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$L^{p,\Phi}$ spaces and their application to elliptic partial differential operators

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Abstract. In this paper we give an application of the results obtained in [4], concerning certain convolution operators leaving $L^{p,\Phi}$ spaces invariant, to homogeneous elliptic partial differential equation.

The results presented here are of the same nature as those of Companato [1], Peetre [5] and others (for references see, for instance, [6]). However, they pertain to spaces $L^{p,\Phi}$ with mixed norms, for $1 \leqslant p_i < \infty$, and to operators of arbitrary order but for functions with compact support only.

1. Let R be the set of real numbers, let m_i be positive integers, and let $1 \leqslant p_i, q_i < \infty$. Let $p, q, \lambda, v, x, r, \varrho$ be vector symbols, for instance $p = (p_1, \ldots, p_n)$, etc. The index i is equal to $1, \ldots, n$ everywhere, unless otherwise stated. Let $\Phi_i = \Phi_i(r_i)$, $r_i > 0$, be positive increasing functions. We assume also that

$$\Phi_i(2r_i) \leqslant k_i \Phi_i(r_i), \tag{1}$$

$$\int_{r_n}^{\infty} \cdots \int_{r_1}^{\infty} \prod_{i=1}^{n} \varrho_i^{-(m_i/p_i)-(1/n)} [\Phi_i(\varrho_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \cdots \frac{d\varrho_n}{\varrho_n} \leqslant c_1 \prod_{i=1}^{n} r_i^{-(m_i/p_i)-(1/n)} [\Phi_i(r_i)]^{1/p_i}, \quad r_i \geqslant 0, \tag{2}$$

with c_1 independent of r_i.

DEFINITION. We shall denote by $L^{p,\Phi}$ the space of locally integrable functions $f = f(x)$, $x = (x_1, \ldots, x_n)$, $x_i \in R^{m_i}$, for which there exists a positive constant M depending on f such that for every x_0, r with $x_i^0 \in R^{m_i}$ and every $r_i > 0$ there exists a number c depending on f, x_0, r for which the following inequality holds:

$$\int_{|x_n - x_n^0| \leqslant r_n} \Big[\ldots\Big(\int_{|x_1 - x_1^0| \leqslant r_1} |f(x)-c|^{p_1}\, dx_1\Big)^{p_2/p_1} dx_2 \ldots\Big]^{p_n/p_{n-1}} dx_n \leqslant M^{p_n} \prod_{i=1}^{n} [\Phi_i(r_i)]^{p_n/p_i}. \tag{3}$$

Then

$$\|\|f\|\|_{L^{p,\Phi}} = \inf M \tag{4}$$

is a seminorm in $L^{p,\Phi}$.

2. Let $P_0(D)$ be any homogeneous elliptic partial differential operator of order k and denote by D^k any derivation of order k,

$$D^k = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \ldots \left(\frac{\partial}{\partial x_N}\right)^{k_N}, \quad k = k_1 + \ldots + k_N, \; m_1 + \ldots + m_n = N.$$

We define a by $\hat{a}(\xi) = \xi^k / P_0(\xi)$ where $\hat{a}$ denotes the Fourier transform, e.g. we have

$$a(x) = (2\pi)^{-N} \int e^{ix\xi} \frac{\xi^N}{P_0(\xi)} \, d\xi.$$

Let u be any function k times continuously differentiable and with compact support. Then we apply the theorem from [4] taking $f = P_0(D)u$ and find that the operator

$$g(x) = T[P_0(D)u](x) = [a * P_0(D)u](x)$$

is continuous from $L^{p,\Phi}$ into $L^{p,\Phi}$. We can verify that

$$|[a * P_0(D)u](x)| = |D^k u(x)|.$$

From the above consideration it follows that

$$\|\|D^k u\|\|_{L^{p,\Phi}} \leqslant c_2 \|\|P_0(D)u\|\|_{L^{p,\Phi}}. \tag{5}$$

We consider a homogeneous partial differential operator of the form

$$P(x, D) = P_0(D) + \sum a_k(x) D^k. \tag{6}$$

Before proving the basic theorem of the paper, we will consider an auxiliary lemma.

Definition. Let M^h denote the space of locally bounded measurable functions g such that

$$|g(x) - g(x_0)| \leqslant c_3 h\left(\prod_{i=1}^{n} |x_i - x_i^0|\right), \quad |x_i - x_i^0| \leqslant \tfrac{1}{2},$$

where c_3 is independent of x and x_0, and where

$$h\left(\prod_{i=1}^{n} r_i\right) = \prod_{i=1}^{n} r_i^{-m_i/p_i} [\Phi_i(r_i)]^{1/p_i} \times \\ \times \left\{ \int_{r_n}^{1} \ldots \int_{r_1}^{1} \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(\varrho_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \ldots \frac{d\varrho_n}{\varrho_n} \right\}^{-1}.$$

LEMMA. *Let $g \in L^\infty \cap M^h$ and let $f \in L^p \cap L^{p,\Phi}$. Then $gf \in L^p \cap L^{p,\Phi}$ and we have for any R_i, $0 < R_i \leqslant 1$, $i = 1, \ldots, n$, the following inequality:*

$$(7) \qquad |||gf|||_{L^{p\Phi}} \leqslant \left[c_4 \sup |g(x)| + c_5 \sup_{|x_i - x_i^0| \leqslant R_i} \frac{|g(x) - g(x_0)|}{h\left(\prod_{i=1}^{n} |x_i - x_i^0|\right)}\right] |||f|||_{L^{p\Phi}} + \left[c_6 \sup |g(x)| + c_7 \sup_{|x_i - x_i^0| \leqslant R_i} \frac{|g(x) - g(x_0)|}{h\left(\prod_{i=1}^{n} |x_i - x_i^0|\right)}\right] |||f|||_{L^p},$$

where c_5, c_6, c_7 are constants that depend on R_i while c_4 is independent of R_i.
Moreover, if

$$(8) \qquad \int_0^1 \ldots \int_0^1 \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(\varrho_i)]^{-1/p_i} \frac{d\varrho_1}{\varrho_1} \ldots \frac{d\varrho_n}{\varrho_n} < \infty$$

then $c_5 \to 0$ as $R_i \to 0$.

Proof. Let x_i^0 and r_i be given. For any σ we then have

$$(9) \qquad J = \Big\{ \int_{|x_n - x_n^0| \leqslant r_n} \Big[\ldots \Big(\int_{|x_n - x_1^0| \leqslant r_1} |g(x) f(x) - g(x_0)\sigma|^{p_1} dx_1 \Big)^{p_2/p_1} dx_2 \ldots \Big]^{p_n/p_{n-1}} dx_n \Big\}^{1/p_n}$$
$$\leqslant \Big\{ \int_{|x_n - x_n^0| \leqslant r_n} \Big[\ldots \Big(\int_{|x_1 - x_1^0| \leqslant r_1} |g(x)|^{p_1} |f(x) - \sigma|^{p_1} dx_1 \Big)^{p_2/p_1} dx_2 \ldots \Big]^{p_n/p_{n-1}} dx_1 \Big\}^{1/p_n} +$$
$$+ |\sigma| \Big\{ \int_{|x_n - x_n^0| \leqslant r_n} \Big[\ldots \Big(\int_{|x_1 - x_1^0| \leqslant r_1} |g(x) - g(x_0)|^{p_1} dx_1 \Big)^{p_2/p_1} dx_2 \ldots \Big]^{p_n/p_{n-1}} dx_n \Big\}^{1/p_n}.$$

Taking into account that $g \in L^\infty \cap M^h$ and Φ_i are increasing functions, we get after some transformations

$$(10) \qquad J \leqslant \sup |g(x)| \times \Big\{ \int_{|x_n - x_n^0| \leqslant r_n} \Big[\ldots \Big(\int_{|x_1 - x_1^0| \leqslant r_1} |f(x) - \sigma|^{p_1} dx_1 \Big)^{p_2/p_1} dx_2 \ldots \Big]^{p_n/p_{n-1}} dx_n \Big\}^{1/p_n} + |\sigma| \prod_{i=1}^{n} r_i^{m/p_i} h\Big(\prod_{i=1}^{n} r_i\Big) \sup_{|x_i - x_i^0| \leqslant r_i} \frac{|g(x) - g(x_0)|}{h\left(\prod_{i=1}^{n} |x_i - x_i^0|\right)}.$$

First, let $r_i > \frac{1}{2} R_i$. Then we take $\sigma = 0$ and get

$$(11)\qquad J \leqslant \sup|g(x)|\,\|f\|_{l,p} \leqslant c_7 \sup|g(x)| \prod_{i=1}^{n} [\Phi_i(r_i)]^{1/p_i}[\Phi_i(R_i)]^{-1/p_i}$$
$$\leqslant c_9 \prod_{i=1}^{n} [\Phi_i(r_i)]^{1/p_i} \sup|g(x)|\,\|f\|_{l,p}.$$

Next, let $r_i \leqslant \frac{1}{2}R_i$. Applying (3) and (4) we get

$$(12)\qquad \Big\{\int\limits_{|x_n-x_n^0|\leqslant r_n} \Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant r_1} |f(x)-\sigma|^{p_1}\,dx_1\Big)^{p_2/p_1} dx_2\ldots\Big]^{p_n/p_{n-1}} dx_n\Big\}^{1/p_n}$$
$$\leqslant \Big\{\int\limits_{|x_n-x_n^0|\leqslant 2r_n} \Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2r_1} |f(x)-\sigma|^{p_1}\,dx_1\Big)^{p_2/p_1} dx_2\ldots\Big]^{p_n/p_{n-1}} dx_n\Big\}^{1/p_n}$$
$$\leqslant |||f|||_{l,p,\Phi} \prod_{i=1}^{n} [\Phi_i(2r_i)]^{1/p_i} \leqslant c_{10} \prod_{i=1}^{n} [\Phi_i(r_i)]^{1/p_i}\, |||f|||_{l,p,\Phi}.$$

Now let us find an estimate for $|\sigma|$. We consider

$$(13)\qquad r_{i,l} = 2^l r_i, \qquad l = 0, 1, \ldots, s,$$

where $r_{i,s-1} \leqslant \frac{1}{2}R_i < r_{i,s}$.

Let us denote by σ_l, $l = 0, 1, \ldots, s$, the successive numbers constructed by the above procedure (in particular, $\sigma_s = 0$).

Applying the idea originating from Morrey (see e.g. Campanato [2] or, in the case of mixed norms, Jaroszewska [3]), we get

$$|\sigma_l-\sigma_{l+1}|^{p_1} \leqslant 2^{p_1}|\sigma_l-f(x)|^{p_1}+2^{p_1}|f(x)-\sigma_{l+1}|^{p_1}.$$

Integrating both sides of this inequality with respect to x_1 on $|x_1-x_1^0| \leqslant r_{1,l}$ and raising them to the power p_2/p_1, we get

$$|\sigma_l-\sigma_{l+1}|^{p_2} \leqslant 2^{p_2}(r_1^l)^{-m_1p_2/p_1} \times$$
$$\times\Big[\Big(\int\limits_{|x_1-x_1^0|\leqslant r_{1,l}} |f(x)-\sigma_l|^{p_1}\,dx_1\Big)^{p_2/p_1}+\Big(\int\limits_{|x_1-x_1^0|\leqslant r_{1,l+1}} |f(x)-\sigma_{l+1}|^{p_1}\,dx_1\Big)^{p_2/p_1}\Big].$$

Repeating this procedure with respect to all variables $x_2, \ldots, x_n$ successively, we obtain

$$(14)\qquad |\sigma_l-\sigma_{l+1}| \leqslant 2\prod_{i=1}^{n} (r_{i,l})^{-m_i/p_i} \times$$
$$\times\Big\{\Big\{\int\limits_{|x_n-x_n^0|\leqslant r_{n,l}} \Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant r_{1,l}} |f(x)-\sigma_l|^{p_1}\,dx_1\Big)^{p_2/p_1}\ldots\Big]^{p_n/p_{n-1}} dx_n\Big\}^{1/p_n}+$$
$$+\Big\{\int\limits_{|x_n-x_n^0|\leqslant r_{n,l+1}} \Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant r_{1,l+1}} |f(x)-\sigma_{l+1}|^{p_1}\,dx_1\Big)^{p_2/p_1} dx_2\ldots\Big]^{p_n/p_{n-1}} dx_n\Big\}^{1/p_n}\Big\}.$$

Hence, by (13), it follows that

$$(15)\quad |\sigma_l-\sigma_{l+1}| \leqslant 2\prod_{i=1}^{n}(2^l r_i)^{-m_i/p_i}\times$$

$$\times\Big\{\Big\{\int\limits_{|x_n-x_n^0|\leqslant 2^l r_n}\Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2^l r_1}|f(x)-\sigma_l|^{p_1}dx_1\Big)^{p_2/p_1}dx_2\ldots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}+$$

$$+\Big\{\int\limits_{|x_n-x_n^0|\leqslant 2^{l+1} r_n}\Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2^{l+1} r_1}|f(x)-\sigma_{l+1}|^{p_1}dx_1\Big)^{p_2/p_1}dx_2\ldots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}\Big\}.$$

Applying (15), we get

$$(16)\quad |\sigma|\leqslant\sum_{l=0}^{s-1}|\sigma_l-\sigma_{l+1}|\leqslant 2\sum_{l=0}^{s-2}\prod_{i=1}^{n}(2^l r_i)^{-m_i/p_i}\times$$

$$\times\Big\{\Big\{\int\limits_{|x_n-x_n^0|\leqslant 2^l r_n}\Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2^l r_1}|f(x)-\sigma_l|^{p_1}dx_1\Big)^{p_2/p_1}\ldots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}+$$

$$+\Big\{\int\limits_{|x_n-x_n^0|\leqslant 2^{l+1} r_n}\Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2^{l+1} r_1}|f(x)-\sigma_{l+1}|^{p_1}dx_1\Big)^{p_2/p_1}\ldots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}\Big\}+$$

$$+2\prod_{i=1}^{n}(2^{s-1}r_i)^{-m_i/p_i}\times$$

$$\times\Big\{\int\limits_{|x_n-x_n^0|\leqslant 2^{s-1} r_n}\Big[\ldots\Big(\int\limits_{|x_1-x_1^0|\leqslant 2^{s-1} r_1}|f(x)-\sigma_{s-1}|^{p_1}dx_1\Big)^{p_2/p_1}\ldots\Big]^{p_n/p_{n-1}}dx_n\Big\}^{1/p_n}+$$

$$+2\prod_{i=1}^{n}(r_{i,s})^{-m_i/p_i}\|f\|_{L^p}.$$

From (15), by (1), (3), (4) and (13), we deduce

$$|\sigma|\leqslant c_{11}\Big\{\sum_{l=0}^{s-1}\prod_{i=1}^{n}(2^l r_i)^{-m_i/p_i}[\Phi_i(2^l r_i)]^{1/p_i}|||f|||_{L^{p,\Phi}}+\prod_{i=1}^{n}R_i^{-m_i/p_i}\|f\|_{L^p}\Big\}$$

with c_{11} independent of R_i.

Then, applying a suitable substitution to the first term of the right-hand side of the above inequality, we obtain

$$(17)\quad |\sigma|\leqslant c_{12}\Big\{\int_{r_n}^{R_n}\ldots\int_{r_1}^{R_1}\prod_{i=1}^{n}\varrho_i^{-m_i/p_i}[\Phi_i(\varrho_i)]^{1/p_i}\frac{d\varrho_1}{\varrho_1}\ldots\frac{d\varrho_n}{\varrho_n}\cdot|||f|||_{L^{p,\Phi}}+$$

$$+\prod_{i=1}^{n}R_i^{-m_i/p_i}\|f\|_{L^p}\Big\}.$$

Now, inequalities (10), (11), (12), and (17) yield (18)

$$(18)\quad J \leqslant \Big\{c_{10} \sup |g(x)| + c_{12} \cdot A \cdot \sup_{|x_i - x_i^0| \leqslant R_i} \frac{|g(x)-g(x_0)|}{h(\prod_{i=1}^{n} |x_i - x_i^0|)}\Big\} \, |||f|||_{L^p,\Phi} \times$$
$$\times \prod_{i=1}^{n} [\Phi_i(r_i)]^{1/p_i} +$$
$$+ \Big\{c_9 \sup |g(x)| + c_{12} \cdot B \cdot \sup_{|x_i - x_i^0| \leqslant R_i} \frac{|g(x)-g(x_0)|}{h(\prod_{i=1}^{n} |x_i - x_i^0|)}\Big\} \, ||f||_{L^p} \times$$
$$\times \prod_{i=1}^{n} [\Phi_i(r_i)]^{1/p_i}$$

where

$$(19)\quad A = h(\prod_{i=1}^{n} r_i) \times$$
$$\times \prod_{i=1}^{n} r_i^{m_i/p_i} [\Phi_i(r_i)]^{-1/p_i} \int_{r_n}^{R_n} \dots \int_{r_1}^{R_1} \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(r_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \dots \frac{d\varrho_n}{\varrho_n}$$
$$\leqslant \sup_{r_i \leqslant \frac{1}{2} R_i} \int_{r_n}^{R_n} \dots \int_{r_1}^{R_1} \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(\varrho_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \dots \frac{d\varrho_n}{\varrho_n} \times$$
$$\times \Big\{ \int_{r_n}^{1} \dots \int_{r_1}^{1} \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(\varrho_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \dots \frac{d\varrho_n}{\varrho_n} \Big\}^{-1} < \infty$$

and

$$(20)\quad B = h(\prod_{i=1}^{n} r_i) \prod_{i=1}^{n} R_i^{-m_i/p_i} r_i^{m_i/p_i} [\Phi_i(r_i)]^{-1/p_i}$$
$$\leqslant \prod_{i=1}^{n} R_i^{-m_i/p_i} \Big\{ \int_{R_n/2}^{1} \dots \int_{R_1/2}^{1} \prod_{i=1}^{n} \varrho_i^{-m_i/p_i} [\Phi_i(\varrho_i)]^{1/p_i} \frac{d\varrho_1}{\varrho_1} \dots \frac{d\varrho_n}{\varrho_n} \Big\}^{-1} < \infty.$$

From (18), (19) and (20) we obtain (7).

THEOREM. *Let the coefficients $a_k(x)$ of the operator $P(x, D)$ given by* (6) *belong to M^k. Let Φ_i satisfy* (2) *and let u be a function k times continuously differentiable with a compact support.*

There exists a number $\delta_0 = \delta_0(P_0(D), \Phi_i)$ such that if $\delta < \delta_0$, where

$$\delta = \begin{cases} \sum \sup |a_k(x)| & \textit{if } (8) \textit{ holds}, \\ \max \Big[\sum \sup \dfrac{|a_k(x)-a_k(x_0)|}{h(\prod_{i=1}^{n} |x_i - x_i^0|)}, \sum \sup |a_k(x)| \Big], & \textit{otherwise}, \end{cases}$$

then we have the inequality

$$|||D^k u|||_{L^{p,\Phi}} \leqslant c_{13}\left(|||P(x, D)u|||_{L^{p,\Phi}} + \sum \|D^k u\|_{L^p}\right)$$

where c_{13} is independent of u.

Proof. We have from (6)

$$|||P_0(D)u|||_{L^{p,\Phi}} \leqslant |||P(x, D)u|||_{L^{p,\Phi}} + \sum |||a_k D^k u|||_{L^{p,\Phi}}.$$

Then, applying (5) to the left-hand side of the above inequality, applying the lemma to the second term of the right-hand side and taking into account the assumptions of the theorem, we get

$$|||D^k u|||_{L^{p,\Phi}} \leqslant c_2 |||P(x, D)u|||_{L^{p,\Phi}} + c_{14}\cdot\delta |||D^k u|||_{L^{p,\Phi}} + c_{15}\sum \|D^k u\|_{L^p}.$$

Hence, with $\delta < 1/c_{14}$, we get the theorem.

References

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