ON CONFORMALLY RELATED
CONFORMALLY RECURRENT METRICS
I. SOME GENERAL RESULTS

BY

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1. Introduction. Let \((M, g)\) be a Riemannian manifold with a (possibly indefinite) metric \(g\).

A tensor field \(T^{i_1 \cdots i_p}_{j_1 \cdots j_q}\) of type \((p, q)\) on \(M\) is called recurrent if

\[
T^{h_1 \cdots h_l}_{i_1 \cdots i_p} T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = T^{h_1 \cdots h_l}_{i_1 \cdots i_p} T^{i_1 \cdots i_p}_{j_1 \cdots j_q},
\]

where the comma denotes covariant differentiation with respect to \(g\).

Relation (1) states that at any point \(x \in M\) such that \(T(x) \neq 0\) there exists a (unique) covariant vector \(u\) (called the recurrence vector of \(T\)) which satisfies the condition

\[
T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x) = u^k T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x).
\]

A Riemannian manifold \((M, g)\) is called recurrent (Ricci-recurrent) if its curvature tensor (Ricci tensor) is recurrent.

According to Adati and Miyazawa [1], an \(n\)-dimensional \((n \geq 4)\) Riemannian manifold \((M, g)\) is called conformally recurrent if its Weyl conformal curvature tensor

\[
C_{hijk} = R_{hijk} - \frac{1}{n-2} (g_{ij} R_{hk} - g_{ik} R_{hj} + g_{hk} R_{ij} - g_{hj} R_{ik}) + \frac{R}{(n-1)(n-2)} (g_{hk} g_{ij} - g_{ik} g_{hj})
\]

is recurrent.

If \(C_{hijk,l} = 0\) everywhere on \(M\) and \(\dim M \geq 4\), then \((M, g)\) is said to be conformally symmetric [3]. Such a manifold is called essentially conformally symmetric [5] if it is neither conformally flat \((C_{hijk} = 0)\) nor locally symmetric \((R_{hijk,l} = 0)\).

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as all recurrent manifolds of dimension \(n \geq 4\).
The existence of essentially conformally recurrent manifolds, i.e., of conformally recurrent manifolds which lie beyond the two classes mentioned above, has been established in [9]. Namely, using the expressions for components of the Weyl conformal curvature tensor and its covariant derivative (see [9], Lemmas 2-3), one can easily verify the following theorem:

**Theorem A.** Let $M$ denote the Euclidean $n$-space ($n \geq 4$) endowed with the metric $g_{ij}$ given by

\begin{equation}
 g_{ij} dx^i dx^j = Q (dx^1)^2 + k_{1\mu} dx^i dx^\mu + 2 dx^1 dx^n, \\
 Q = [(\exp (x^1)) c_{1\mu} + k_{1\mu}] x^i x^\mu,
\end{equation}

where $i, j = 1, 2, \ldots, n$, $\lambda, \mu = 2, 3, \ldots, n - 1$, $(k_{1\mu})$ is a symmetric and non-singular matrix, and $(c_{1\mu})$ is a symmetric matrix satisfying

$$(c_{1\mu}) \neq \frac{1}{n - 2} (k_{1\mu}) \quad \text{and} \quad k^{\lambda\mu} c_{1\mu} = 1 \text{ with } (k^{\lambda\mu}) = (k_{1\mu})^{-1}.$$

Then $(M, g)$ is an essentially conformally recurrent Ricci-recurrent manifold satisfying

\begin{equation}
 C_{hijk,lm} - C_{hijk,mi} = 0
\end{equation}

and its recurrence vector is non-zero everywhere on $M$.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold whose metric $g$ need not be definite. If $\bar{g}$ is another metric on $M$ and there exists a function $P$ on $M$ such that $\bar{g} = (\exp (2P)) g$, then $g$ and $\bar{g}$ are said to be conformally related or conformal to each other, and such a change of metric $g \rightarrow \bar{g}$ is called a conformal change. If $P = \text{const}$, then the conformal change of metric is called trivial or a homothety.

Conformally related conformally symmetric metrics have been studied by Adati and Miyazawa. Their main result ([2], Theorem 4.1) can be formulated as follows:

**Theorem B.** Let $M$ be a conformally symmetric manifold with positive definite metric $g$. If $\bar{g}$ is a conformally symmetric metric on $M$ such that $g$ and $\bar{g}$ are conformally related, then both $g$ and $\bar{g}$ are conformally flat or the conformal change of $g$ is a homothety.

Similar problems were studied by Miyazawa for positive definite conformally recurrent metrics (see [7], Theorem 1.1) as well as for conformally recurrent metrics with the same recurrence vectors (see [8], Theorem 3).

The purpose of the present paper is to investigate (without additional assumptions) conformally related conformally recurrent metrics. More precisely, we shall prove the following theorems:
THEOREM 1. Suppose that $M$ admits two conformally recurrent metrics $g$ and $\bar{g}$ conformally related by $\bar{g} = (\exp(2p))g$. Let $a_j$ and $\bar{a}_j$ be the recurrence vectors of $C$ and $\bar{C}$, respectively.

Then:

(a) $p_i C^h_{ikj} + p_k C^h_{ijk} + p_i C^h_{ik} = 0$ everywhere on $M$.

(b) At each given point of $M$ we have $C^h_{ijk} = 0 = \bar{C}^h_{ijk}$ or $\bar{a}_j = a_j - 4p_j$ and $p^r p_r = 0$, where $p_j = \partial_j p$.

THEOREM 2. Suppose that $(M, g)$ is conformally recurrent. If $p$ is a function on $M$ satisfying condition (a), then $\bar{g} = (\exp(2p))g$ is conformally recurrent.

THEOREM 3. Let $(M, g)$ be conformally symmetric. If $\bar{g}$ is a conformally symmetric metric on $M$ such that $g$ and $\bar{g}$ are conformally related, then both $g$ and $\bar{g}$ are conformally flat or the conformal change of $g$ is a homothety.

We shall also show that there exist conformally related and non-homothetic essentially conformally recurrent metrics.

All manifolds under consideration are assumed to be connected and of class $C^\infty$. The Riemannian metrics are not assumed to be definite.

2. Preliminary results. In the sequel we shall need the following lemmas:

LEMMA 1. The Weyl conformal curvature tensor satisfies the following well-known relations:

\begin{align}
C_{hijk} &= -C_{ihjk} = -C_{hikj} = C_{jkh}, \\
C_{hijk} + C_{hjki} + C_{hkij} &= 0, \quad C^r_{ijr} = C^r_{irk} = C^r_{rjk} = 0.
\end{align}

LEMMA 2. If $c_j$ and $T_{ij}$ are numbers satisfying

\begin{equation}
c_i T_{ij} + c_j T_{ij} + c_j T_{ij} = 0,
\end{equation}

then each $c_j$ is zero or each $T_{ij}$ is zero.

The proof is trivial.

LEMMA 3. If $c_j$, $p_j$, and $B_{hijk}$ are numbers satisfying

\begin{align}
c_i B_{hijk} + p_h B_{hijk} + p_i B_{bijk} + p_j B_{bijk} + p_k B_{bijk} &= 0, \\
B_{hijk} &= B_{jkh} = -B_{hik}, \quad B_{hijk} + B_{bijk} + B_{bijk} = 0,
\end{align}

then each $b_j = c_j + 2p_j$ is zero or each $B_{hijk}$ is zero.

Proof. Suppose that one of the $b$'s, say $b_q$, is not zero. Then (9) with $l = h = k = q$ gives $b_q B_{qijq} = 0$, since $B_{qijq} = 0 = B_{qijq}$, and, consequently, $B_{qijq} = 0$ for all $i$ and $j$. Setting $k = h = q$ in (9) and applying
\( B_{qij} = 0 \), we get

\[
\tag{11} p_q (B_{ijq} + B_{qij}) = 0 .
\]

Assume \( p_q = 0 \). Then \( c_q = b_q \neq 0 \) and, therefore, (9) with \( l = h = q \) yields \( c_q B_{qijk} = 0 \) since \( B_{qijq} = 0 \). Hence \( B_{qijk} = 0 \) for all \( i, j, \) and \( k \). Setting now \( l = q \) in (9) and using the first equation of (10), we obtain \( c_q B_{hij} = 0 \), whence \( B_{hij} = 0 \) for all \( h, i, j, \) and \( k \).

Suppose now that \( p_q \neq 0 \). Then (11), in view of the first equation of (10), implies

\[
\tag{12} B_{qij} = B_{qij} .
\]

But the second equation of (10) yields \( B_{qijk} + B_{qik} + B_{qij} = 0 \), whence, in view of (12), \( B_{qijk} + B_{qik} + B_{qij} = 0 \). Applying to \( B_{qijk} \) the condition (12), we obtain easily \( B_{qijk} = 0 \) for all \( j, k, \) and \( i \). Setting now \( h = q \) in (9) and taking into account \( B_{qijk} = 0 \), we get \( p_q B_{ijq} = 0 \), which, evidently, completes the proof.

**Lemma 4.** If \( c_i, p_j, \) and \( D_{hk} \) are numbers satisfying

\[
\tag{13} c_i D_{ij} + p_i D_{i} + p_j D_{ik} + p_k D_{ij} = 0 ,
\]

\[
\tag{14} D_{ijk} = - D_{ikj} , \quad D_{ijk} + D_{jki} + D_{kij} = 0 ,
\]

then each \( b_j = c_j + 2p_j \) is zero or each \( D_{ij} \) is zero.

**Proof.** Suppose that one of the \( b \)'s, say \( b_q \), is not zero. Then, by an argument similar to that in the proof of Lemma 3, we obtain \( D_{qij} = 0 \) and

\[
\tag{15} p_q (D_{ij} + D_{ij}) = 0 .
\]

Assume \( p_q = 0 \). Then \( c_q = b_q \neq 0 \) and, therefore, (13) with \( l = k = q \) yields \( c_q D_{qij} = 0 \) since \( D_{qij} = 0 = D_{qij} \). Hence \( D_{qij} = 0 \) for all \( i \) and \( j \). On the other hand, setting \( l = i = q \) in (13) and using the first equation of (14), we get \( c_q D_{qij} = 0 \), whence \( D_{qij} = 0 \) for all \( j \) and \( k \). Now, (13) with \( l = q \) implies \( c_q D_{qij} = 0 \). Thus \( D_{qij} = 0 \) for all \( i, j, \) and \( k \).

If now \( p_q \neq 0 \), then (15), in view of the first equation of (14), gives

\[
\tag{16} D_{ijq} = D_{qij} .
\]

On the other hand, the second equation of (14) implies \( D_{qij} + D_{jqi} + D_{kqj} = 0 \), which, because of (16), yields \( 2D_{qij} - D_{kqj} = 0 \). But the last result, in view of (16) and (14), gives \( D_{qij} = 0 \) for all \( j \) and \( k \). Setting now \( i = q \) in (13) and applying \( D_{qij} = 0 \), we obtain easily \( p_q D_{ij} = 0 \). Thus the lemma is proved.
Lemma 5. Let \( \bar{g}_{ij} = (\exp(2p))g_{ij} \). Then we have \([6], \text{p. } 89 \text{ and } 90\)

\[
\begin{align*}
\{ \bar{k} \mid \{k \mid \{ij \} + \delta_i^j p_j + \delta_j^k p_i - p^k g_{ij}, \\
\end{align*}
\]

\begin{align}
\bar{C}^{h}_{ijk} &= C^{h}_{ijk}, \tag{18} \\
\bar{R}_{ij} &= R_{ij} + (n-2)(p_{i,j} - p_{j,i}) + (p^r_r + (n-2)p^r p_r)g_{ij}, \tag{19}
\end{align}

where \( p_j = \delta_j p \) and \( p^k = g^{kr} p_r \).

Proposition 1. Let \( g \) and \( \bar{g} \) be two conformally recurrent metrics on \( M \). If \( g \) and \( \bar{g} \) are conformally related by \( \bar{g} = (\exp(2p))g \), then \( p_r C^r_{ijk} = 0 \). Proof. It is sufficient to prove our assertion in the open subset of \( M \) where \( C^h_{ijk} \neq 0 \).

Differentiating (18) covariantly and using (17), we get

\[
\bar{C}^{h}_{ijkl} = C^h_{ijkl} + \delta^h_i p_r C^r_{ijk} - 2p_i C^h_{ijk} - p^h C^r_{ijk} - p_i C^h_{ijkl} - p_l C^h_{ilk} - \\
    p_k C^h_{iji} + g_{il} p^l C^h_{rijk} + g_{ij} p^l C^h_{irk} + g_{ik} p^l C^h_{jlr},
\]

where the semicolon denotes covariant differentiation with respect to \( \bar{g} \).

Since \( g \) and \( \bar{g} \) are both conformally recurrent by assumption, relation (2) implies \( C^h_{ijkl} = a_i C^h_{ijk} \) and \( \bar{C}^h_{ijkl} = \bar{a}_i \bar{C}^h_{ijk} \), where \( a_j \) and \( \bar{a}_j \) denote the recurrence vectors of \( C \) and \( \bar{C} \), respectively.

Hence (20) can be written as

\[
(\bar{a}_i - a_i) C^h_{ijk} = g_{il} p_r C^r_{ijk} + g_{il} p^r C^h_{rijk} + g_{ij} p^l C^h_{irk} + \\
    g_{ik} p^l C^h_{jlr} - 2p_i C^h_{ijk} - p^h C^r_{ijk} - p_i C^h_{ijkl} - p_l C^h_{ilk} - p_k C^h_{ili},
\]

whence, by Lemma 1 and contraction with \( g^{h} \), we obtain

\[
d_r C^r_{ijk} = (n-3) p_r C^r_{ijk}, \tag{22}
\]

where \( d_r = \bar{a}_i - a_i \).

Transvecting now (21) with \( p^h \), we get

\[
d_l p_r C^r_{ijk} = -p_i p_r C^r_{ijk} - p^r p_r C^r_{ijk} - p_i p_r C^r_{ijk} - \\
    -p_j p_r C^r_{ilk} - p_k p_r C^r_{jil} - g_{il} T_{ik} + g_{kl} T_{ij}, \tag{23}
\]

where \( T_{ij} = p^r p^s C^r_{ij}. \)

But (23), by transvection with \( p^k \), implies

\[
p^s p_r C^r_{ijk} + p_s C^r_{jil} + p_i T_{ij} + p_j T_{il} = -d_l T_{ij}. \tag{24}
\]

Permuting the indices \( i, j, l \) in (24) cyclically, adding the resulting equations to (24) and using (7), we get

\[
(\bar{d}_r + 2p_l) T_{ij} + (\bar{d}_r + 2p_i) T_{ij} + (\bar{d}_r + 2p_j) T_{ji} = 0,
\]
which, evidently, is of the form (8). If, according to Lemma 2, \( d_j = -2p_j \), then (22) implies \((n-1)p_rC'_{ijk} = 0\). Hence \( T_{ij} = 0 \) at each point.

Substituting the last result into (24), we obtain

\[
p'r(p_sC'_{jil} + p_sC'_{jul}) = 0.
\]

Suppose now that at a given point we have \( p'r = 0 \). Then (23) implies

\[
(d_l + p_l)p_rC'_{ijk} + p_lp_rC'_{iik} + p_kp_rC'_{ikl} + p_kp_rC'_{uli} = 0.
\]

Hence, in view of Lemma 4, we have \( d_j = -3p_l \) or \( p_rC'_{ijk} = 0 \).

If \( d_j = -3p_l \), then (22) yields \( p_rC'_{ijk} = 0 \).

Assume now \( p_rC'_{ijk} + p_rC'_{iik} = 0 \). Then, in view of Lemma 1, we have \( p_rC'_{ijk} = p_rC'_{iik} \), which, in virtue of \( p_rC'_{ijk} + p_rC'_{iik} + p_rC'_{uli} = 0 \), gives \( 2p_rC'_{iik} = 0 \) and, consequently, \( 3p_rC'_{iik} = 0 \). Thus our assertion is proved.

3. Main results. We are now in a position to prove Theorems 1-3.

Proof of Theorem 1. It is sufficient to prove our assertion at points where \( C'_{ijk} \neq 0 \).

Since \( p_rC'_{ijk} = 0 \), (21) yields

\[
(d_l + 2p_l)C'_{hij} + p_hC'_{ijk} + p_lC'_{ijk} + p_jC'_{hik} + p_kC'_{hjl} = 0,
\]

whence, using Lemma 3, we get \( d_l = d_j = -4p_j \).

Hence (25) takes the form

\[
p_hC'_{ijk} + p_lC'_{hjl} + p_jC'_{hik} + p_kC'_{hjl} = 0.
\]

Permuting the indices \( l, j, k \) in (26) cyclically, adding the resulting equations to (26) and using Lemma 1, we obtain easily (a). But the last result, together with \( p_rC'_{ijk} = 0 \), implies \( p'r = 0 \), which completes the proof.

Proof of Theorem 2. Assume now (a). Then, by (6), we have

\[
(p_lC'_{hik} + p_kC'_{hjl} + p_lC'_{hik}) + (p_hC_{iik} + p_iC_{hij} + p_lC_{ikj}) = 0,
\]

whence

\[
-4p_iC'_{ijk} = -2p_iC'_{ijk} - p_hC_{iik} - p_jC_{hij} - p_kC_{hjl} = 0.
\]

But as an immediate consequence of (a) we have \( p_rC'_{ijk} = 0 \), which, together with (27) and (20), yields \( C_{ijk} + 4p_iC'_{ijk} = C_{ijk} \). Hence \( C_{ijk} = (a_j - 4p_j)C_{ijk} \) at points where \( C_{ijk} \neq 0 \), which, evidently, completes the proof.
Proof of Theorem 3. Since $\mathcal{C}$ and $\overline{\mathcal{C}}$ are parallel, we may assume that $\mathcal{C}_{ijk}^h \neq 0 \neq \overline{\mathcal{C}}_{ijk}^h$ everywhere. Hence, because of (2), $\bar{a}_j = a_j = 0$. But, in view of Theorem 1, the last relation implies $p_j = 0$, which completes the proof.

**Proposition 2.** For each $n \geq 4$, there exist $n$-dimensional pairwise conformally related and non-homothetic metrics $g$, $g_1$, $g_2$, $g_3$ such that $g$ satisfies (5) and

(i) $g$, $g_1$ are essentially conformally recurrent,

(ii) $g_2$ is recurrent,

(iii) $g_3$ is essentially conformally symmetric.

**Proof.** As one can easily verify, in the metric (4) the only Christoffel symbols not identically zero are

$$\text{[\begin{array}{c} \lambda \\ 11 \end{array}] = -\frac{1}{2} k^{1\mu} Q_{\cdot \mu}, \quad \text{[\begin{array}{c} n \\ 11 \end{array}] = \frac{1}{2} Q_{\cdot 1}, \quad \text{[\begin{array}{c} n \\ 11 \end{array}] = \frac{1}{2} Q_{\cdot 1},}$$

where the dot denotes partial differentiation with respect to coordinates.

It can be also found [9] that the only components of $R_{ij}$, $R_{ij,k}$, and $C_{hijk}$ not identically zero are those related to

$$R_{11} = n - 2 + \exp(x^1), \quad R_{11,1} = \exp(x^1),$$

$$C_{11\mu 1} = \left( c_{\mu} - \frac{1}{n-2} k_{\mu} \right) \exp(x^1).$$

Moreover, we can easily show that $a_j = \delta_j^1$ and that condition (a) is satisfied for $p = cx^1$, where $c$ is an arbitrary constant. In view of Theorem 2, the metric $\tilde{g}_{ij} = (\exp(2cx^1))g_{ij}$ is conformally recurrent. $\overline{\mathcal{C}}_{ijk}^h \neq 0$ everywhere and $\bar{a}_j = a_j - 4p_j = (1 - 4c) \delta_j^1$.

Hence $\tilde{g}$ is recurrent if and only if its Ricci tensor $\tilde{R}_{ij}$ satisfies

$$\tilde{R}_{ij,j} = \bar{a}_k \tilde{R}_{ij}. \quad (28)$$

In view of (17), (19), $p_{i,j} = 0$, and $p^r p_r = p^r_r = 0$, we get

$$\tilde{R}_{ij,k} = R_{ij,k} - 2p_k R_{ij} - p_i R_{jk} - p_j R_{ik} + 4(n-2)p_i p_j p_k. $$

Thus (28) can be written as

$$R_{ij,k} + 2p_k R_{ij} - p_i R_{jk} - p_j R_{ik} - a_k R_{ij} + (n - 2) a_k p_i p_j = 0. \quad (29)$$

If now $c = 1$, then (29) is satisfied and the metric $g_3 = (\exp(2x^1))g$ is recurrent. If $c = -1$, then we have $R_{11,1} + 2p_1 R_{11} - p_1 R_{11} - p_1 R_{11} + a_1 R_{11} + + (n - 2) a_1 p_1 p_1 = 3(n-2) \neq 0$. Hence $g_1 = (\exp(4x^1))g$ is essentially conformally recurrent. Finally, if $c = \frac{1}{4}$, then $\bar{a}_j = (1 - 4c) \delta_j^1 = 0$ and
$g_3 = (\exp(\frac{1}{2}ax^1))g$ is conformally symmetric. Since $\tilde{R}_{11,1} \neq 0$, $g_3$ is essentially conformally symmetric. This completes the proof.

As an immediate consequence of Theorems 1 and 2, we have the following

**Corollary.** Let $(M, g)$ be conformally recurrent and $p$ a function on $M$. Then $\tilde{g} = (\exp(2p))g$ is conformally recurrent if and only if $p$ satisfies condition (a).

**Remark.** Conformally symmetric manifolds with positive definite metrics are necessarily conformally flat or locally symmetric (see [4], Theorem 2). On the other hand, as proved in [5] (see Theorems 2 and 4), for each $q \in \{1, 2, \ldots, n-1\}$ there exist essentially conformally symmetric manifolds with metrics of index $q$.

Theorem 3 deals therefore with a more general class of Riemannian manifolds than Theorem B. Moreover, it implies Theorem B.

Theorem 3 can be also deduced from Miyazawa's Lemmas 1 and 2 of [8]. Theorem 1.1 of [7] and Theorem 3 of [8] are consequences of Theorem 1.

REFERENCES


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