CONTINUOUS MEASURES AND ANALYTIC SETS*

BY

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0. Introduction. Let \( \Gamma \) be a countably infinite abelian group, and \( G \) its dual group. A subset \( S \) of \( \Gamma \) is called a \( w \)-set in \( \Gamma \) if there is a continuous complex-valued measure \( \mu \) in \( G \) such that \( |\hat{\mu}| \geq 1 \) everywhere in \( S \). (The name refers to the work of Wiener [11], apparently the first on Fourier–Stieltjes transforms of continuous measures; by definition, \( \hat{\mu}(\gamma) = \int \gamma d\mu \).) Regarding the space \( M_c(G) \) as a subset of a dual space \( C^*(G) \), it is of type \( F_{\sigma\delta} \) in the \( w \)-topology. This gives the easy half of our main result.

THEOREM. In the metric space \( 2^\Gamma \), the class \( w\Gamma \) of all \( w \)-sets is an analytic set but not a Borel set.

A previous work on classes contained in \( 2^\Gamma \) concerns the class \( R \) of Riesz sets [9]; the class \( R \), which plays the role of negligible sets, is \( co \)-analytic but not Borel. The present theorem depends on the harmonic analysis on certain non-locally compact topological groups established by Varopoulos [10], pp. 112–131 (see also [7]). For certain groups \( \Gamma \) (type I) this dependence is explicit, and for the remaining groups \( \Gamma \), the analysis of [10] is adapted by a ruse.

Remark. Every Sidon set is a \( w \)-set [2]. On the other hand, it is easy to prove that Sidon sets are a class of type \( F_{\sigma\delta} \) in \( 2^\Gamma \). So our Theorem shows that not every \( w \)-set is Sidon. This is known and can be proved in many ways (cf., e.g., [6]).

1. Preliminaries. We shall first reduce the main result to certain special cases.

LEMMA 1. Let \( \varphi \) be a homomorphism of \( \Gamma \) onto a group \( \Gamma_1 \), and \( S_1 \subseteq \Gamma_1 \). Then \( S_1 \) is a \( w \)-set in \( \Gamma_1 \) if and only if \( \varphi^{-1}(S_1) \) is a \( w \)-set in \( \Gamma \).

Proof. Suppose that \( \mu \in M_c(G_1) \) and \( |\hat{\mu}| \geq 1 \) on \( S_1 \). (We have written \( G_1 \) for the dual of \( \Gamma_1 \).) Then the dual mapping \( \varphi^* \) is a homeomorphism of \( G_1 \) into \( G \), and

\[
(\varphi^*\mu)^*(s) = \hat{\mu}(\varphi s),
\]

whence \(|(\varphi^*\mu)^*| \geq 1\) on \( \varphi^{-1}(S_1) \), and of course \( \varphi^*\mu \) belongs to \( M_c(G) \). In the opposite direction, we begin with \( \mu \in M_c(G) \) and replace \( \mu \) by \( \lambda = \mu \cdot \hat{\mu} \), so that \( |\lambda| \geq 1 \) on \( \Gamma \) and \( |\lambda| \geq 1 \) on \( \varphi^{-1}(S_1) \). Let \( A \) be the kernel of \( \varphi \),

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let \( \hat{\lambda} \) be a Banach limit over \( \Lambda \), and observe that there is a measure \( \sigma \) such that

\[
\hat{\sigma}(\chi) \equiv \lim_{\gamma} \hat{\lambda}(\chi + \gamma).
\]

What we need to know about \( \sigma \) is that

\[
\sigma(E) = \lambda(E \cap \Lambda^\perp)
\]

for each Borel set \( E \), and \( \hat{\sigma}(\chi) \) is contained in the convex hull of the set \( \hat{\lambda}(\chi + \Lambda) \) for each \( \chi \). Identifying \( \Lambda^\perp \) with \( G_1 \), we conclude that \( S_1 \) is a w-set in \( \Gamma_1 \).

We now divide the groups \( \Gamma \) into two classes:

(I) For every integer \( m = 1, 2, 3, \ldots \), \( \Gamma/m\Gamma \) is finite.

(II) For some integer \( m \geq 1 \), \( \Gamma/m\Gamma \) is infinite.

2. Groups of type I. In this case the subgroup of \( G \) defined by the equation \( mg = 0 \) is finite for each \( m \). Therefore, \( G \) is an I-group and contains a perfect Kronecker set \( K \) (see [3], pp. 566–570, and [8], pp. 99–102). We suppose in fact that \( K \) is a Cantor set. We define a map from closed sets \( E \) of \( K \) into \( 2^\Gamma \) as follows:

\[
\chi \in B(E) \iff |\chi - 1| < 1/3 \quad \text{on} \quad E.
\]

The mapping is lower semi-continuous in the following sense: if \( \lim E_n = E \) in the Hausdorff metric, then

\[
B(E) \subseteq \liminf B(E_n).
\]

As for real-valued lower semi-continuous functions, whenever \( U \) is open in \( 2^\Gamma \), the inverse image \( \{ E : B(E) \in U \} \) is then of type \( F_\sigma \) in \( 2^K \), and in particular the inverse image is Borelian. We shall show that \( B(E) \) is a w-set if and only if \( E \) is uncountable, or in different terms: \( B(E) \) is w\( \Gamma \)-\( E \) is uncountable. By a theorem of Hurewicz [4], the class of uncountable closed sets in \( 2^K \) is analytic but not Borelian, whence w\( \Gamma \) is not Borelian. Clearly, \( B(E) \) is a w-set when \( E \) is uncountable, since \( E \) supports a continuous probability measure. For the more difficult implication, we first summarize the necessary results from [7] and [10].

Suppose that \( X \) is a 0-dimensional compact metric space and that \( S(X) \) is the (metric) group of all unimodular, continuous functions on \( X \).

(a) Every continuous character of \( S(X) \) is in the subgroup generated algebraically by the evaluations at the elements of \( X \).

(b) Bochner’s theorem is valid for \( S(X) \): every continuous positive-definite function on \( S(X) \) is represented as an integral over continuous characters on \( S \). Strictly speaking, the continuous characters have to be turned into a measurable set; this quibble does not affect the remaining argument. See also [1].

(c) Let \( H \) be an abelian group provided with an invariant pseudo-metric \( d; \)
let \((Y, \mu)\) be a finite measure space and \(S^*(Y)\) the group of unimodular, \(\mu\)-measurable functions on \(Y\). Let \(T\) be an algebraic homomorphism from \(H\) into \(S^*(Y)\). One of the following two cases must occur:

\((c_1)\) There is a measurable subset \(Y_1 \subseteq Y\), \(\mu(Y_1) > 0\), so that \(T\) is continuous as a mapping from \(H\) to \(S^*(Y)\).

\((c_2)\) For every neighborhood \(V\) of the identity of \(H\), the convex hull \(\text{co}(T(V))\) contains the function 0 in its closure (in \(L^1(\mu)\), for example).

We can now prove that when \(B(E)\) belongs to \(w\Gamma\), then \(E\) must be uncountable. Suppose, then, that \(\mu\) is a continuous signed measure such that \(\hat{\mu} \geq 1\) in \(B(E)\). In \(\Gamma\) we introduce a pseudo-metric \(d\) by the formula

\[d(\gamma, 0) \equiv \sup\{|\gamma(g) - 1|: g \in E\}.
\]

We apply \((c)\) to the measure space \((G, |\mu|)\). Now \(B(E)\) is a neighborhood of 0 in \(\Gamma\) (using the pseudo-metric \(d\)) and for every function \(f\) in the convex hull of \(B(E)\) we have

\[\int |f| d\mu \geq \text{Re} \int f d\mu \geq 1
\]

(since the last inequality is true for characters in \(B(E)\)). Therefore, alternative \((c_2)\) must be rejected, and \((c_1)\) accepted. We now define

\[p(\chi) = \int \hat{\chi} |d\mu|, \quad \chi \in \Gamma,
\]

so that \(p\) is positive definite on \(\Gamma\), and continuous for the pseudo-metric \(d\).

Since \(K\) is a 0-dimensional Kronecker set, \(E\) is one as well, and \(p\) determines (by uniform approximation) a continuous positive-definite function on \(S(E)\). Comparison with \((b)\) in the summary above shows that

\[p(\chi) = \int \hat{\chi} d\lambda
\]

with a measure \(\lambda\) concentrated in the algebraic subgroup generated by \(E\). If \(E\) were countable, then we would have \(|\mu| Y_1 = 0\), whence

\[p(1) = |\mu| Y_1 = 0;
\]

this concludes the proof for groups of type I.

3. **Groups of type II.** In this case there is a prime \(p\) such that \(\Gamma/p\Gamma\) is infinite, for the inequality

\[o(\Gamma/m_1 m_2 \Gamma) \leq o(\Gamma/m_1 \Gamma) \cdot o(\Gamma/m_2 \Gamma)
\]

is valid for all pairs of integers \(m_1, m_2 \geq 1\). Then \(\Gamma/p\Gamma\) is an infinite sum \(\mathbb{Z}_p^\infty\), and, by Lemma 1, we can assume that \(\Gamma\) is one of these groups.

In this case \(G\) contains a perfect \(K_p\)-set \(F_1\) (see [3] and [8]); this means that every continuous function on \(F_1\) to the group of \(p\)-th roots of unity is the restriction to \(F_1\) of a continuous character of \(G\). Now \(F_1\) is homeomorphic
to a Cantor set, and therefore to a union of three disjoint Cantor sets. Thus $F_1$ can be represented as a product

$$F_1 = F \times \{1, 2, 3\},$$

$F$ being also a Cantor set. To each closed set $E$ of $F$ we attach objects $\alpha(E)$, $\beta(E)$, and $B(E)$.

(i) $\alpha(E)$ is the subgroup of $G$ generated algebraically by $E \times \{1, 2, 3\}$, and $\beta(E)$ is the closure of $\alpha(E)$ in $G$.

(ii) $B(E)$ is the subset of $\Gamma$ defined by this condition: for each $x$ in $E$, $\gamma(x \times i) = 1$ for at least two numbers $i = 1, 2, 3$. $B(E)$ takes the place of the set $B(E)$ used before. (Attempting to follow the method used for groups of type I, we would consider the characters on a certain group — but that group is discrete.) The analytical part of the proof is contained in

**Lemma 2.** There exists a sequence $(\lambda_n)$ of probability measures in $B(F)$ such that $\lambda_n(g) \to 0$ on $\beta(F) \setminus \alpha(F)$.

**Proof.** For each $k = 1, 2, 3, \ldots$ let $(A_1, \ldots, A_r)$ be a partition of $F$ into disjoint closed sets of diameter $< k^{-1}$. Let $\gamma(i, j)$ be a continuous character of $G$ such that $\gamma(i, j) = \omega_p = \exp(2\pi ip^{-1})$ in $A_j \times i$ and 1 in the remainder of $F \times \{1, 2, 3\} = F_1$ ($i = 1, 2, 3, 1 \leq j \leq r$). This recipe determines the value $(\gamma(i, j), g)$ for each $g$ in $\beta(F)$. Finally, let

$$\lambda_k = \prod_{j=1}^{r} \left\{ \delta(0) + \delta(\gamma(1, j)) + \delta(\gamma(2, j)) + \delta(\gamma(3, j)) \right\}.$$

A constant $c_p < 1$ is defined by

$$c_p^2 = (7 + \cos 2\pi/p)/8.$$

Suppose that $g \in \beta(F)$ and $\limsup |\lambda_k(g)| \geq \eta > 0$, while $c_p^M < \eta$ for some integer $M \geq 1$. For infinitely many $k = 1, 2, 3, \ldots$, fewer than $M$ of the factors of $\lambda_k$ have modulus $< c_p$ at $g$. Since the value of $\gamma(i, j)$ is always a $p$-th root of 1, the equations $(\gamma(i, j), g) = 1$ for $i = 1, 2, 3$ are valid for every $j$ with at most $M$ exceptions. Let $\Gamma_k$ be the subgroup of $\Gamma$ generated by the characters $\gamma(i, j)$ introduced at the $k$-th step. Then there is an identity

$$(\gamma, g) = \prod_{1}^{M} \gamma(g_n)^{e_n} \quad \text{for every } \gamma \in \Gamma_k,$$

with elements $g_n$ of $F_1$ and numbers $e_n = 0, 1, \ldots, p - 1$. This holds for infinitely many integers $k$, but since $g \in \beta(F)$, it is clear that a single relation of this kind must hold for every $\gamma$ in $\Gamma$, i.e., $g \in \alpha(F)$.

We can now complete the main theorem for II. The mapping $E \to B(E)$ is continuous, in fact homeomorphic from $2^E$ to $2^F$. When $E$ is uncountable, we take a continuous probability measure $\nu$ in $E$ and set

$$\mu = \nu \otimes (\delta(0) + \delta(1) + \delta(2)),$$
whence \( \text{Re} \hat{\mu} \geq 1 \) in \( B(E) \). Suppose, in the opposite direction, that \( B(E) \) is in \( w\Gamma \) and \( \hat{\mu} \geq 1 \) in \( B(E) \) for some continuous measure \( \mu \) in \( \Gamma \). Since \( B(E) + \beta(E) \perp B(E) \), this will remain true for a continuous measure concentrated in \( \beta(E) \). We apply Lemma 2, replacing \( F \) by \( E \) throughout. For the sequence \( (\lambda_k) \) of probability measures in \( B(E) \),

\[
\int \lambda_k(g) \mu(dg) = \int \hat{\mu}(X) \lambda_k(d\chi) \geq 1
\]

and Lemma 2 confirms that \( |\mu| \) has positive mass in \( \alpha(E) \), whence \( \alpha(E) \) — and consequently \( E \) itself — must be uncountable. The theorem of Hurewicz cited earlier then shows that \( w\Gamma \) cannot be a Borel set in \( 2\Gamma \).

In the proof just concluded, the mapping of \( E \) to \( B(E) \) is a homeomorphism, but for groups of type I it is possibly discontinuous. (That has no effect on the succeeding argument.) To remove this defect, let \( \sigma \) be a continuous measure on \( K \), and \( t \) a number in \( (0, 1/3) \) such that

\[ \sigma\{g : |\chi(g)| - 1 = t\} = 0 \quad \text{for each} \ \chi \ \text{in} \ \Gamma. \]

There is a closed set \( K_1 \subseteq K \) such that \( \sigma(K_1) > 0 \) and \( \chi - 1 \neq t \) in \( K_1 \) for each \( \chi \). We then define \( B'(E) \) for \( E \subseteq K_1 \) by the inequality \( |\chi - 1| < t \) in \( E \). This is a homeomorphism from \( 2^{K_1} \) into \( 2\Gamma \) and the remaining steps are the same.

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