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# Periodic solutions of $x''+f(x)x'^{2n}+g(x)=\mu p(t)$

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**1.** We shall deal with the second-order non-linear differential equation

$$x''+f(x)x'^{2n}+g(x)=\mu p(t)\,, \tag{1}$$

where $n\geqslant 1$ is an integer, $\mu$ is a parameter, functions $f(x)$, $g(x)$ and $p(t)$ are continuous in their respective arguments $(-\infty<x,t<\infty)$ and, for a positive $\omega$, $p(t+\omega)\equiv p(t)$.

In the present note we state the sufficient conditions for the existence of periodic solutions of (1) with period $\omega$ (Theorem 1). The cases $n=1$ and $f(x)\equiv 0$ are considered separately (Theorems 2 and 3). Equation (1) with $f(x)\equiv 0$ has been considered by Kulig [2]. Theorem 3 gives a generalization of her result.

**2.** Define the function $G(x)$ by

$$G(x)=\int_0^x g(u)\,du\,.$$

THEOREM 1. *Assume that*

$$xg(x)>0 \quad \textit{for } x\neq 0\,, \tag{2}$$

$$\lim_{|x|\to\infty} G(x)=\infty\,, \tag{3}$$

$$\limsup_{x\to-\infty} g(x)/x=a\,, \tag{4}$$

$$0<b\leqslant f(x)\leqslant c<\infty \quad \textit{for all } x\,. \tag{5}$$

*Then for every $p(t)$ continuous and periodic with period $\omega$, there is $\mu_0>0$ such that for all $|\mu|<\mu_0$ (1) has at least one periodic solution with period $\omega$.*

Before proceeding to the proof of Theorem 1, replace (1) by the equivalent system

$$x'=y\,,\quad y'=-f(x)y^{2n}-g(x)+\mu p(t) \tag{6}$$

and consider the autonomous system

$$x'=y\,,\quad y'=-f(x)y^{2n}-g(x) \tag{7}$$

corresponding to (6). It will be assumed in the sequel that (1) has the property of uniqueness of solutions.

The proof of Theorem 1 will be based on the following lemma, due to Bernstein and Halanay [1].

LEMMA. *If* (7) *has a periodic solution with period* $\omega_0$ *and if* $\omega \neq \omega_0$, *then for* $|\mu|$ *small enough* (6) *has a periodic solution with period* $\omega$.

For the completeness of the proof, we shall prove Lemma. Our proof differs from the argumentation used in [1].

Proof of Lemma. Let $\Gamma$ be the simple-closed curve in the $(x, y)$-plane representing the periodic solution of (7). Let $\Omega$ be the domain containing the origin, bounded by $\Gamma$. Denote by $x = x(t; \mu, x_0, y_0)$, $y = y(t; \mu, x_0, y_0)$ the solution of (6) satisfying

$$x(0; \mu, x_0, y_0) = x_0, \quad y(0; \mu, x_0, y_0) = y_0$$

and define in the $(x, y)$-plane the vector fields $C$, $C_\mu$:

$$C: (x, y) \to v(x, y) = (-x, -y),$$

$$C_\mu: (x, y) \to v_\mu(x, y) = \big(x(\omega; \mu, x, y) - x, y(\omega; \mu, x, y) - y\big).$$

To prove the lemma it remains to show that $C_\mu$ has a singular point. Since $\Gamma$ is compact and $x(\omega; \mu, x, y)$, $y(\omega; \mu, x, y)$ are continuous in $(\mu, x, y)$, from $\omega \neq \omega_0$ it follows that for $|\mu|$ small enough $C$ and $C_\mu$ are never in opposition on $\Gamma$. Hence (cf. [3], p. 184)

$$\operatorname{Ind}(C_\mu, \Gamma) = \operatorname{Ind}(C, \Gamma), \tag{8}$$

where $\operatorname{Ind}(C, \Gamma)$ ($\operatorname{Ind}(C_\mu, \Gamma)$) denotes the index of $\Gamma$ relative to the field $C$ ($C_\mu$). (For the definition of the index see for example [3], p. 337.) Since $\operatorname{Ind}(C, \Gamma) = 1$, (8) implies that $C_\mu$ vanishes in a certain point of $\Omega$, which completes the proof of Lemma.

**3.** Proof of Theorem 1. From Lemma it is clear that to prove the theorem it is sufficient to show that (7) has periodic solutions with distinct periods. To this end we shall prove that (7) has a non-periodic solution, say $x = x(t)$, $y = y(t)$, defined for all $t$, such that in every neighbourhood of $\big(x(0), y(0)\big)$ there is $(x_0, y_0)$ such that the solution of (7) through $(x_0, y_0)$ is periodic. The desired property of (7) will follow from the continuous dependence of solutions on initial conditions.

Let $x = x(t; x_0, y_0)$, $y = y(t; x_0, y_0)$ be the solution of (7) satisfying the initial condition $x(0; x_0, y_0) = x_0$, $y(0; x_0, y_0) = y_0$. The system (7) has the following properties:

(I) *For an arbitrary point* $(x_0, y_0)$ *there is* $T(x_0, y_0)$ *such that*

$$y\big(T(x_0, y_0); x_0, y_0\big) = 0, \quad x\big(T(x_0, y_0); x_0, y_0\big) > 0.$$

(II) *There are points* $(x_1, y_1), y_1 > 0$ *such that* $y(t; x_1, y_1) > 0$ *for all* $t < 0$ *for which* $x(t; x_1, y_1)$, $y(t; x_1, y_1)$ *exist.*

We shall prove (I) for $y_0 > 0$. For $y_0 < 0$ the proof is similar.

Consider in the $(x, y)$-plane the set of points

$$K(x_0, y_0) = \{(x, y): y \geqslant 0, V(x, y) \leqslant V(x_0, y_0)\},$$

where $V(x, y) = y^2/2 + G(x)$. From the formula

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' = -f(x)|y^{2n+1}|\operatorname{sgn} y$$

and (5) it follows that for $t > 0$ sufficiently small

$$\big(x(t; x_0, y_0), y(t; x_0, y_0)\big) \in K(x_0, y_0).$$

By (3), $K(x_0, y_0)$ is bounded. Since the origin is not a cluster point of $x = x(t; x_0, y_0)$, $y = y(t; x_0, y_0)$, $t \to \infty$, there is $T(x_0, y_0) < \infty$ such that

$$\Big(x\big(T(x_0, y_0); x_0, y_0\big), y\big(T(x_0, y_0); x_0, y_0\big)\Big) \in \operatorname{Fr} K(x_0, y_0),$$

i.e. property (I) holds.

To show (II), put $k = -(a+1)/b$ and define the functions $U(x, y)$, $W(x, y)$ by

$$U(x, y) = y^{2n} - kx, \qquad W(x, y) = \frac{\partial U}{\partial x}y + \frac{\partial U}{\partial y}\big(-f(x)y^{2n} - g(x)\big).$$

A simple calculation gives

$$W\big(x, (kx)^{1/2n}\big) = (kx)^{1/2n}\Big(-k - 2n(kx)^{(2n-1)/2n}\big(f(x) + g(x)/(kx)\big)\Big).$$

By (4) and (5), there is $M < 0$, such that $W\big(x, (kx)^{1/2n}\big) < 0$ for $x \leqslant M$. Thus, if $x_1 \leqslant M$, $y_1 = (kx_1)^{1/2n}$, then, for $t < 0$, $U\big(x(t; x_1, y_1), y(t; x_1, y_1)\big) > 0$, which proves (II).

Since the integral curves of (7) are situated in $(x, y)$-plane symmetrically with respect to the $x$-axis, (I) implies that for arbitrary $s < 0$ the solution $x = x(t; s, 0)$, $y = y(t; s, 0)$ is periodic. Moreover, from (I) and (II) it follows that (7) has non-periodic solutions.

It is easy to see that solutions with initial points belonging to the set $S = \{(x, y): U(x, y) = 0 \text{ for } x \leqslant M,\ y^{2n} = kM \text{ for } 0 \leqslant x \leqslant M\}$ are all non-periodic.

For $x_0 < 0$, $y_0 \geqslant 0$ put $Y(x_0, y_0) = y\big(\tau(x_0, y_0); x_0, y_0\big)$, where $\tau(x_0, y_0) \in \big(0, T(x_0, y_0)\big)$ is the root of the equation

$$x(\tau; x_0, y_0) = 0.$$

The uniqueness of solutions of (7) implies that

$$Y(s_1, 0) > Y(s_2, 0) \quad \text{for } s_1 < s_2, \tag{9}$$

$$Y(x_1, y_1) < Y(x_2, y_2) \quad \text{for } (x_i, y_i) \in S\ (i = 1, 2),\ x_1 < x_2. \tag{10}$$

Let $q = \sup\{s \in (-\infty, 0): Y(s, 0)\}$ and let

$$x = x(t), \quad y = y(t) \tag{11}$$

be the solution of (7) starting from $(0, q)$. (9) implies that (11) is not periodic. Let $R$ be the set $\{(x, y): x \leqslant 0, 0 \leqslant y \leqslant y_S, (x, y_S) \in S\}$. Since $Y\big((-\infty, 0)\times\{0\}\big)$ is connected, the inequality $Y(s, 0) < Y\big(0, (kM)^{1/2n}\big)$ and (10) imply that

$$\big(x(t), y(t)\big) \in \operatorname{Int} R \quad \text{for } t < 0 . \tag{12}$$

By (4) and (5), for $(x, y) \in R$ we have

$$|y| + |-f(x)y^{2n} - g(x)| \leqslant |y| + c|Ax+B| , \tag{13}$$

where $A$ and $B$ are suitably chosen constants. From (12) and (13) it follows that (11) exists for all $t < 0$, which completes the proof of Theorem 1.

**4.** For the differential equation

$$x'' + f(x)x'^2 + g(x) = \mu p(t) , \tag{14}$$

obtained from (1) by putting $n = 1$ the conclusion of Theorem 1 holds under the weaker assumptions. Namely we have the following

THEOREM 2. *Assume condition* (2) *and* (3) *of Theorem* 1. *Let* $f(x)$ *satisfy*

$$f(x) \geqslant 0 \quad \textit{for all } x , \tag{15}$$

$$\int_{-\infty}^{0} \exp F(u)\,du < \infty , \tag{16}$$

*where* $F(x) = \int_0^x f(s)\,ds$. *Then* (14) *has at least one periodic solution with period* $\omega$, *for arbitrary* $p(t)$ $\big(p(t+\omega) \equiv p(t)\big)$, *provided* $|\mu|$ *is sufficiently small.*

Proof. As previously, the proof reduces to showing that the system

$$x' = y , \quad y' = -f(x)y^2 - g(x) , \tag{17}$$

which corresponds to (14) for $\mu = 0$ has a family of periodic solutions with non-constant periods.

We shall prove first that in a certain neighbourhood of origin (17) has only periodic solutions.

System (17) has a first integral

$$H(x, y) = y^2/\big(2a(x)\big) + b(x) ,$$

where

$$a(x) = \exp\big(-2F(x)\big) , \quad b(x) = \int_0^x g(s)/a(s)\,ds .$$

By (2), (3) and (15), $b(x)$ is increasing and unbounded in $(0, \infty)$ and decreasing in $(-\infty, 0)$. It is easily seen that for

$$0 < C < \lim_{x\to-\infty} b(x) = K ,$$

the level curves $H(x, y) = C$ are closed. Hence, for $0 < C < K$, the solutions $x = x(t, C)$, $y = y(t, C)$ of (17) determined by the initial conditions $x(0, C) = 0$, $y(0, C) = \sqrt{2C}$ are periodic.

The period $T(C)$ of the solution $x = x(t, C)$, $y = y(t, C)$ is given by the formula

$$T(C) = \sqrt{2} \int_{-A}^{B} \frac{dx}{\sqrt{a(x)\big(C-b(x)\big)}}, \tag{18}$$

where $A, B > 0$, $b(-A) = b(B) = C$.

Define (see [4]) the function $X = X(x)$ by

$$\tfrac{1}{2}X^2(x) = b(x) , \quad xX(x) > 0 \quad \text{for } x \neq 0 \tag{19}$$

and let $\varphi(X)$ be the inverse of $X(x)$.

Put $R(X) = X\sqrt{a(\varphi(X))}/g(\varphi(X))$. $R(X)$ is defined and positive for $0 < |X| < \sqrt{2K}$. Making the change of variables (19) in (18) gives

$$T(C) = \int_{-\sqrt{2C}}^{\sqrt{2C}} \frac{R(X)}{\sqrt{2C-X^2}} dX .$$

A slight modification of the reasoning in [4] proves that $T(C) = \omega_0$ for all $0 < C < K$ if and only if the function $\frac{2\pi}{\omega_0}R(X)-1$ is odd, i.e. if and only if $V(X) = R(X)+R(-X)$ is constant for $0 < |X| < \sqrt{2K}$.

Since $R(X) > 0$, $V(X)$ may be constant only if $R(X)$ is bounded. We shall prove that, for $X < 0$, $R(X)$ cannot be bounded. From this it will follow that (17) has solutions with non-constant periods.

Assume the contrary and let $N > 0$ be a constant such that $R(X) < N$ for $X < 0$. Since $X(x) = -\sqrt{2b(x)}$, $g(\varphi(X)) = b'((\varphi(X))a(\varphi(X))$, this inequality implies that

$$0 < -\frac{\sqrt{2b(x)}}{b'(x)\sqrt{a(x)}} < N \quad \text{for } x < 0 . \tag{20}$$

From (20) and $b(0) = 0$ it follows that

$$2\sqrt{b(x)} < \frac{\sqrt{2}}{N} \int_{x}^{0} \frac{1}{\sqrt{a(s)}} ds .$$

Thus, by (16), $b(x)$ is bounded. Therefore there is a sequence $\{x_n\}$ such that $x_n \to -\infty$, $b'(x_n) \to 0$, $b(x_n) \to K$. Since $0 < a(x_n) < 1$, we would have $-\sqrt{2b(x_n)}/\big(b'(x_n)\sqrt{a(x_n)}\big) \to \infty$, a contradiction. Thus the proof of Theorem 2 is completed.

**5.** The system of differential equations

$$x' = y\,, \quad y' = -g(x) + \mu p(t) \tag{21}$$

corresponding to $x'' + g(x) = \mu p(t)$, may be considered as a special case of (6).

Since the lemma remains valid if (6), (7) are replaced by (21) and the system

$$x' = y\,, \quad y' = -g(x)\,, \tag{22}$$

the conclusion of Theorem 1 will hold for (21), if (5) will be replaced by any condition which assures the non-constancy of periods of solutions of (22). This suggests the following

THEOREM 3 (compare [2], Th. 1). *Assume* (2) *and* (3). *Let one of the following conditions*

$$\begin{gathered}\lim_{x\to-\infty} g(x)/x = 0\,, \quad \lim_{x\to\infty} g(x)/x = 0\,, \\ \lim_{x\to-\infty} G(x)/x^2 = 0\,, \quad \lim_{x\to\infty} G(x)/x^2 = 0\end{gathered} \tag{23}$$

*hold. Then if $p(t)$ is periodic with period $\omega$, then for $|\mu|$ sufficiently small* (21) *has a periodic solution with period $\omega$.*

Proof. Let $x = x(t, C)$, $y = y(t, C)$ be the solution of (22) satisfying $x(0, C) = 0$, $y(0, C) = \sqrt{2C}$ and let $T(C)$ be the period of this solution. Put $T(C) = T_1(C) + T_2(C)$, where $T_1(C)$, $T_2(C)$ satisfy

$$\begin{gathered}x\big(T_1(C), C\big) = 0\,, \quad y\big(T_1(C), C\big) = -\sqrt{2C}\,, \\ x\big(-T_2(C), C\big) = 0\,, \quad y\big(-T_2(C), C\big) = -\sqrt{2C}\,.\end{gathered}$$

From (2), (3) and anyone of conditions (23) it follows that one of the half-periods $T_1$ and $T_2$ tends to infinity as $C \to \infty$ (see 5, pp. 62 and 65). So $T(C)$ is unbounded, which completes the proof.

**6.** The lemma, and hence Theorem 1, 2 and 3, were proved under the additional assumption of the uniqueness of solutions of (1), (14) and (21). This assumption is quite immaterial and can be easily removed.

To see this, notice that (1), (14) or (21) may be approximated by equations with the property of uniqueness of solutions, for which the assertions of theorems are true. By an appropriate passing to the limit, one can show that the limit equations (1), (14) and (21) satisfy Theorems 1, 2 and 3.

## References

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