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LOCAL MODELS OF THE GREATEST CHARACTERISTIC EXPONENT OF DIFFERENTIAL EQUATIONS DEPENDING ON PARAMETERS

HOANG HUU DUONG and NGUYEN VAN MINH

Mathematical Institute, Hanoi University
Hanoi, Vietnam

§ 1. Introduction

Let us consider the differential equation

$$\frac{dx}{dt} = A(\lambda)x$$

where $x \in \mathbf{R}^n$, $A(\lambda)$ is an $n \times n$-matrix depending differentiably on a parameter λ that belongs to a differentiable manifold Λ of finite dimension. We put

$$f(\lambda) = \lim_{t \to +\infty} \frac{1}{t} \ln \|e^{A(\lambda)t}\|.$$

Then $f(\lambda)$ is equal to the greatest real part of all eigenvalues of $A(\lambda)$ [1]. In general, $f(\lambda)$ is a continuous but not necessarily differentiable function of λ.

In the case where $\dim(\Lambda)$ is less than 2, V. I. Arnol'd [2] has classified singularities of $f(\lambda)$ by using versal deformations of matrices. Our purpose is to consider the problem in the case where $\dim(\Lambda)$ is arbitrary and the equation has the form

$$a_0(\lambda)y^{(n)} + a_1(\lambda)y^{(n-1)} + \ldots + a_n(\lambda)y = 0,$$

where $a_i \in C^\infty(\Lambda)$.

Our method makes use of the technique of stratification, the Weierstrass Preparation Theorem, Mather Division Theorem, Thom Transversality Theorem and a lemma on a family of Morse functions. Our paper consists of several steps:

- stratifying the space of polynomials,
- describing the behaviour of polynomials in a neighbourhood of a stratum,

– determining equations and calculating the codimension of the strata,
– using Thom Transversality Theorem and lemma on a family of Morse functions to find the local models of $f(\lambda)$.

It will be proved that in the "general case" $f(\lambda)$ admits only a finite number of local models which can be written down in a list. Note that in the case of $a_0(\lambda) = 1$ the finiteness of the number of these local models was proved geometrically by L. V. Levantovski [5].

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§ 2. Stratification of the space of polynomials

Let us consider the set P of non-constant polynomials

$$P = \{x_0 t^n + x_1 t^{n-1} + \ldots + x_n, \sum_{i=0}^{n-1} x_i^2 \neq 0\}.$$

We denote by $S_l(k, k_1, \ldots, k_r)$ the set of polynomials of degree $n-l$ (i.e. with $x_0 = \ldots = x_{l-1} = 0$, $x_l \neq 0$) such that the maximum real part of its roots is attained at a real root of multiplicity k and r pairs of complex roots of multiplicities $k_1, \ldots, k_r$, respectively. Then we have

$$P = \bigcup_{l=0}^{n-1} \bigcup_{k,k_j} S_l(k; k_1, \ldots, k_r).$$

These sets $S_l(k; k_1, \ldots, k_r)$ are disjoint. Besides they form a stratification of P, whose strata are $S_l(k; k_1, \ldots, k_r)$ for all possible $k, k_1, \ldots, k_r$ (that means, the closure of each stratum is composed of itself and of the finite union of all strata of lower dimension).

§ 3. Description of the behaviour of polynomials in the neighbourhood of a stratum

3.1. We use the following notation:

C_p^∞ = ring of germs of real C^∞-differentiable functions at a point p,
$\mathscr{C}_p^\infty$ = ring of germs of complex valued C^∞-differentiable functions at a point p,
$M_p = \{u \in C_p^\infty \mid u(p) = 0\}$,
$\mathscr{M}_p = \{v \in \mathscr{C}_p^\infty \mid v(p) = 0\}$,
$C_p^\infty[t]$ = ring of polynomials whose coefficients belong to C_p^∞.

Let us consider the polynomial,

$a_0(x)t^n + a_1(x)t^{n-1} + \ldots + a_n(x)$ where $a_i \in C_p^\infty$ for $i = 0, 1, \ldots, n$.

LEMMA 1. (a) *Suppose that the equation* $\sum_{i=1}^{n} a_i(p)t^{n-i} = 0$ *has one real root* α *of multiplicity* k. *Then there are* $b_i \in M_p$ $(i = 1, 2, \ldots, k)$, $c_j \in C_p^\infty$ $(j = 0, 1, \ldots, n-k)$ *such that in a neighbourhood of* $p \in R^l$ *we have*

$$\sum_{i=0}^{n} a_i t^{n-i} = \Big((t-\alpha)^k + \sum_{j=1}^{k} b_k (t-\alpha)^{k-j}\Big) \sum_{i=0}^{n-k} c_i t^{n-k-i}. \tag{1}$$

(b) *Suppose that the equation* $\sum_{i=0}^{n} a_i(p)t^{n-i} = 0$ *has one pair of conjugate complex roots* $\alpha \pm i\omega$ $(\omega \neq 0)$ *of multiplicity* k. *Then there are* $b_j \in \mathcal{M}_p$ $(j = 1, \ldots, k)$, $c_j \in C_p^\infty$ $(j = 0, 1, \ldots, n-2k)$ *such that in a neighbourhood of* $p \in \boldsymbol{R}^l$ *we have*

$$\sum_{i=0}^{n} a_i t^{n-i} = \Big(\tau^k + \sum_{j=1}^{k} b_j \tau^{k-j}\Big)\Big(\bar{\tau}^k + \sum_{j=1}^{k} \bar{b}_j \bar{\tau}^{k-j}\Big) \sum_{j=0}^{n-2k} c_j t^{n-2k-j} \tag{2}$$

where $\tau = t - \alpha + i\omega$.

Proof. Let us consider the polynomial

$$P(z_0, z_1, \ldots, z_n, w) = z_0 w^n + z_1 w^{n-1} + \ldots + z_{n-1} w + z_n$$

where $(z_0, z_1, \ldots, z_n) \in C^{n+1}$, $w \in C$.

Suppose that $P(a_0(p), \ldots, a_n(p), \beta) = 0$, where β is real or complex and that $P(a_0(p), \ldots, a_n(p), w) \not\equiv 0$. It follows from Weierstrass Theorem that in a neighbourhood of $(a_0(p), \ldots, a_n(p), \beta) \in C^{n+1} \times C$ we have

$$P(z_0, \ldots, z_n, w) = \{(w-\beta)^k + p_1(z_0, \ldots, z_n)(w-\beta)^{k-1} + \ldots$$
$$\ldots + p_k(z_0, \ldots, z_n)\} \varphi(z_0, \ldots, z_n, w),$$

where $k \geqslant 1$ is the multiplicity of β, $p_i(a_0(p), \ldots, a_n(p)) = 0$, $\varphi(z_0, \ldots, z_n, w) \neq 0$; p_i, φ are holomorphic in this neighbourhood.

(a) Suppose $\beta = \alpha \in \boldsymbol{R}$. According to the Mather Division Theorem [3], p_i, φ are real C^∞-differentiable functions if $(z_0, \ldots, z_n) \in \boldsymbol{R}^{n-1}$, $w \in \boldsymbol{R}$. Besides, p_i, φ satisfy the conditions

$$p_i(a_0(p), \ldots, a_n(p)) = 0, \quad \varphi(a_0(p), \ldots, a_n(p)) \neq 0.$$

So we put $p_i(a_0(x), \ldots, a_n(x)) = b_i(x)$, $b_i \in M_p$. It is easy to see that in this case $\varphi(a_0(x), \ldots, a_n(x))$ has the form $\sum_{i=0}^{n-k} c_i(x) t^{n-k-i}$, where k is the multiplicity of β, c_i are C^∞-differentiable functions of x $(i = 0, 1, \ldots, n-k)$.

(b) Suppose that $\beta = \alpha \pm i\omega \in \boldsymbol{C}$, $\omega \neq 0$, $(z_0, \ldots, z_n, w) \in \boldsymbol{R}^{n+1} \times \boldsymbol{R}$. According to the fundamental theorem of algebra,

$$z_0 t^n + z_1 t^{n-1} + \ldots + z_{n-1} t + z_n$$
$$= \{(w-\alpha-i\omega)^k + p_1(z_0, \ldots, z_n)(w-\alpha-i\omega)^{k-1} + \ldots$$
$$\ldots + p_k(z_0, \ldots, z_n)\} \cdot \{(w-\alpha-i\omega)^{k-1} + \overline{p_1}(z_0, \ldots, z_n)(w-\alpha+i\omega)^{k-1} + \ldots$$
$$\ldots + \overline{p_k}(z_0, \ldots, z_n)\} q(z_0, \ldots; z_n, w),$$

where $q(z_0, \ldots, z_n, w)$ can be written in the form of a polynomial in w and $q(a_0(p), \ldots, a_n(p), \beta) \neq 0$, the functions p_i are holomorphic ($i = 1, 2, \ldots, k$).

If we put $z_i = a_i(x)$, $a_i \in C_p^\infty$ ($i = 0, \ldots. n$), then $p_i(a_0(x), \ldots, a_n(x)) = b_i(x)$, $b_i \in C_p^\infty$ satisfy

$$\sum_{i=0}^{n} a_i(x) t^{n-i} = \left(\tau^k + \sum_{j=1}^{k} b_j \tau^{k-j}\right)\left(\bar{\tau}^k + \sum_{j=1}^{k} \bar{b}_j \bar{\tau}^{k-j}\right) \sum_{j=0}^{n-2k} c_j t^{n-2k-j},$$

where $\tau = t - \alpha - i\omega$, $b_i \in \mathscr{M}_p$, $c_j \in \mathscr{C}_p^\infty$.

Remark. In this lemma if $a_0 = 1$, then $c_0 = 1$ in both cases.

3.2. Let us consider the equation $\sum_{j=0}^{n} a_j(p) t^{n-j} = 0$, where $\sum_{j=0}^{n-1} a_j^2(p) \neq 0$. Suppose that the roots of this equation are as follows:

– real roots: α_1 of multiplicity k_1,

. . . .

α_r of multiplicity k_r,

– complex roots: $\beta_1 \pm i\omega_1$ of multiplicity l_1,

. . . .

$\beta_s \pm i\omega_s$ of multiplicity l_s
($\omega_j \neq 0$, $j = 1, 2, \ldots, s$).

Then, by successive application of Lemma 1 we obtain:

LEMMA 2. *In a neighbourhood of p each element of $C_p^\infty[t]$ admits the factorization*

$$\sum_{j=1}^{n} a_j t^{n-j} = \prod_{j=1}^{r} P_j(t) \prod_{k=1}^{s} Q_k(t) \prod_{k=1}^{s} \bar{Q}_k(t) R(t),$$

where

$$P_j(t) = (t-\alpha_j)^{k_j} + \sum_{h=1}^{k_j} u_{jh}(t-\alpha_j)^{k_j-h},$$

$$u_{jh} \in M_p \quad \text{for } h = 1, \ldots, k_j,\ j = 1, 2, \ldots, r,$$

$$Q_{k(t)} = (t-\beta_k - i\omega_k)^{l_k} + \sum_{h=1}^{l_k} v_{kh}(t-\beta_k - i\omega_k)^{l_k - h},$$

$$v_{kh} \in \mathcal{M}_p \quad \text{for } h = 1, \ldots, l_k \text{ and } k = 1, \ldots, s,$$

$$\bar{Q}_k(t) = (t-\beta_k + i\omega_k)^{l_k} + \sum_{h=1}^{l_k} \bar{v}_{kh}(t-\beta_k + i\omega_k)^{l_k - h},$$

$$R(t) = \sum_{i=0}^{n-K-2L} q_i t^i, \quad K = \sum_{i=1}^{r} k_i, \quad L = \sum_{k=1}^{s} l_k,$$

$$q_i \in M_p \quad \text{for } i = 1, 2, \ldots, n-K-2L, \; q_0 \in C_p^\infty \setminus M_p.$$

Let $\mathscr{P}$ be the mapping

$$\Lambda \ni \lambda \to x_0(\lambda) t^n + \ldots + x_n(\lambda) \in \boldsymbol{R}^{n+1} \setminus \{0\} \times \ldots \{0\} \times \boldsymbol{R}.$$

Suppose that

$$\mathscr{P}(\lambda_0) = x_0(\lambda_0) t^n + \ldots + x_n(\lambda_0) \in S_l(k; k_1, \ldots, k_r)$$

and that α is the greatest real part of its roots. The following lemma will describe the behaviour of polynomials in a neighbourhood of $\mathscr{P}(\lambda_0)$.

LEMMA 3. *There exists a neighbourhood of λ_0 which $\mathscr{P}(\lambda)$ has the form*

$$\mathscr{P}(\lambda) = P(\tau) \prod_{j=1}^{r} Q_j(\tau_j) \prod_{j=1}^{r} \bar{Q}_j(\tau_j) R(t) S(t),$$

where

(3) $$P(\tau) = (\tau - a_1)^k - \sum_{j=2}^{k} a_j (\tau - \alpha_1)^{k-j} \text{ and } \tau = t - \alpha \quad \text{if } k \geq 2$$

(4) $$P(\tau) = \tau - a_1 \quad \text{if } k = 1,$$

(5) $$Q_j(\tau_j) = (\tau_j - b_{j1})^{k_j} - \sum_{h=2}^{k_j} b_{jk}(\tau_j - b_{j1})^{k_j - h} \quad \text{and}$$

$$\tau_j = t - \alpha + i\omega_j \quad \text{if } k_j \geq 2,$$

(6) $$Q_j(\tau_j) = \tau_j - b_{j1} \quad \text{if } k_j = 1,$$

the polynomial

(7) $$S(t) = \sum_{j=0}^{l} c_j t^{l-j}$$

is constant at λ_0,

$$R(t) = R_1(t) R_2(t), \ldots, R_s(t),$$

where the factors $R_i(t)$ have the form

$$R_j(t) = (t-\gamma_j)^{m_j} + \sum_{i=1}^{m_j} d_i (t-\gamma_j)^{m_j-i} \quad \textit{for } j = 1, \ldots, s,$$

a_j, b_{jh}, c_j, d_i are differentiable functions of λ in the considered neighbourhood, γ_j are real or complex differentiable functions satisfying

$$\operatorname{Re}(\gamma_j(\lambda_0)) < \alpha.$$

Proof. At first we notice that the polynomial $\tau^n + x_1\tau^{n-1} + \ldots + x_n$ can be written in the form

$$(\tau - y_1)^n - \sum_{i=2}^{n} y_i (\tau - y_1)^{n-i} \quad \text{if } n \geqslant 2 \tag{8}$$

where $y_1 = -\dfrac{x_1}{n}$, $y_2 = -x_2 + c_n^2\left(\dfrac{x}{n}\right)^2, \ldots, y_n = x_n + \ldots$

The mappings $\boldsymbol{R}^n \ni (x_1, \ldots, x_n) \xrightarrow{\Gamma} (y_1, \ldots, y_n) \in \boldsymbol{R}^n$ and Γ^{-1} are both polynomial mappings. So Γ is diffeomorphism. Note that polynomials of the form (8) have $\tau = y_1$ as a root of multiplicity n iff $y_2 = y_3 = \ldots = y_n = 0$.

Now, applying Lemma 2 we obtain Lemma 3.

§ 4. Equations and codimension of strata

Let us consider the family

$$x_0 t^n + x_1 t^{n-1} + \ldots + x_n, \quad (x_0, \ldots, x_n) \in \boldsymbol{R}^{n+1} \setminus \{0\} \times \ldots \times \{0\} \times \boldsymbol{R}.$$

According to Lemma 3, if $(x_0^0, \ldots, x_n^0) \in S_l(k; k_1, \ldots, k_r)$, then the stratum $S_l(k; k_1, \ldots, k_r)$ is defined in a neighbourhood of $(x_0^0, \ldots, x_n^0)$ by the equations

$$\begin{aligned}
&a_h = 0 && \text{for } h = 2, \ldots, k,\\
&\operatorname{Re}(b_{jh}) = \operatorname{Im}(b_{jh}) = 0 && \text{for } h = 2, \ldots, k_j,\ j = 1, \ldots, r,\\
&a_1 = \operatorname{Re}(b_{j1}) && \text{for } j = 1, \ldots, r,\\
&c_h = 0 && \text{for } h = 0, \ldots, l-1.
\end{aligned}$$

So it follows that $\operatorname{codim}(S_l(k; k_1, \ldots, k_r))$ is equal to

$$l + k + 2\sum_{j=1}^{r} k_j - r - 1.$$

§ 5. Application of the Thom Transversality Theorem and a lemma on a family of Morse functions. Local models of $f(\lambda)$

Let Λ be a differentiable manifold of finite dimension. In view of Thom's Transversality Theorem the set of mappings

$$\Lambda \ni \lambda \to x_0(\lambda)t^n + \ldots + x_n(\lambda) \in \mathbf{R}^{n+1} \setminus \{0\} \times \ldots \times \{0\} \times \mathbf{R}$$

which are transversal to every $S_l(k; k_1, \ldots, k_r)$ forms an everywhere dense set in $C^\infty(\Lambda, \mathbf{R}^{n+1} \setminus \{0\} \times \ldots \times \{0\} \times \mathbf{R})$ with Whitney C^∞-topology. We say that $x_i(\lambda)$ $(i = 1, \ldots, n)$ are *generic*.

LEMMA 4. *Let P be transversal to the stratification at λ_0 defined above. Then there exists a system of local coordinates in Λ around λ_0 such that, if $P(\lambda_0) \in S_l(k; k_1, \ldots, k_r)$ we have*

$$f_0 P(\lambda) = \alpha + g(\lambda) + \max_{1 \leqslant j \leqslant r} (\mu, \lambda_{j1} + v_j, \xi - \alpha - g(\lambda)),$$

where $l + k + 2\sum_{j=1}^{r} k_j - r - 1$ first coordinates of λ are denoted by $\lambda_2, \ldots, \lambda_k$, $\lambda_{j1}, \lambda_{j2\eta-1}, \lambda_{j2\eta}, \lambda_{01}, \ldots, \lambda_{0l}$ $(j = 1, \ldots, r; \eta = 2, \ldots, k_j)$, μ is the greatest real part of all roots of the polynomial

$$\tau^k - \sum_{i=2}^{k} \lambda_i \tau^{k-i} \quad \text{if} \quad k \geqslant 2, \ \mu = 0 \ \text{if} \ k = 1,$$

v_j is the greatest real part of all roots of the polynomial

$$\tau^{k_j} - \sum_{\eta=2}^{k_j} (\lambda_{j2\eta-1} + i\lambda_{j2\eta}) \tau^{k_j - \eta} \quad \text{if } k_j \geqslant 2, \ v_j = 0 \ \text{if } k_j = 1,$$

ξ is the greatest real part of all roots of the polynomials

$$\lambda_{01} t^l + \ldots + \lambda_{0l} t + 1 \quad \text{if } (\lambda_{01}, \lambda_{0l}) \neq (0, \ldots, 0), \ \xi = \alpha - 1$$
$$\text{if } (\lambda_{01}, \ldots, \lambda_{0l}) = (0, \ldots, 0).$$

Proof. According to Lemma 3 we can represent $P(\lambda)$ in a neighbourhood of $\lambda_0 \in \Lambda$ as the product

$$P(\lambda) = P(\tau) \prod_{j=1}^{r} Q_j(\tau_j) \prod_{j=1}^{r} \bar{Q}_j(\tau_j) R(t) S(t),$$

where the expressions $P(\tau)$, $Q_j(\tau_j)$, $R(t)$, $S(t)$ are defined as in Lemma 3 (see (3)–(7)). So

$$f_0 \mathscr{P}(\lambda) = \max_{1 \leqslant j \leqslant r} (f P, f_0 Q_j, f_0 S).$$

According to (4)–(8) we have

$$f_0 P = \alpha + a_1 + \mu(a_2, \ldots, a_k), \quad \text{where } \mu(a_2, \ldots, a_k)$$

equals zero if $k = 1$ and equals the greatest real part of all roots of the equation $P(\tau) = 0$ if $k \geqslant 2$.

$$f_0 Q_j = \alpha + \mathrm{Re}(b_{j1}) + \nu_j(\mathrm{Re}(b_{j2}), \mathrm{Im}(b_{j2}), \ldots, \mathrm{Re}(b_{jk_j}), \mathrm{Im}(b_{jk_j})),$$

where $\nu_j(\ldots)$ equals zero if $k_j = 1$ and equals the greatest real part of all roots of the equation $Q_j(\tau_j) = 0$ if $k_j \geqslant 2$ $f_0 S = \xi(c_0, \ldots, c_l)$ where $\xi(\ldots)$ is the greatest real part of all roots of the equation $\sum_{j=1}^{l} c_j t^{l-j} = 0$ $(c_0 \neq 0)$ if $(c_1, \ldots, c_l) \neq (0, \ldots, 0)$ and $\xi = \alpha - 1$ if $(c_1, \ldots, c_l) = (0, \ldots, 0)$. Since $c_0(\lambda)$ is different from zero in the neighbourhood under consideration we can put $c_0(\lambda) = 1$.

On the other hand, the condition of transversality of P to $S_l(k; k_1, \ldots, k_r)$ at λ_0 is equivalent to the fact that φ_0 is a submersion in a neighbourhood U of λ_0, where

$$\boldsymbol{R}^{n+1} \setminus \{0\} \times \ldots \times \{0\} \times \boldsymbol{R} \supset U(P(\lambda_0)) \xrightarrow{\varphi} \boldsymbol{R}^{l+k+2\sum_{i=1}^{r} k_i - r - 1}$$

is defined by

$$x \to \begin{cases} a_\nu & \text{for } \nu = 2, \ldots, k, \\ \mathrm{Re}(b_{jh}),\ \mathrm{Im}(b_{jh}) & \text{for } h = 2, \ldots, k_j,\ j = 1, \ldots, r, \\ \mathrm{Re}(b_{j1}) - a_1, & \\ c_h & \text{for } h = 1, \ldots, l. \end{cases}$$

Note that in this case

$$U(\mathscr{P}(\lambda_0)) \cap S_l(k; k_1, \ldots, k_r) = \varphi^{-1}(0).$$

As

$$\Lambda \supset U(\lambda_0) \xrightarrow{\varphi_0 \mathscr{P}} \boldsymbol{R}^{l+k+2\sum_{i=1}^{r} k_i - r - 1}$$

is a submersion at $\lambda_0 \in \Lambda$, there exists a system of coordinates around λ_0 whose the $l + k + 2\sum_{i=1}^{r} k_i - r - 1$ first coordinates are denoted by $\lambda_2, \ldots, \lambda_k$, $\lambda_{j1}, \lambda_{j2\eta-1}, \lambda_{j2\eta}, \lambda_{01}, \ldots, \lambda_{0l}$ $(j = 1, \ldots, r,\ \eta = 2, \ldots, k_j)$ such that in this system φ_0 becomes the canonical projection onto $l + k + 2\sum_{i=1}^{r} k_i - r - 1$ first coordinates

$$\begin{aligned}
&a_\nu = \lambda_\nu && \text{for } \nu = 2, \ldots, k,\\
&\mathrm{Re}(b_{j1}) - a_1 = \lambda_{j1} && \text{for } j = 1, \ldots, r,\\
&\mathrm{Re}(b_{j\eta}) = \lambda_{j2\eta-1} && \text{for } j = 1, \ldots, r,\ \eta = 2, \ldots, k_j,\\
&\mathrm{Im}(b_{j\eta}) = \lambda_{j2\eta} && \text{for } j = 1, \ldots, r,\ \eta = 2, \ldots, k_j,\\
&c_{l-h} = \lambda_{0h} && \text{for } h = 0, \ldots, l-1.
\end{aligned}$$

According to Lemma 2, $s_1 = \mathrm{Re}(b_{j1}) = g(\lambda)$ where g is a C^∞-differentiable function, $g(0) = 0$ and we can write

$$\begin{aligned}
f_0\,\mathscr{P} &= \max_{1 \leqslant j \leqslant r} (f_0\,P, f_0\,Q_j, f_0\,S)\\
&= \alpha + g(\lambda) + \max_{1 \leqslant j \leqslant r} \big(\mu,\ \lambda_{j1} + \nu_j,\ \xi - \alpha - g(\lambda)\big),
\end{aligned}$$

where μ, ν_j, ξ are defined in Lemma 4.

Remark (1). From the conditions of transversality it follows that if $\mathscr{P}$ is transversal to the stratification in question, then $\mathscr{P}(\lambda)$ belongs to only those $S_l(k;\, k_1, \ldots, k_r)$ which satisfy

$$s = l + k + 2\sum_{i=1}^{r} k_i - r - 1 \leqslant \dim(\Lambda).$$

(2) We can transform $g(\lambda)$ into a simple form by using the following lemma in which we set $m = \dim(\Lambda) - s$.

LEMMA 5 (on a family of Morse functions). *There exists an open everywhere dense set of functions in $C^\infty(\boldsymbol{R}^m \times \boldsymbol{R}^s, \boldsymbol{R})$ which can be reduced in a neighbourhood of $(0, 0) \in \boldsymbol{R}^m \times \boldsymbol{R}^s$ by a change of coordinates of type $(x', y) \to (x, y)$ to one of the following forms*

(1) $g(x, y) = \mathrm{const} + x_1$,

(2) $g(x, y) = \mathrm{const} + h(y) + \sum_{i=1}^{m} \varepsilon_i x_i^2$, $\varepsilon_i = \pm 1$,

where $h(y)$ is differentiable, $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_s)$[(1)].

The proof of Lemma 5 is analogous to the Morse lemma [3].

From Lemma 4 and Lemma 5 we obtain

THEOREM. *Let us consider the following family of differential equations*

$$a_0(\lambda)\, y^{(n)} + a_1(\lambda)\, y^{(n-1)} + \ldots + a_n(\lambda)\, y = 0, \qquad \lambda \in \Lambda.$$

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(1) Here $(y_1, \ldots, y_s) = (\lambda_{j1}, \lambda_{j2\eta-1}, \lambda_{j2\eta}, \lambda_{0h}, \ldots, \lambda_\nu)$, where $j = 1, \ldots, r$, $\eta = 2, \ldots, k_j$, $h = 0, \ldots, l-1$, $\nu = 2, \ldots, k$.

If the coefficients $a_i(\lambda)$ are generic, then for every $\lambda_0 \in \Lambda$ there exists a local systems of coordinates $\lambda = (x, y)$ around λ_0 such that $f(\lambda)$ has one of the following local models:

(I) $\alpha + x_1 + \max(\mu, y_1 + v_1, \ldots, y_r + v_r, \xi - \alpha - x_1)$,

(II) $\alpha + \sum_{i=1}^{m} \varepsilon_i x_i^2 + h(y)$

$$+\max\left(\mu, y_1 + v_1, \ldots, y_r + v_r, \xi - \alpha - h(y) - \sum_{i=1}^{m} \varepsilon_i x_i^2\right),$$

(III) $\alpha + h(y) + \max(\mu, y_1 + v_1, \ldots, y_r + v_r, \xi - \alpha - h(y))$,

where $a_0(\lambda_0)t^n + \ldots + a_n(\lambda_0) \in S_l(k; k_1, \ldots, k_r)$, μ, v_j, ξ are defined as in Lemma 4 and the coordinates (x, y) and $h(y)$ are chosen as in Lemma 5.

Applying this theorem for each case of $\dim(\Lambda)$ we obtain the following lists.

List of local models for the case of $\dim(\Lambda) = 1$

Codim	Strata	Local models of $f(\lambda)$
0	$S_0(1)$, $S_0(0; 1)$	$\alpha + \lambda$, $\alpha \pm \lambda^2$
1	$S_0(2)$	$\alpha + g(\lambda) + \mathrm{Re}(\sqrt{\lambda})$
1	$S_0(1; 1)$, $S_0(0; 1, 1)$	$\alpha + g(\lambda) + \lvert\lambda\rvert$
1	$S_1(1)$, $S_1(0; 1)$	$\begin{cases} \max(\alpha + g(\lambda), 1/\lambda) & \text{if } \lambda > 0, \\ \alpha + g(\lambda) & \text{if } \lambda < 0, \end{cases}$ where $g(\lambda)$ is a differentiable function, $g(0) = 0$

List of local models of $f(\lambda)$ for the case of $\dim(\Lambda) = 2$

Codim	Strata	Local models of $f(\lambda)$
0	$S_0(1)$, $S_0(0; 1)$	$\alpha + x$, $\alpha \pm x^2 \pm y^2$
1	$S_0(2)$	$\alpha + x + \begin{cases} \sqrt{y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$ $\alpha \pm x^2 + h(y) + \begin{cases} \sqrt{y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$
1	$S_0(1; 1)$ $S_0(0; 1, 1)$	$\alpha + x + y + \lvert y\rvert$ $\alpha \pm x^2 + \lvert y\rvert + h(y)$
1	$S_1(1)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + x & \text{if } y \leq 0 \end{cases}$
1	$S_1(0; 1)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha \pm x^2 + h(y) & \text{if } y \leq 0 \end{cases}$

List of local models of $f(\lambda)$ for the case of $\dim(\Lambda) = 2$ (continued)

Codim	Strata	Local models of $f(\lambda)$
2	$S_0(3)$	$\alpha + g(x, y) + \mu(x, y)$
2	$S_0(2; 1)$	$\alpha + g(x, y) + \max(\sqrt{x}, y)$ if $x \geqslant 0$ $\alpha + x + \lvert y\rvert$ or $\alpha \pm x^2 + \varphi(y) + \lvert y\rvert$ if $x < 0$
2	$S_0(1; 1, 1)$, $S_0(0; 1, 1, 1)$	$\max(0, x, y) + \alpha + g(x, y)$
2	$S_0(0; 2)$	$\alpha + g(x, y) + \lvert \mathrm{Re}(\sqrt{x+iy})\rvert$
2	$S_1(2)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + g(x, 0) + \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} & \text{if } y = 0 \end{cases}$ $\begin{cases} \alpha + y + \sqrt{x} \\ \text{or } \alpha \pm y^2 + \sqrt{x} \end{cases}$ if $x > 0$, $y < 0$ $\begin{cases} \alpha + x \\ \alpha \pm x^2 \pm y^2 \end{cases}$ if $x < 0$, $y < 0$
2 2	$S_1(1; 1)$ $S_1(0; 1, 1)$	$\begin{cases} 1/y & \text{if } y > 0 \\ \alpha + g(x, y) + \lvert x\rvert & \text{if } y < 0 \end{cases}$
2	$S_2(1)$, $S_2(0; 1)$	$\max(\alpha + g(x, y), \xi(x, y))$ where $h(y)$, $\varphi(y)$, $g(x, y)$ are differentiable functions, $\mu(x, y)$ and $\xi(x, y)$ are the greatest real part of all the roots of the polynomials $\tau^3 - x\tau - y$, $xl^2 + yt + 1$ respectively, $h(0) = \varphi(0) = g(0, 0) = 0$

List of local models of $f(\lambda)$ for the case of $\dim(\Lambda) = 3$

Codim	Strata	Local models of $f(\lambda)$
1	$S_0(2)$	$\alpha + y + \begin{cases} \sqrt{z} & \text{if } z \geqslant 0 \\ 0 & \text{if } z < 0 \end{cases}$ $\alpha \pm x^2 \pm y^2 + h(z) + \begin{cases} \sqrt{z} & \text{if } z \geqslant 0 \\ 0 & \text{if } z < 0 \end{cases}$
1 1	$S_0(1, 1)$ $S_0(0; 1, 1)$	$\alpha + y + z + \lvert z\rvert$ $\alpha \pm x^2 \pm y^2 + h(z) + \lvert z\rvert$
1	$S_1(1)$	$\begin{cases} 1/z & \text{if } z > 0 \\ \alpha + x & \text{if } z \leqslant 0 \end{cases}$
1	$S_1(0; 1)$	$\begin{cases} 1/z & \text{if } z > 0 \\ \alpha \pm x^2 \pm y^2 + h(z) & \text{if } z < 0 \end{cases}$
2	$S_0(3)$	$\alpha + x + \mu(y, z)$ $\alpha \pm x^2 + h(y, z) + \mu(y, z)$

2	$S_0(2;1)\}$	$\begin{cases}\alpha+g(x,y,z)+\max(\sqrt{y},z) & \text{if } y\geqslant 0\\ \alpha+x+\lvert z\rvert \text{ or } \alpha\pm x^2\pm y^2+h(z)+\lvert z\rvert & \text{if } y<0\end{cases}$

List of local models of $f(\lambda)$ for the case of $\dim(\Lambda)=3$ (continued)

Codim	Strata	Local models of $f(\lambda)$
0	$S_0(1)$, $S_0(0;1)$	$\alpha+x$, $\alpha\pm x^2\pm y^2\pm z^2$
2 2	$S_0(1;1,1)$ $S_0(0;1,1,1)$	$\alpha+x+\max(0,y,z)$ $\alpha\pm x^2+h(y,z)+\max(0,y,z)$
2	$S_0(0;2)\}$	$\alpha+x+\lvert\mathrm{Re}(\sqrt{y+ix})\rvert$ $\alpha\pm x^2+h(y,z)+\lvert\mathrm{Re}(\sqrt{y+iz})\rvert$
2	$S_1(2)$	$\begin{cases}1/z & \text{if } z>0\\ \alpha+y+\sqrt{x} \text{ or } \alpha\pm y^2\pm z^2+\sqrt{x} & \text{if } z\leqslant 0,\ x\geqslant 0\\ \alpha\pm x^2\pm y^2\pm z^2 \text{ or } \alpha+x & \text{if } z\leqslant 0,\ x<0\end{cases}$
2 2	$S_1(1;1)$ $S_1(0;1,1)$	$\begin{cases}1/z & \text{if } z>0\\ \alpha+x+\lvert y\rvert \text{ or } \alpha\pm x^2+h(y,z)+\lvert y\rvert & \text{if } z<0\end{cases}$
2	$S_2(1)$, $S_2(0;1)$	$\max(\alpha+g(x,y,z),\ \xi(y,z))$
3	$S_0(4)$	$\alpha+g(x,y,z)+\mu(x,y,z)$
3	$S_0(3;1)$	$\alpha+g(x,y,z)+\max(\mu(x,y),z)$
3	$S_0(2;1,1)$	$\alpha+g(x,y,z)+\begin{cases}\max(\sqrt{x},y,z) & \text{if } x\geqslant 0\\ \max(0,y,z) & \text{if } x<0\end{cases}$
3	$S_0(1;1,1,1)$ $S_0(0;1,1,1,1)$	$\alpha+g(x,y,z)+\max(0,x,y,z)$
3	$S_0(1;2)$, $S_0(0;1,2)$	$\alpha+g(x,y,z)+\max(\lvert\mathrm{Re}(\sqrt{x+iy})\rvert,z)$
3	$S_1(3)$	$\begin{cases}\max(1/x,\ \alpha+g(x,y,z)+\mu(y,z)) & \text{if } x\neq 0\\ \alpha+g(0,y,z)+\mu(y,z) & \text{if } x=0\end{cases}$
3	$S_1(2;1)$	$\alpha+g(x,y,z)+\begin{cases}\max(\sqrt{y},z,1/x-\alpha-g(x,y,z)) & \text{if } y\geqslant 0\\ \max(0,x,1/x-\alpha-g(x,y,z)) & \text{if } y<0\end{cases}$
3	$S_1(1;1,1)$, $S_1(0;1,1,1)$	$\begin{cases}\alpha+g(x,y,z)+\max(0,y,z,1/x-\alpha-g(x,y,z)) & \text{if } x\neq 0\\ \alpha+g(0,y,z)+\max(0,y,z) & \text{if } x=0\end{cases}$
3	$S_1(0,2)$	$\begin{cases}1/z & \text{if } z>0\\ \alpha+g(x,y,z)+\lvert\mathrm{Re}(\sqrt{x+iy})\rvert & \text{if } z<0\end{cases}$

List of local models of $f(\lambda)$ for the case of $\dim(\Lambda) = 3$ (continued)

Codim	Strata	Local models of $f(\lambda)$
3	$S_2(2)$	$\alpha+g(x, y, z)+\begin{cases}\max(\sqrt{x}, \xi(y, z)-\alpha-g(x, y, z)) & \text{if } x \geqslant 0\\ \max(0, \xi(y, z)-\alpha-g(x, y, z)) & \text{if } x < 0\end{cases}$
3	$S_2(1; 1)$, $S_2(0; 1, 1)$	$\alpha+g(x, y, z)+\max(0, x, \xi(y, z)-\alpha-g(x, y, z))$
3	$S_3(1)$, $S_3(0; 1)$	$\alpha+g(x, y, z)+\max(0, \xi(x, y, z)-\alpha-g(x, y, z))$ where $h(z)$, $h(y, z)$, $g(x, y, z)$ are differentiable functions, $h(0) = h(0, 0) = g(0, 0, 0) = 0$, $\mu(y, z)$, $\mu(x, y, z)$, $\xi(y, z)$, $\xi(x, y, z)$ are the greatest real parts of all roots of the following polynomials: t^3-yt-z, t^4-xt^2-yt-z, yt^2+zt+1, xt^3+yt^2+zt+1, respectively.

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