequently, $K_\varrho\subset f(K_1)$ and finaly $f\in S_\varrho^c$.

Remark. From Theorem 1 we obtain that integral (1.2) transforms S_R^* into S_ϱ^c with connection between R and ϱ given by (1.7) or (1.10).

It follows from (1.10) that $\varrho(1) = \frac{4}{9}+\frac{16}{27}\ln 2 = 0.855\ldots$

THEOREM 2. *If $\varrho > \varrho(1)$, then S_ϱ^c is the class of bounded convex functions.*

Proof. If $\varrho = \varrho(a) > \varrho(1)$, then ϱ corresponds a, $0 < a < 1$. It follows that for $f \in S_\varrho^c$

$$|f(z)| \leqslant F(|z|) = \int_0^{|z|} \frac{G(t)}{t}\, dt.$$

From (1.6) we get the following inequality:

$$|f(z)| \leqslant M[1-(1-|z|)^{1-a}] \quad \text{which yields} \quad |f(z)| \leqslant M.$$

Let us denote by H^p $(p > 0)$ the class of functions $f(z)$ analytic in $|z| < 1$ and such that the integral

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

is bounded in $0 \leqslant r < 1$.

THEOREM 3. *Suppose that $R > \frac{4}{9}\sqrt{3} = 0.76\ldots$*
Then $g \in S^*$ implies g belongs to H^1.

Proof. Let $f(z) = \int_0^z \frac{g(\xi)}{\xi} d\xi$ with $g \in S_R^*$. We point out that $f \in S_\varrho^c$ with $\varrho > \varrho(1)$ so that f is bounded.

Now

$$\int_0^{2\pi} |g(re^{i\theta})|\, d\theta = r\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta$$

and right-hand side of the above equality denotes the length of convex curve onto which $|z| = r$ is mapped by f. Thus

$$\int_0^{2\pi} |g(re^{i\theta})|\, d\theta \quad \text{is bounded in} \quad 0 \leqslant r < 1.$$

The proof is complete.

Let us denote $M(r, T) = \operatorname{Max}_{|z|=r} |T(z)|$. From the estimations (1.5), (1.11) we obtain

COROLLARY. *Suppose that $g \in S_R^*$, $f \in S_\varrho^c$. Then for $r \to 1^-$*

$$M(r, g) = O\left(\frac{1}{(1-r)^a}\right),$$

$$M(r,f) = \begin{cases} O\left(\dfrac{1}{(1-r)^{a-1}}\right) & \textit{provided } \varrho < \varrho(1)\ (1 < a < 2), \\[2mm] O\left(\log \dfrac{1}{1-r}\right) & \textit{provided } \varrho = \varrho(1)\ (a = 1). \end{cases}$$

From the result of Corollary we can obtain an asymptotic behaviour of the coefficients a_n in Taylor expansion (1.1) if $z+\sum_{n=2}^{\infty} a_n z^n$ belongs to S_R^* or S_ϱ^c (see [5]).

References

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