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Boundary value problem for functional differential equations

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Abstract. By using the contingent equations technique we have proved the existence of solutions of the boundary value problem for non-linear functional differential equations. We have assumed that the right-hand side of functional equations satisfies the Carathéodory type conditions and comparative contingent equations satisfy the uniqueness conditions. We have also proved the existence of a solution of certain contingent equations.

1. Introduction. Let R^m be a real Euclidean m-space with the norm $|\cdot|$ and denote by $cf(R^m)$ a family of all non-empty, compact and convex subsets of R^m.

Let C_h denote a space of all continuous functions $[a-h, a]\to R^m$, $h>0$, with the norm of uniform convergence $\|x\| = \sup\{|x(t)|: t\in [a-h, a]\}$. For a continiuous function x: $[a-h, b]\to R^m$ and $t\in [a, b]$ define $x_t\in C_h$ by $x_t(\tau) = x(t+\tau-a)$, $\tau\in [a-h, a]$.

Let

$$f\colon\ [a, b]\times C_h \ni (t, a)\to f(t, a)\in R^m$$

and

$$F\colon\ [a, b]\times C_h \ni (t, a)\to F(t, a)\in cf(R^m)$$

be given.

For $b > a+h$ consider the non-linear functional differential equation

$$x' = f(t, x_t), \qquad a\leqslant t\leqslant b \tag{1.1}$$

and the non-linear functional differential equation with multi-valued right-hand side

$$x'\in F(t, x_t), \qquad a\leqslant t\leqslant b. \tag{1.2}$$

Equations (1.1) and (1.2) will be considered with the boundary condition

$$Mx_a+Nx_b = 0, \tag{1.3}$$

where M and N are $m\times m$ matrices.

By a *solution* $x(\cdot)$ of the boundary value problem (1.1), (1.3) (resp. (1.2), (1.3)) we mean any absolutely continuous function on $[a-h, b]$, satisfying (1.1) (resp. (1.2)) almost everywhere on $[a, b]$ and such that condition (1.3) holds.

Problem (1.1), (1.3) presents a non-linear version of a boundary value problem posed by Cooke [1]. This problem has been studied by R. Fennell and P. Waltman [2]. Using an approximation method, the above-mentioned authors have proved the existence of solutions of (1.1), (1.3) in the case of $f(t, a)$ continuous and bounded or Lipschitzian.

In this paper it will be shown that an application of the contingent technique to (1.1), (1.3) due to A. Lasota and Z. Opial [4] allows us to obtain the existence theorems under the more general assumption that f satisfies Carathéodory type conditions. The results of [2] are a special case of our theorems.

2. Preliminaries. Let B denote a Banach space with the norm $\|\cdot\|$. For $u \in B$, $A \subset B$ we write

$$\delta(u, A) = \inf\{\|u-v\|: v \in A\}, \qquad |A| = \sup\{\|u\|: u \in A\}.$$

Denote by $c(B)$ the family of all non-empty, convex subsets of B.

Let $H: B \to c(B)$. The map H will be called *compact* if for any bounded subset D of B the closure of the set $\bigcup_{u \in D} H(u)$ is compact in B. The map H will be called *homogeneous* if for every $u \in B$ and any real λ, $H(\lambda u) = \lambda H(u)$. The map H will be called *upper semi-continuous* if the set $\{(u, v): u \in B, v \in H(u)\}$ is closed in $B \times B$. The map H will be called *completely continuous* if it is compact and upper semi-continuous.

It is easy to see that a homogeneous map H is compact if and only if the closure of the set $\bigcup_{\|u\|=1} H(u)$ is compact.

In the space $cf(R^m)$ we introduce the Hausdorff distance by

$$d(C, D) = \max\{\sup_{p \in D} \delta(p, C), \sup_{p \in C} \delta(p, D)\}, \qquad C, D \in cf(R^m).$$

A map $F: B \to cf(R^m)$ will be called *continuous* if it is continuous in the Hausdorff topology.

We say that a map F of the compact interval $[a, b] \subset R^1$ into $cf(R^m)$ is *Lebesgue measurable* if, for each closed subset A of R^m, the set $\{t \in A: F(t) \cap A \neq \emptyset\}$ is Lebesgue measurable [5].

We say that a map $F(t, u)$ (resp. $f(t, u)$) of $[a, b] \times B$ into $cf(R^m)$ (resp. R^m) *satisfies the Carathéodory condition* if F (or f) is measurable in t for each $u \in B$ and continuous in u for each $t \in [a, b]$.

3. Existence and uniqueness. We make the following assumptions.

(i) $F(t, a)$ satisfies the Carathéodory conditions, is homogeneous with respect to a and

$$\sup_{\|a\|=1} |F(t, a)| \leqslant \varphi(t),$$

where $\varphi(t)$ is integrable on $[a, b]$;

(ii) $f(t, a)$ satisfies the Carathéodory conditions and

$$\lim_{n\to\infty} \frac{1}{n} \int_a^b \sup_{\|a\|\leqslant n} \delta\big(f(t, a), F(t, a)\big)dt = 0;$$

(iii) M and N are $m\times m$ matrices such that the matrices $M+N$ and $I-(M+N)^{-1}N$ are non-sigular.

THEOREM 3.1. *Suppose that* (i), (ii) *and* (iii) *hold. If* $x = 0$ *is the only solution of problem* (1.2), (1.3), *then there exists at least one solution of problem* (1.1), (1.3).

Proof. Let $T = -(M+N)^{-1}$. Let the mappings G and H of $B = C_{[a-h,b]}$ into B and $c(B)$ be defined by

(3.1) $G(x)(t)$

$$= \begin{cases} (I+TN)^{-1}TN\Big(\int_a^{b-a+t} f(s, x_s)ds + TN\int_a^b f(s, x_s)ds\Big) & \text{for } a-h\leqslant t\leqslant a, \\ \int_a^t f(s, x_s)ds + TN\int_a^b f(s, x_s)ds & \text{for } a< t\leqslant b, \end{cases}$$

and

(3.2) $H(x)(t)$

$$= \begin{cases} (I+TN)^{-1}TN\Big(\int_a^{b-a+t} u(s)ds + TN\int_a^b u(s)ds\Big) & \text{for } a-h\leqslant t\leqslant a, \\ \int_a^t u(s)ds + TN\int_a^b u(s)ds & \text{for } a< t\leqslant b, \end{cases}$$

respectively, where $u(s)\in F(s, x_s)$ is an arbitrary measurable function.

We will prove that x is a fixed point of the map G if and only if x is a solution of (1.1), (1.3). Let x: $[a-h, b]\to R^m$ be the solution of (1.1), (1.3). Integrating (1.1) over $[a, t]$ yields

$$x(t) = \int_a^t f(s, x_s)ds + x(a), \quad a\leqslant t\leqslant b. \tag{3.3}$$

Substituting (3.3) into (1.3), we have

$$Mx_a(t) + N\Big(\int_a^{b-a+t} f(s, x_s)ds + x(a)\Big) = 0, \quad a-h\leqslant t\leqslant a. \tag{3.4}$$

Hence for $t = a$ we obtain

$$x(a) = TN\int_a^b f(s, x_s)ds. \tag{3.5}$$

By (3.4) and the definition of T

$$(3.6)\qquad x_a^-(t) = (I+TN)^{-1}TN\Big(\int_a^{b-a+t} f(s, x_s)ds + x(a)\Big), \qquad a-h \leqslant t \leqslant a.$$

(3.3), (3.5) and (3.6) imply that x is the fixed point of h.

Suppose now that a function $x \in C_{[a-h,b]}$ is the fixed point of G. Then

$$(3.7)\qquad x(t) = \begin{cases} (I+TN)^{-1}TN\Big(\int_a^{b-a+t} f(s, x_s)ds + TN\int_a^b f(s, x_s)ds\Big), & a-h \leqslant t \leqslant a, \\ \int_a^t f(s, x_s)ds + TN\int_a^b f(s, x_s)ds, & a < t \leqslant b. \end{cases}$$

Differentiating the second part of (3.7) gives

$$x'(t) = f(t, x_t), \qquad a \leqslant t \leqslant b.$$

For $a-h \leqslant t < a$ from the first part of (3.7) we obtain

$$(3.8)\qquad Mx(t) = -N\Big(\int_a^{b-a+t} f(s, x_s)ds + TN\int_a^b f(s, x_s)ds\Big).$$

By (3.8) and (3.7) x satisfies (1.3). In the same manner one can prove that x is a fixed point of a map H if and only if x is a solution of (1.2), (1.3).

Thus, in order to apply Theorem 1.1 in [3] it remains to verify that

1° G is completely continuous;

2° $\lim\limits_{\|x\|\to\infty} \dfrac{\delta\big(G(x), H(x)\big)}{\|x\|} = 0$;

3° $x = 0$ is the unique vector of $C_{[a-h,b]}$ satisfying the condition $x \in H(x)$.

4° H is homogeneous and completely continuous.

By (ii), G and H satisfy 2°; moreover, G is completely continuous. If $x \in H(x)$, then x is a solution of (1.2), (1.3); hence $x = 0$.

For 4° it suffices to show that H is completely continuous since the homogeneouity of H is an immediate consequence of the definition H and (i).

First of all we show that H is compact. By (i) $|u(t)| \leqslant \varphi(t)$, where $\varphi(t)$ is an integrable function on $[a, b]$. Hence the function $H(x)$ is uniformly bounded and equicontinuous on $[a, b]$. Hence by the Arzelà theorem the closure of $\bigcup\limits_{\|x\|=1} H(x)$ is compact.

Let $\{x^n\}$, $\{y^n\}$ be the sequences of $C_{[a-h,b]}$ such that $y^n \in H(x^n)$ for $n = 1, 2, \ldots$, and $x^n \to x^0$, $y^n \to y^0$ as $n \to \infty$. We will prove that $y^0 \in H(x^0)$, which will imply that H is upper semi-continuous.

By (3.2)

$$(3.9)\qquad y^n(t) = \begin{cases} (I+TN)^{-1}TN\int\limits_a^{b-a+t} u^n(s)ds + TN\int\limits_a^b u^n(s)ds, & a-h \leqslant t \leqslant a, \\ \int\limits_a^t u^n(s)ds + TN\int\limits_a^b u^n(s)ds, & a < t \leqslant b, \end{cases}$$

where

$$u^n(s) \in F(s, x_s^n) \qquad (n = 1, 2, \ldots).$$

From (i) it follows that $|u^n(t)| \leqslant \tilde{\varphi}(t)$ for $t \in [a, b]$, where $\tilde{\varphi}(t)$ is integrable on $[a, b]$. By Lemma 2.1 in [3] there exists a function z^0 and sequences $\{z^n\}$ of the form

$$(3.10)\qquad z^n(t) = \sum_{k=n}^{a_n} \lambda_{kn} u^k(t),$$

where

$$\sum_{k=n}^{a_n} \lambda_{kn} = 1, \qquad \lambda_{kn} \geqslant 0,\ a_n > n,$$

converging to z^0 almost everywhere on $[a, b]$ as $n \to \infty$.

We write

$$F_\varepsilon(t, a) = \{q \in R^m:\ \delta(q, F(t, a)) \leqslant \varepsilon,\ \varepsilon > 0\}.$$

Obviously F_ε is compact and convex. Since $F(t, a)$ is continuous in a, for each $t \in [a, b]$ there exists a positive integer $n(t, \varepsilon)$ such that, for $n > n(t, \varepsilon)$, $F(t, x_t^n) \subset F_\varepsilon(t, x_t^0)$ for $t \in [a, b]$. $u^n(t) \in F_\varepsilon(t, x_t^0)$ for $n > n(t, \varepsilon)$ implies that

$$(3.11)\qquad z^n(t) \in F_\varepsilon(t, x_t^0) \quad \text{for } n > n(t, \varepsilon),\ a \leqslant t \leqslant b.$$

The condition $y^n \in H(x^n)$ can be written in the form (3.9) Multiplying (3.9) by λ_{kn} and summing over k, we get

$$\sum_{k=n}^{a_n^{n_n}} \lambda_{kn} y^n(t) = \begin{cases} (I+TN)TN\left(\int\limits_a^{b-a+t} z^n(s)ds + TN\int\limits_a^b z^n(s)ds\right), & a-h \leqslant t \leqslant a, \\ \int\limits_a^t z^n(s)ds + TN\int\limits_a^b z^n(s)ds, & a < t \leqslant b. \end{cases}$$

From the inequality

$$\left\|y^0 - \sum_{k=n}^{a_n} \lambda_{kn} y^k\right\| = \left\|\sum_{k=n}^{a_n} \lambda_{kn}(y^0 - y^k)\right\| \leqslant \sup \|y^k - y^0\|$$

it follows that the sequence $\sum\limits_{k=n}^{a_n} \lambda_{kn} y^k$ converges to y^0 uniformly on $[a, b]$.

By the continuity of the operators TN, $(I+TN)^{-1}$ and the Fatou Lemma we obtain, passing to the limit in (3.9),

$$y^0(t) = \begin{cases} (I+TN)^{-1}TN\Big(\int\limits_a^{b-a+t} z^0(s)\,ds + TN\int\limits_a^b z^0(s)\,ds\Big), & a-h \leqslant t \leqslant a, \\ \int\limits_a^t z^0(s)\,ds + TN\int\limits_a^b z^0(s)\,ds, & a < t \leqslant b. \end{cases}$$

This and condition (3.11) complete the proof.

Now we shall formulate the condition for the existence and uniqueness of solutions of the boundary value problem (1.1), (1.3).

Namely, we assume that

(iv) f is measurable in t for each $a \in C_h$ and satisfies the conditions

$$f(t, a)-f(t, \beta) \in F(t, a-\beta), \qquad \int\limits_a^b |f(t, 0)|\,dt < +\infty.$$

THEOREM 3.2. *Suppose f, F, M and N satisfy* (i), (iv), (iii), *respectively. If $x = 0$ is the only solution of problem* (1.2), (1.3), *then there exists exactly one solution of the problem* (1.1), (1.3).

Proof. Indeed, (i) and (iv) imply (i) and (ii). Hence the existence of a solution of problem (1.1), (1.3) is a consequence of Theorem 3.1.

To show the uniqueness, assume that x_1 and x_2 are two solutions of (1.1), (1.3). By (iv) and the linearity of M and N the function $x = x_1 - x_2$ satisfies (1.2) and (1.3). By the hypothesis of the theorem, $x_1 - x_2 = 0$.

In the case where the matrix M is invertible, the boundary condition (1.3) can be written in the form ($P = M^{-1}N$)

$$x_a + Px_b = 0. \tag{3.12}$$

An easy calculation shows that in this case $(I+TN)^{-1}NT = c-P$, $TN = -(I+P)^{-1}P$. If $\|P\| < 1$, where $\|P\|$ denotes any norm of P, then $I+P$ is invertible. Theorems 3.1 and 3.2 imply the following

THEOREM 3.3. *Let f: $[a, b] \times C_h \to R^m$, F: $[a, b] \times C_h \to cf(R^m)$ and let P be a matrix such that $\|P\| < 1$.*

If $x = 0$ is the only solution of problem (1.2), (3.12), *then*

1° *if the functions F and f satisfy conditions* (i) *and* (ii), *then the problem* (1.1), (3.12) *has at least one solution*;

2° *if the function F and f satisfy conditions* (i) *and* (iv), *then problem* (1.1), (3.12) *has exactly one solution.*

4. Application of differential inequalities. In this section we will prove some lemmas concerning the existence of solutions of the boundary value problem for certain contingent equations. As a consequence of those lemmas and the previous results we will obtain the existence and uniqueness theorems for the equations with deviated arguments. Our results generalize the theorems due to Fennell and Waltman [2].

Lemma 4.1. *Let M and N be $m \times m$ matrices such that $M+N$ is non-singular. Then the problem*

$$x' = 0, \quad Mx_a + Nx_b = 0 \tag{4.1}$$

has only the trivial solution $x = 0$.

Proof. The equation $x' = 0$ has only the constant solutions. Thus the existence of $(M+N)^{-1}$ implies that (4.1) has only the trivial solution.

Lemma 4.2. *Let M and N be $m \times m$ matrices such that $(M+N)^{-1}$ exists and is bounded. If the constant $K > 0$ satisfies*

$$e^{K(b-a)} < \frac{1 - \|TN\|}{\|TN\|}, \tag{4.2}$$

then the problem

$$|x'(t)| \leqslant K\|x_t\|, \quad Mx_a + Nx_b = 0 \tag{4.3}$$

has exactly one solution $x = 0$.

Proof. Let x be a solution of (4.3). By integrating (4.3) over $[a, t]$ we get

$$|x(t) - x(a)| \leqslant K \int_a^t \|x_s\| ds. \tag{4.4}$$

Thus

$$|x(t)| \leqslant |x+(a)| + K \int_a^t \|x_s\| ds,$$

which implies

$$u(t) \leqslant u(a) + K \int_a^t u(s) ds,$$

where

$$u(s) = \sup\{|x(\tau)|: \tau \in [s-h, s]\}.$$

By the Gronwall inequality

$$u(t) \leqslant u(a) e^{K(t-a)}. \tag{4.5}$$

Since $\|x_s\| = u(s)$, from (4.4) it follows that

$$|x(t) - x(a)| \leqslant K \int_a^t u(a) e^{K(s-a)} ds = u(a)(e^{K(t-a)} - 1).$$

Hence for $\tau \in [a-h, a]$ we have the inequality

$$\begin{aligned} |x(t) - x_a(\tau)| &\leqslant |x(t) - x(a)| + |x(a) - x_a(\tau)| \\ &\leqslant u(a)(e^{K(b-a)} - 1) + 2u(a) = u(a)(e^{K(b-a)} + 1). \end{aligned}$$

We write $z(t,\tau) = x(t)-x_a(\tau)$. By the last inequality, $|z| \leqslant u(a)(e^{K(b-a)}+1)$. Moreover, $x_b(\tau) = x_a(\tau)+z(b-a+\tau,\tau)$ for $a-h \leqslant \tau \leqslant a$. Hence the boundary condition may be written in the form

$$Mx_a+Nx_a+Nz = 0,$$

i.e.,

$$x_a = TNz \qquad (T = -(M+N)^{-1}).$$

Thus

$$\|x_a\| \leqslant \|TN\|\cdot\|z\| \leqslant \|TN\|(e^{K(b-a)}+1)u(a).$$

This implies that $u(a) = 0$; hence by (4.5) $x = 0$.

THEOREM 4.1. *If a function $f\colon [a,b]\times C_h \to R^m$ is bounded and satisfies the Carathéodory conditions, and M and N satisfy* (iii), *then problem* (1.1), (1.3) *has at least one solution.*

Proof. Define the operator $F_0\colon [a,b]\times C_h \to cf(R^m)$ by

$$F_0(t,a) = \{0\}.$$

Since f is bounded,

$$K = \sup\{|f(t,a)|\colon t\in[a,b],\ a\in C_h\} < +\infty.$$

Obviously

$$\sup\delta\big(f(t,a),F_0(t,a)\big) \leqslant K.$$

It is easy to see that the mappings F_0, f satisfy assumptions (i) and (ii). Hence Theorem 4.1 is a consequence of Theorem 3.1 and Lemma 4.1.

The last theorem immediately implies

THEOREM 4.2 (Fennell, Waltman [2]). *Let $f\colon [a,b]\times C_h \to R^m$ be continuous and bounded and let M and N be $m\times m$ matrices such that $M+N$ is non-singular. If $\|(M+N)^{-1}N\| < 1$, then problem* (1.1), (1.3) *has a solution.*

In the next two theorems we replace the boundedness of f by the Lipschitz condition.

THEOREM 4.3. *Let f satisfy the Carathéodory conditions and the inequality*

$$|f(t,a)| \leqslant K\|a\|+\varphi(t), \tag{4.6}$$

where $\varphi(t)$ is an integrable and non-negative function.

If M and N satisfy (iii) *and if* (4.2) *holds, then problem* (1.1), (1.3) *has at least one solution.*

Proof. We write

$$F_0(t,a) = \{q\in R^m\colon |q| \leqslant K\|a\|\}.$$

It is easy to see that F_0 is homogeneous in a and satisfies the Carathéodory conditions. By Lemma 4.2 the problem

$$x'\in F_0(t,x_t), \qquad Mx_a+Nx_b = 0$$

has only the trivial solution $x = 0$.

By (4.6)

$$\delta\big(f(t, a), F(t, a)\big) \leqslant K\|a\| + \varphi(t),$$

which implies that F_0 satisfies conditions (i) and (ii). Now a straightforward application of Theorem 3.1 completes the proof.

In a similar manner, from Theorem 3.2 we obtain

THEOREM 4.4. *Let the hypothesis of Theorem* 4.3 *hold with* (4.6) *replaced by*

$$|f(t, a) - f(t, \beta)| \leqslant K\|a - \beta\|, \qquad \int_a^b |f(t, 0)|\,dt < +\infty.$$

Then problem (1.1), (1.3) *has exactly one solution.*

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