THE GEOMETRY OF A SEMI-DIRECT EXTENSION
OF A HEISENBERG TYPE NILPOTENT GROUP

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The notion of nilpotent group of type H introduced by Kaplan [5] has
attracted considerable attention ([6], [7], [9], [10], [14], [15]). These groups
include the nilpotent subgroups \( N \) of semisimple rank one Lie groups \( G \)
which appear in the Iwasawa decomposition \( G = NAK \).

Following ideas of Cygan and Kaplan and Putz [8] we go a step further
and we look at the groups \( S = NA \), where \( N \) is a type H group and \( A \) a
group of dilations of \( N \), equipped with a suitable left-invariant metric. We
thus obtain a generalization of rank one symmetric spaces.

The aim of this paper is to describe the group \( I(S) \) of isometries of \( S \).
As expected it turns out that only in the classical case, i.e., where \( S = G/K \), \( I(S) \)
is large. In all the remaining cases \( I(S) \) appears to be as small as possible,
i.e., the semi-direct product of the group \( A(S) \) of automorphisms of \( S \) which
preserve the inner product (cf. definitions below) and the group \( S \) itself
(Theorem 4.4).

The main idea of the proof is to describe the set \( \{ d\eta_e : \eta \in I(S), \eta(e) = e \} \).
Our reasoning is based on the fact that \( d\eta_e \) is orthogonal and it preserves the
connection \( \nabla \), the curvature tensor \( R \), and its covariant derivative \( \nabla R \).
For the non-classical group \( S \) these conditions imply that \( d\eta_e \) must be an
automorphism of \( s \) in the following way (\( s, \eta, \) and \( \alpha \) denote Lie algebras of
\( S, N \) and \( A \), respectively).

The main point is Theorem 4.2 which states that for non-classical cases
\( \nabla R(x) = 0 \) iff \( x \in \alpha \). To prove this we consider two cases:

(i) \( \eta = O^\alpha \times O, \) \( O \) being the octonions, \( n > 1 \);
(ii) \( \dim Z \neq 0, 1, 3, 7, Z \) being the center of \( \eta \)
(c.f. Propositions 4.2 and 4.3). Now, from Theorem 4.2 we conclude that
\( d\eta_e(\alpha) \subset \alpha, d\eta_e(V) \subset V, d\eta_e(Z) \subset Z \) and, finally, that \( d\eta_e \) is an automorphism
(Theorem 4.3).
Our choice of invariants $V$, $R$, and $VR$ is somewhat arbitrary and it is quite likely that by selecting other invariants one could obtain a simpler proof.

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1. **Introduction.** Let $\mathfrak{n}$ be a nilpotent 2-step algebra with an inner product. Denote by $V$ the orthogonal complement to its center $Z$. Then, for every $v \in V$, $ad_v$ maps $V$ into $Z$ and we have the orthogonal decomposition of $V$ given by

$$V = \text{Ker} \ ad_v \oplus D(v).$$

$\mathfrak{n}$ is said to be of Heisenberg type (shortly, of type $H$) if for every unit vector $v \in V$ the mapping $ad_v : D(v) \to Z$ is a surjective isometry.

Every Lie algebra of type $H$ arises as follows [5]. Let $U$ and $V$ be vector spaces with positive definite quadratic forms $|\cdot|^2$. By definition, a composition of these quadratic forms is a bilinear map $\mu : U \times V \to V$ which satisfies

$$|\mu(u, v)| = |u||v|, \quad u \in U, \ v \in V,$$

and for a $u_0$

$$\mu(u_0, v) = v, \quad v \in V.$$

Let $Z$ be the orthogonal complement to $Ru_0$ and let $\pi : U \to Z$ be the orthogonal projection. Define a bilinear map $\Phi : V \times V \to U$ by

$$\langle u, \Phi(v, v') \rangle = \langle \mu(u, v), v' \rangle.$$ (1.1)

Then $\pi \Phi$ is skew-symmetric [5] and $\mathfrak{n} = V \times Z$ with the bracket

$$[(v, z), (v', z')] = (0, \pi \Phi(v, v')),$$

and the inner product

$$\langle (v, z), (v', z') \rangle = \langle v, v' \rangle + \langle z, z' \rangle$$

is an algebra of type $H$.

Let $\varrho$ be a function defined on nonnegative integers by the condition: if $n = (\text{odd}) \ 2^{4p+q}$, $0 \leq q \leq 3$, then

$$\varrho(n) = 8p + 2^q.$$ 

An algebra of type $H$ with $\dim V = n$ and $\dim Z = m - 1$ exists if and only if $m \leq \varrho(n)$ (see [2]). In particular, the equality $m = n$ yields $n = 1, 2, 4, 8$.

Let $N$ be a connected and simple connected Lie group whose Lie algebra is $\mathfrak{n}$. If we identify $N$ with $\mathfrak{n}$ by the exponential map, the multiplication in $N$ is given by

$$(v, z)(v', z') = (v + v', z + z' + \frac{1}{2} \pi \Phi(v, v')).$$
We denote by $A$ the multiplicative group of $R^+$. Let

$$S = NA$$

be a semi-direct product of $N$ and $A$, $A$ acting on $N$ as dilations $\delta_a(v, z) = (av, a^2 z)$. Thus we identify $S$ with $V \times Z \times A$ and

$$(v, z, a)(v', z', a') = (v + av', z + a^2 z' + \frac{1}{2} a a \pi \Phi(v, v'), aa').$$

$S$ has a Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ with the bracket

$$[v + z + rh_0, v' + z' + r' h_0] = r v' - r' v + 2rz' - 2r' z + \pi \Phi(v, v'),$$

where $\mathfrak{a}$ is the Lie algebra of $A$ and $h_0 \in \mathfrak{a}$ is such that $[h_0, v] = v$.

In the Lie algebra $\mathfrak{s}$ we select an inner product

$$\langle v + z + rh_0, v' + z' + r' h_0 \rangle_S = \langle v, v' \rangle + \langle z, z' \rangle + 4rr',$$

and the left-invariant metric it defines on $S$ we denote also by $\langle \cdot, \cdot \rangle_S$.

In the next part we shall prove that the above construction includes the noncompact rank one symmetric spaces (that is, hyperbolic spaces) as particular cases.

2. Spaces $(S, \langle \cdot, \cdot \rangle_S)$ as a generalization of hyperbolic spaces. Let $G$ denote a connected semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $B$ the Killing form of $\mathfrak{g}$, $\theta$ the Cartan involution, and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ the Cartan decomposition. Let $K$ be the connected subgroup of $G$ with the Lie algebra $\mathfrak{l}$ and let $\varphi: G \rightarrow G/K$ be given by $\varphi(g) = gK$. The space $G/K$ with $G$-invariant Riemannian structure is a symmetric space and it does not depend on the choice of the Cartan decomposition and $G$-invariant metric $Q$. If $G$ is one of the groups $\text{SO}_0(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$, $F_{4(-20)}$ (see [1], [3], [16]), and $Q_k = B \circ (d\varphi_g|_\mathfrak{l})^{-1}$, we get all, up to isometry, noncompact rank one symmetric spaces [3].

Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. For the hyperbolic spaces the subalgebra $\mathfrak{a}$ is one-dimensional; hence for a root $\alpha$ the positive part of the set of restricted roots is either $\{\alpha\}$ or $\{\alpha, 2\alpha\}$. The root spaces $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$ corresponding to $-\alpha$ and $-2\alpha$ are orthogonal relatively to the inner product $\langle \cdot, \cdot \rangle_\theta = -B(\cdot, \theta \cdot)$. Then

$$\mathfrak{n} = \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$$

is a nilpotent algebra with the center $\mathfrak{g}^{-2\alpha}$ and we have the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{l}.$$

$\mathfrak{n}$ and $\mathfrak{a}$ are $\langle \cdot, \cdot \rangle_\theta$-orthogonal ([17], p. 163–168).

Let now $N$ and $A$ be the connected subgroups of $G$ with Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$, respectively. The map $(n, a, k) \rightarrow nak$ is a diffeomorphism of $N \times A \times K$ on $G$, and $S = NA$ is a closed subgroup of $G$ [17]. Obviously, $S$ with the
metric $f^* Q$ induced from $G/K$ by the diffeomorphism $f = \varphi|_S$ is a symmetric space.

**Proposition 2.1.** $f^* Q$ is a left-invariant metric given by

$$(f^* Q)_e(x_1 + y_1, x_2 + y_2) = \frac{1}{2} \langle x_1, x_2 \rangle_0 + \langle y_1, y_2 \rangle_0,$$

where $x_i \in \mathfrak{n}$ and $y_i \in \mathfrak{a}$ (see [8]).

**Proof.** Let $L(s) : S 	o S$ and $\tau(s) : G/K \to G/K$ be defined by

$$L(s_1) = ss_1 \quad \text{and} \quad \tau(s)(gK) = sgK.$$ 

Then $\tau(s) \circ f = f \circ L(s)$, whence

$$d\tau(s) \circ df = df \circ dL(s).$$ 

Let $X$ and $X'$ be left-invariant vector fields on $S$. Then

$$d\tau(s)(df(X)) = df(dL(s)(X)) = df(X')$$

and

$$(f^* Q)_e(X, X') = Q_{sk}(df_e(X_s), df_e(X'_s))$$

$$= Q_{sk}(d\tau(s)_k \circ df_e(X_s), d\tau(s)_k \circ df_e(X'_s))$$

$$= Q_k(df_e(X_s), df_e(X'_s))$$

$$= B\left(\frac{1}{2}(X_e - \theta X_e), \frac{1}{2}(X'_e - \theta X'_e)\right)$$

$$= -\frac{1}{2} \langle X_e, \theta X_e \rangle_0 + \frac{1}{4} \langle X_e, X'_e \rangle_0.$$ 

Now, putting in the above formula left-invariant vector fields belonging to $\mathfrak{n}$ or $\mathfrak{a}$, we obtain immediately (2.1).

If $n$ and $N$ and, consequently, $s$ and $S$ are of the forms just described, we call them classical and we shall often write $n \in \mathfrak{g}$, $N \in \mathfrak{g}$, $S \in \mathfrak{g}$, $s \in \mathfrak{g}$.

**Proposition 2.2.** If $S \in \mathfrak{g}$, then $(S, f^* Q)$ is a particular case of construction (1.2).

**Proof.** $n$ with the inner product

$$\langle \cdot, \cdot \rangle_n = \frac{1}{m_a + 4m_{2a}} \langle \cdot, \cdot \rangle_0$$

is an algebra of type $H$ (see [10]), where $m = \dim g^{-a}$ and $m_{2a} = \dim g^{-2a}$.

Let $h \in a$ be such that $\alpha(h) = 1$ and let $v \in g^{-a}$, $z \in g^{-2a}$. Then

$$[v + z + lh, v' + z' + l'h] = -lv' + l'v - 2lz' + 2l'z + [v, v']$$

and (cf. [8])

$$\langle h, h \rangle_0 = B(h, h) = 2(m_a + 4m_{2a}).$$

Hence

$$\langle v + z + lh, v' + z' + lh' \rangle_0 = (m_a + 4m_{2a}) (\langle v + z, v' + z' \rangle_n + 2ll').$$
According to Proposition 2.1, $S$ with the left-invariant metric given by
\[
\langle v + z + lh, v' + z' + lh' \rangle_S = \frac{1}{2} (m_z + 4m_{2z}) (\langle v + z, v' + z' \rangle_u + 4l l')
\]
is a symmetric space. Since a connected and simply connected Lie group is uniquely determined by its algebra, construction (1.2) includes $(S, f^* Q)$, $S \in \mathcal{G}$.

3. Some properties of the composition of quadratic forms. From now on we assume that $u, u' \in U$, $z, z' \in Z$, $v, v' \in V$. In the sequel we need the following properties of the composition of quadratic forms:

(3.1) \[ \langle \mu(u, v), \mu(u', v') \rangle = |u|^2 \langle v, v' \rangle, \]
(3.2) \[ \langle \mu(u, v), \mu(u', v) \rangle = |v|^2 \langle u, u' \rangle. \]

Let $A_u: V \to V$ and $B_v: U \to U$ be the mappings defined by
\[ A_u(v) = \mu(u, v) \quad \text{and} \quad B_v(u) = \mu(u, v). \]

(3.3) $A_u$ is an isomorphism. If $|u| = 1$, then $A_u$ is an orthogonal mapping.
(3.4) $B_v$ is a monomorphism. If $|v| = 1$, then $B_v$ is an orthogonal mapping.

(3.5) \[ \mu(z, \mu(z, v)) = -|z|^2 v. \]

For the proof of (3.1)--(3.5) see [5] and [12].

(3.6) \[ \mu(z, \mu(z', v)) = -\mu(z', \mu(z, v)) - 2 \langle z, z' \rangle v. \]

We easily obtain (3.6) applying (3.5) to $z + z'$ and $v$.

(3.7) \[ D(v) = \{ \mu(z, v): z \in Z \}. \]

Indeed, it is sufficient to notice that $\mu(z, v) \in D(v)$ and $\text{dim} \{ \mu(z, v): z \in Z \} = \text{dim} Z$.

(3.8) \[ \langle \mu(u, v), \mu(u', v') \rangle + \langle \mu(u, v'), \mu(u', v) \rangle = 2 \langle v, v' \rangle \langle u, u' \rangle. \]

Applying (3.2) to $u$ and $v + v'$ we get (3.8).

(3.9) \[
\pi \Phi(v, \mu(z, v')) = -\pi \Phi(v', \mu(z, v)) + 2 \langle v, v' \rangle z, \\
\pi \Phi(\mu(z, v'), v) = -\pi \Phi(\mu(z, v), v') - 2 \langle v, v' \rangle z.
\]

To see (3.9) we use (1.1) and (3.8).

(3.10) \[ \pi \Phi(v, \mu(z, v)) = |v|^2 z. \]
(3.11) \[ \pi \Phi(\mu(z, v), \mu(z, v')) = -|z|^2 \pi \Phi(v, v') + 2 \langle z, \pi \Phi(v, v') \rangle z. \]

We change the places of $\mu(z, v)$ and $v'$ according to (3.9) and use (3.5).
Then we obtain (3.10) and (3.11).

\begin{equation}
\pi \Phi(\mu(z, v), \mu(z, v')) = 0 \quad \text{iff} \quad \pi \Phi(v, v') = 0.
\end{equation}

This follows immediately from (3.11) and (3.5).

4. The group of isometries of \((S, \langle \cdot, \cdot \rangle_8)\). The left-invariant Riemannian connection on a Lie group with a left-invariant metric satisfies (see [13])

\begin{equation}
\langle \nabla_x y, u \rangle = \frac{1}{2}(\langle [x, y], u \rangle - \langle [y, u], x \rangle + \langle [u, x], y \rangle),
\end{equation}

where \(x, y, u\) belong to the Lie algebra. Let \(e_0 = \frac{1}{2} h_0\) (see (1.3)). Using (4.1) we obtain

\begin{align*}
\nabla_{e_0} x &= 0, \quad \nabla_{e_0} e_0 = -\frac{1}{2} r v, \\
\nabla_r e_0 &= -r z, \quad \nabla_r v = \nabla_v z = -\frac{1}{2} \mu(z, v), \\
\nabla_z' &= \langle z, z' \rangle e_0, \quad \nabla_v v' = \frac{1}{2} \langle v, v' \rangle e_0 + \frac{1}{2} \pi \Phi(v, v').
\end{align*}

Now we calculate \(R\) and \(\nabla R\) in the required cases:

\begin{align*}
R(e_0, v, v') &= -\frac{1}{2}(\langle v, v' \rangle e_0 + \pi \Phi(v, v')), \\
(4.4) \quad R(e_0, v, e_0) &= \frac{1}{2} v, \\
(4.5) \quad R(e_0, z, e_0) &= z, \\
(4.6) \quad R(v, e_0, z) &= -\frac{1}{2} \mu(z, v), \\
(4.7) \quad R(z, v, v') &= -\frac{1}{2} \langle v, v' \rangle z + \frac{1}{2} \langle \mu(z, v), v' \rangle e_0 + \frac{1}{2} \pi \Phi(v, \mu(z, v')), \\
\end{align*}

\begin{align*}
R(v, v', v'') &= -\frac{1}{4} \langle v', v'' \rangle v + \frac{1}{4} \langle v, v'' \rangle v' - \frac{1}{4} \mu(\pi \Phi(v', v''), v) + \\
&\quad + \frac{1}{4} \mu(\pi \Phi(v, v''), v') + \frac{1}{2} \mu(\pi \Phi(v, v'), v''), \\
(4.9) \quad R(z, v, z') &= -\frac{1}{4} \mu(z', \mu(z, v)), \\
(4.10) \quad R(z, e_0, v) &= -\frac{1}{2} \mu(z, v), \\
(4.11) \quad R(z, z', v) &= -\frac{1}{2} \mu(z', \mu(z, v)) - \frac{1}{2} \langle z, z' \rangle v, \\
(4.12) \quad R(z, v, e_0) &= -\frac{1}{2} \mu(z, v), \\
(4.13) \quad \nabla R(e_0, x, y, t) &= 0, \\
(4.14) \quad \nabla R(z, z', v, v') &= 0.
\end{align*}

By (4.3) and (4.7) we get

\begin{align*}
\nabla R(z, z', v, v') &= \left( -\frac{1}{4} \langle v, v' \rangle \langle z, z' \rangle + \frac{1}{8} \langle \mu(z, v), \mu(z', v') \rangle + \\
&\quad + \frac{1}{8} \langle \mu(z', v), \mu(z, v) \rangle e_0 - \frac{1}{4} \langle \mu(z, v), v' \rangle z + \\
&\quad + \frac{1}{4} \langle z, z' \rangle \pi \Phi(v, v') + \frac{1}{8} \pi \Phi(\mu(z, v), \mu(z', v')) + \\
&\quad + \frac{1}{8} \pi \Phi(v, \mu(z', \mu(z, v'))). \right)
\end{align*}
Applying (3.6) and (3.9) to $\pi \Phi(v, \mu(\varepsilon, \mu(\varepsilon)))$, and (3.8) to the expression by $e_0$ we get (4.14).

The map $u \to A_u$ (cf. (3.3)) defines a structure of a Clifford module over the Clifford algebra $C(Z, -|\cdot|^2)$. From the classification of these objects (see [4], [7], and [12]) one can see that for $\dim Z = 0, 1, 3, 7$ the type $H$ algebra can be regarded as

$$n = F^n \times F_0$$

with the bracket

$$[(q, p), (q', p')] = (0, 2\text{Im } \bar{q} q')$$

and the inner product

$$\langle(q, p), (q', p')\rangle = 4\text{Re}(\bar{q} q' + \bar{p} p'),$$

where $F$ denotes the algebra of real ($R$), complex ($C$), quaternionic ($H$) or Cayley numbers ($O$), and

$$F_0 = \{p \in F: \bar{p} = -p\},$$

$$q = (q_1, \ldots, q_n), \quad \bar{q} = (\bar{q}_1, \ldots, \bar{q}_n),$$

$$qq' = \sum_{i=1}^{n} q_i q'_i.$$ The suitable composition of quadratic forms $\mu: F \times F^n \to F^n$ is $\mu(p, q) = 2qp$. For $\dim Z = 0, 1, 3$ and $\dim Z = 7, n = 1$, in such a manner we obtain the classical algebras. They have the following property:

**Proposition 4.1.** Let

$$O(v) = \{\mu(u, v): u \in U\}.$$ If $n \in \mathcal{C}$, then for every $v$ we have

$$\langle q', \mu(p, q) \rangle = 8\text{Re}(\bar{q}' q(p)) = 8\text{Re}(\bar{q} q'(p) q(p)).$$

Hence $\bar{q}' q = 0$ and

$$\langle\mu(p', q'), \mu(p, q)\rangle = 16\text{Re}(\bar{q}' p' \cdot qp) = 16\text{Re}(\bar{p}' \bar{q}' q(\bar{p}' q) p) = 0.$$ If $F = O$ and $n = 1$, then $O(q) = F$, and (4.17) holds trivially.

**Remark.** If $F = O$ and $n > 1$, then we can only say that (4.17) is satisfied by some $v$; for example, $v = (q_1, 0, \ldots, 0)$. At the same time we have
THEOREM 4.1. If (4.17) holds for some \( v \), then \( \dim Z = 0, 1, 3, 7 \).

Proof. Let \( v \in V \) be such that, for every \( v' \) orthogonal to \( O(v) \), \( O(v') \) is orthogonal to \( O(v) \). First we show that \( \mu(U \times O(v)) \subset O(v) \), that is \( A_u(O(v)) \subset O(v) \) for \( |u| = 1 \).

Consider the orthogonal decomposition \( V = O(v) \oplus V_1 \). Let \( w \in V_1 \). Then \( A_u(w) \in O(w) \) and \( O(w) \perp O(v) \). Hence \( A_u(w) \in V_1 \). This implies that \( V_1 \) and \( O(v) \) are invariant subspaces of \( A_u \).

Let \( \bar{\mu} : U \times O(v) \to O(v) \) has all the properties of the composition of quadratic forms and \( \dim U = \dim V \). Hence \( \dim Z = 0, 1, 3, 7 \).

Proposition 4.1 and Theorem 4.1 imply that if \( n \notin \mathcal{O} \), then either \( n \) is of the form (4.15) for \( F = O \) and \( n > 1 \) or \( \dim Z \neq 0, 1, 3, 7 \), and then \( n \) satisfies the condition

\[(4.18) \quad \text{For every } v \neq 0 \text{ there is } v' \text{ orthogonal to } O(v) \text{ such that } O(v') \text{ and } O(v) \text{ are not orthogonal to each other.}
\]

We will show that in both cases \( VR(x) = 0 \) if and only if \( x \in \mathcal{O} \), where \( VR(x) \) denotes the tensor field arising from \( VR \) by fixing \( x \) at the first place. The proof is based on a common property contained in Proposition 4.2 for the first and in Proposition 4.3 for the second case.

The case \( \dim Z = 7 \). In this case we have the orthogonal decomposition of \( V = V_1 \oplus \cdots \oplus V_n \), \( n > 1 \), such that \( \mu(U \times V_l) \subset V_l \) and \( \pi \Phi(w_i, w_j) = 0 \) for \( w_i \in V_i, w_j \in V_j, i \neq j \). We denote by \( v_i \) the \( i \)-th component of \( v \) according to the above decomposition.

First we show the following lemmas:

**Lemma 4.1.** (a) If \( w_i \in V_i, w_j \in V_j, i \neq j \), then

\[
VR(v, z, w_i, w_j) = \frac{1}{2} \mu(\pi \Phi(\mu(z, v_i), w_i), w_j) + \frac{1}{2} \langle v_i, w_i, r \rangle \mu(z, w_j) + \\
+ \frac{1}{2} \mu(\pi \Phi(v_i, w_i), \mu(z, w_j)) + \frac{1}{2} \langle z, \pi \Phi(v_i, w_i) \rangle w_j + \\
+ \frac{1}{2} \langle \mu(z, v_j), w_j \rangle w_i + \frac{1}{2} \mu(\pi \Phi, w_j), w_j) + \\
+ \frac{1}{2} \mu(\pi \Phi(\mu(z, v_j), w_j), w_i) + \frac{1}{2} \langle v_j, w_j, r \rangle \mu(z, v_i).
\]

(b) If \( w_i, w_i' \in V_i, w_j \in V_j, i \neq j \), then

\[
VR(z, w_i, w_i', w_j) = \frac{1}{2} \mu(\pi \Phi(w_i, w_i'), \mu(z, w_j)) + \\
+ \frac{1}{2} \mu(\pi \Phi(\mu(z, w_i), w_i'), w_j) + \\
+ \frac{1}{2} \langle w_i, w_i', r \rangle \mu(z, w_j) + \frac{1}{2} \langle z, \pi \Phi(w_i, w_i') \rangle w_j.
\]

Proof. (a) By (4.7), \( R(z, w_i, w_j) = 0 \). In view of (4.2) and (4.8) we obtain

\[-R(V_0 z, w_i, w_j) = \frac{1}{2} R(\mu(z, v_i), w_i, w_j) + \frac{1}{2} R(\mu(z, v_i), w_i, w_j).\]
\[ = \frac{1}{4} \mu (\pi \Phi (\mu (z, v_i), w_i), w_j) + \frac{1}{8} \langle \mu (z, v_j), w_j \rangle w_i + \]
\[ + \frac{1}{8} \mu (\pi \Phi (\mu (z, v_j), w_j), w_i). \]

Moreover, by (4.10) and (4.11) we have

\[ -R(z, V_v w_i, w_j) = -R(z, V_v w_i, w_j) = \frac{1}{4} \langle v_i, w_i \rangle \mu (z, w_j) + \]
\[ + \frac{1}{4} \mu (\pi \Phi (v_i, w_i), \mu (z, w_j)) + \frac{1}{4} \langle z, \pi \Phi (v_i, w_i) \rangle w_j \]

and by (4.9) and (4.12) we get

\[ -R(z, w_i, V_v w_j) = -R(z, w_i, V_v w_j) = \frac{1}{8} \mu (\pi \Phi (v_j, w_j), \mu (z, w_i)) + \frac{1}{8} \langle v_j, w_j \rangle \mu (z, w_i). \]

Putting this together we obtain the assertion.

(b) We use (a) and the Bianchi identity

\[ \nabla R(z, w_i, w_i', w_j) = \nabla R(w_i, z, w_i', w_j) - \nabla R(w_i', z, w_i, w_j). \]

**Lemma 4.2.** (a) For every \( q \neq 0 \) in \( O \) there are \( p \in O_0 \) and \( q_1, q_2 \in O \) such that

\[ (q_1, p)(q_2, q) = (p, q_1)(q_2, q) \neq 0. \]

(b) For every \( p \neq 0 \) in \( O_0 \) there are \( q_1, q_2, q_3 \in O \) such that

\[ (q_1, p)(q_2, q_3) - (q_1, p)(q_2, q_3) \neq 0. \]

**Proof.** We identify \( O \) with \( H + H_0 \) with the multiplication defined by

\[ (bc)e = (b \cdot c)e, \quad b(ce) = (cb)e, \quad (be)(ce) = -\bar{c}b \quad \text{for } b, c \in H. \]

For \( q = b + ce, q \neq 0 \) in \( O \), the selection of \( q_1, p, \) and \( q_2 \) is shown in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>( q_1 )</th>
<th>( p )</th>
<th>( q_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b \neq 0 )</td>
<td>( e )</td>
<td>( i )</td>
<td>( q_2 \in H ) is such that ( bq_2 = j )</td>
</tr>
<tr>
<td>( b = 0, c \notin R )</td>
<td>( e )</td>
<td>( p \in H_0 ) is such that ( cp - pc \neq 0 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( b = 0, c \in R )</td>
<td>( 1 )</td>
<td>( i )</td>
<td>( j )</td>
</tr>
</tbody>
</table>

(b) For \( p = b + ce, b \in H_0 \), the selection of \( q_1, q_2, \) and \( q_3 \) is shown in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b \neq 0 )</td>
<td>( q_1 \in H ) is such that ( q_1 b - bq_1 \neq 0 )</td>
<td>( q_2 \in H ) is such that ( c \bar{q}_2 - \bar{q}_2 c \neq 0 )</td>
<td>( e )</td>
</tr>
<tr>
<td>( b = 0, c \notin R )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( e )</td>
</tr>
<tr>
<td>( b = 0, c \in R )</td>
<td>( i )</td>
<td>( j )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>
A simple calculation shows that with these values of \( p, q_1, q_2, \) and \( q_3, \)
(4.19) and (4.20) do not vanish.

**Proposition 4.2.** If \( n = O^n \times O, \ n > 1, \) then
(a) for every \( v \neq 0 \) there are \( z, v', v'' \) such that
\[
VR(v, z, v', v'') \neq 0.
\]
(b) for every \( z \neq 0 \) there are \( v', v'', v''' \) such that
\[
VR(z, v', v'', v''') \neq 0.
\]

**Proof.** (a) If \( v = (q_1, \ldots, q_n), q_i \neq 0, \) we put \( z = p \in O_0 \) and
\[
v' = (0, \ldots, q_i, \ldots, 0), \quad v'' = (0, \ldots, q_j, \ldots, 0), \quad i \neq j.
\]
Then Lemma 4.1 implies
\[
(VR(v, z, v', v''))_j = 2 ((q_j p)(\bar{q}_i q_i) - q_j ((p\bar{q}_i) q_i)).
\]

By Lemma 4.2 the right-hand side does not vanish for suitable \( p, q_i, \)
and \( q_j. \)

(b) If we put \( z = p \in O_0 \) and
\[
v' = (0, \ldots, q_i, \ldots, 0), \quad v'' = (0, \ldots, q_i, \ldots, 0), \quad v''' = (0, \ldots, q_j, \ldots, 0),
\]
then by Lemma 4.1 we have
\[
VR(z, v', v'', v''') = 4 ((q_j p)(\bar{q}_i q_i) - q_j ((p\bar{q}_i) q_i)).
\]
The assertion follows now from Lemma 4.2.

The case \( \dim Z \neq 0, 1, 3, 7. \) Condition (4.18) is equivalent to

(4.21) For every \( v \neq 0 \) there is \( v' \) orthogonal to \( O(v) \) such that
\( D(v) \subset \ker \text{ad}_v. \)

For the proof of this equivalence it is sufficient to notice that \( O(v) \) is an
orthogonal sum of \( \text{lin} \{v\} \) and \( D(v). \)

Before showing in this case the analogy of Proposition 4.2 we prove a
few lemmas.

**Lemma 4.3.** The following conditions are equivalent:
(i) \( \pi \Phi (v, v') = 0 \) and \( v \) is orthogonal to \( v'. \)
(ii) \( v' \) is orthogonal to \( O(v). \)
(iii) \( v \) is orthogonal to \( O(v'). \)
The simple proof is omitted.

**Lemma 4.4.** If \( v' \) is orthogonal to \( O(v) \) and \( \pi \Phi (v', \mu(z, v)) \neq 0, \) then
\[
VR(v, z, v', \mu(z, v)) \neq 0, \quad VR(z, v, v', \mu(z, v)) \neq 0.
\]
Proof. First we prove
\begin{equation}
\mathcal{VR}(v, z, v', \mu(z, v)) = \frac{3}{8} \mu(\pi \Phi(\mu(z, v), v'), \mu(z, v)), \mu(z, v')).
\end{equation}
(4.22)

We can assume that \(|v| = |z| = |v'| = 1\). By (4.7) and Lemma 4.3 we get
\[
\mathcal{VR}(v, z, v', \mu(z, v)) = \frac{1}{4} \mu(\pi \Phi(\mu(z, v), v'), \mu(z, v)) - \mathcal{VR}(z, v, \pi \Phi(v, \mu(z, v))).
\]

Now applying (3.10), (4.8), and (4.9) we obtain the first equality of (4.22). Formulas (4.7)–(4.9) and (3.10) applied to \(\mathcal{VR}(v', z, v, \mu(z, v))\) imply
\[
\mathcal{VR}(v', z, v, \mu(z, v)) = \frac{1}{4} \mu(\pi \Phi(\mu(z, v'), v), \mu(z, v)) - \\
- \frac{1}{8} \mu(z, \mu(\pi \Phi(v', \mu(z, v)), v)).
\]

Now we transform the first summand according to (3.9), the second according to (3.6) and we obtain
\[
\mathcal{VR}(v', z, v, \mu(z, v)) = -\frac{3}{8} \mu(\pi \Phi(\mu(z, v), v'), \mu(z, v)).
\]

The last result combined with the Bianchi identity
\[
\mathcal{VR}(z, v, v', \mu(z, v)) = \mathcal{VR}(v, z, v', \mu(z, v)) - \mathcal{VR}(v', z, v, \mu(z, v))
\]
gives the second equality of (4.22).

Now it is sufficient to notice that
\[
\langle \mu(\pi \Phi(\mu(z, v), v), \mu(z, v), v') \rangle = |\pi \Phi(\mu(z, v), v')|^2.
\]

**Lemma 4.5.** If \(n\) satisfies (4.21), then for every \(z \in Z\) we have
\begin{equation}
(4.23) \quad \text{There are } w, w' \in V \text{ such that } w' \text{ is orthogonal to } O(w) \text{ and } \pi \Phi(v', \mu(z, v)) \neq 0.
\end{equation}

Proof. By assumption, (4.23) holds for a \(z_0\). We can assume \(|z_0| = 1\). Let \(v\) and \(v'\) be such that \(v'\) is orthogonal to \(O(v)\) and \(\pi \Phi(v', \mu(z, v)) \neq 0\). It is sufficient to prove (4.23) when \(|z| = 1, z \notin \text{lin } \{z_0\}\) and \(\pi \Phi(v', \mu(z, v)) = 0\). We put
\[
w = \mu(z - z_0, v) \quad \text{and} \quad w' = \mu(z - z_0, v').
\]

By (3.1), \(w\) is orthogonal to \(w'\) and, by (3.12), \(\pi \Phi(w, w') = 0\). Hence \(w'\) is orthogonal to \(O(w)\). Moreover, \(\langle z, z - z_0 \rangle \neq 0\) and
\[
\pi \Phi(\mu(z - z_0, v'), \mu(z, \mu(z - z_0, v)))
\]
\[
= -\pi \Phi(\mu(z - z_0, v'), \mu(z - z_0, \mu(z, v))) - \\
- 2 \langle z, z - z_0 \rangle \pi \Phi(\mu(z - z_0, v'), v)
\]
\[ 2 \langle z, z - z_0 \rangle \pi \Phi(\mu(z - z_0, v), v') = 2 \langle z, z - z_0 \rangle \pi \Phi(v', \mu(z_0, v)) \neq 0. \]

From Lemmas 4.4 and 4.5 we conclude

**Proposition 4.3.** If \( \dim Z \neq 0, 1, 3, 7 \), then

(a) for every \( v \) there are \( z, v', \) and \( v'' \) such that
\[ \nabla R(v, z, v', v'') \neq 0; \]

(b) for every \( z \) there are \( v', v'' \), and \( v''' \) such that
\[ \nabla R(z, v', v'', v''') \neq 0. \]

This settles the second case.

**Theorem 4.2.** If \( s \notin s' \), then, for every \( x \notin a \), \( \nabla R(x) \neq 0. \)

**Proof.** Let \( x = v + z + re_0 \). If \( v \neq 0 \), then, by Propositions 4.2 (a) and 4.3 (a), \( \nabla R(v, z, v', v'') \neq 0 \) for suitable \( z, v', v'' \). Hence, also \( \nabla R(x, z, v', v'') \neq 0 \) in virtue of (4.13) and (4.14).

If \( v = 0 \) and \( z \neq 0 \), then by Propositions 4.2 (b) and 4.3 (b) there are \( v', v'', v''' \) such that \( \nabla R(z, v', v'', v''') \neq 0 \). Hence \( \nabla R(x, v', v'', v''') \neq 0. \)

Now we shall describe \( I(S) \) for non-classical \( S. \)

**Theorem 4.3.** If \( s \notin s' \) and \( L: s \to s \) is an orthogonal mapping such that

\[ L(\nabla R(x, y)) = \nabla R(L(x), L(y)), \quad x, y \in s, \tag{4.24} \]

then \( L \) is an automorphism of \( s. \)

**Proof.** Obviously,

\[ L(R(x, y, t)) = R(L(x), L(y), L(t)), \tag{4.25} \]

\[ L(\nabla R(x, y, t, w)) = \nabla R(L(x), L(y), L(t), L(w)), \tag{4.26} \]

and

\[ \nabla R(L(e_0), x, y, t) = L(\nabla R(e_0, L^{-1}(x), L^{-1}(y), L^{-1}(t))) = 0. \]

Hence \( L(e_0) = e_0, \) \( e = \pm 1, \) and \( L(n) \subset n. \) Now we prove the following:

(a) \( L(V) \subset V \) and \( L(Z) \subset Z, \)

(b) \( L([x, y]) = e [x, y], \) \( x, y \in s, \)

(c) \( L(\nabla R(x, y)) = e \nabla R(L(x), L(y)), \) \( x, y \in s. \)

(a) Let \( L(v) = v' + z'. \) Then by (4.4) and (4.5) we have
\[ L(R(e_0, v, e_0)) = \frac{1}{2} (v' + z'), \quad R(L(e_0), L(v), L(e_0)) = \frac{1}{2} v' + z'. \]

Hence \( z' = 0, \) \( L(V) \subset V, \) and \( L(Z) \subset Z. \)

(b), (c). By (4.6) and the last result we have
\[ L(R(v, e_0, z)) = -\frac{1}{2} L(\mu(z, v)), \]

\[ R(L(v), L(e_0), L(z)) = -\frac{1}{2} e \mu(L(z), L(v)). \]
Hence

\begin{equation}
L(\mu(z, v)) = \varepsilon \mu(L(z), L(v)).
\end{equation}

Combining (4.27) and (1.1) we get

\begin{equation}
L(\pi\Phi(v, v')) = \varepsilon\pi\Phi(L(v), L(v')),
\end{equation}

which implies immediately (b), while (b) with (4.1) gives (c). But in view of

(4.24) we have \( \varepsilon = 1 \), and \( L \) is an automorphism.

Remark. It is worth to notice here that the orthogonal automorphisms
of \( s \) (without any assumptions on \( S \)) preserve \( V \) and \( Z \) and are identities on
\( a \). This means that they are completely determined by the orthogonal
automorphisms of \( n \), and these have been investigated in [14].

Denote by \( A(S) \) the group of automorphisms of \( S \) preserving the inner
product \( \langle \cdot, \cdot \rangle_S \). We summarize the results above in the following

**Theorem 4.4.** If \( s \notin \mathfrak{C} \), then \( I(S) \) is a semi-direct product \( A(S) \times S \) (\( S \) acting
by left translations).

**Corollary 4.1.** If \( s \notin \mathfrak{C} \), then \( S \) is not a generalized symmetric space (for
definition see [11]).

**References**


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