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THE HIGH-PRECISION SUMMATION OF SERIES WHOSE TERMS ARE RATIONAL NUMBERS

0. Introduction. In [4] Sale gave a method of calculation of e to an arbitrary prescribed precision. The method consists in digit by digit evaluation of a partial sum of sufficiently high degree of the series $\sum_{k=0}^{\infty} 1/k!$. In the present paper we deal with a more general problem of summing series whose terms are rationals. Our method, which in the sequel is called *method Q*, works under certain mild assumptions.

Assume we are given the convergent series

$$S = \sum_{k=0}^{\infty} \frac{a_k}{b_k} \quad (a_k, b_k \text{ non-zero integers})$$

and the natural number p. Define the number n by the formula

$$n = \min_i \left\{ i \colon \left| \sum_{k=i+1}^{\infty} \frac{a_k}{b_k} \right| < 10^{-(p+1)} \right\}.$$

The decimal representation of the number

$$S_n = \sum_{k=0}^{n} \frac{a_k}{b_k},$$

with fractional part limited to p initial digits, approximates the value of S with error less than $2 \cdot 10^{-(p+1)}$, i.e.,

$$|S - S_n| < 2 \cdot 10^{-(p+1)}.$$

By method Q one can calculate arbitrarily many initial digits of the decimal expansion of the number S_n and, therefore, one can find the value of S to any prescribed accuracy.

Method Q can also be applied to the calculation of products of rational factors, as described at the end of Section 1.

1. Method Q. The partial sum S_n can be easily rewritten in the following form, similar to that used in Horner's scheme for polynomials:

$$S_n = \frac{l_0}{m_0}\left(1+\frac{l_1}{m_1}\left(1+\ldots+\frac{l_{n-1}}{m_{n-1}}\left(1+\frac{l_n}{m_n}\cdot 1\right)\ldots\right)\right),$$

where

$$\frac{l_0}{m_0} = \frac{a_0}{b_0}, \quad \frac{l_i}{m_i} = \frac{b_{i-1}a_i}{a_{i-1}b_i} \quad (i = 1, 2, \ldots, n),$$

and

$$m_i > 0 \quad (i = 0, 1, \ldots, n).$$

This form will be called the *bracket form* S_n or, permutably, the *bracket expression*. It is convenient to assume for the time being that the fractions l_i/m_i are irreducible. Notice that in practice this assumption is often not fulfilled.

Let the operator $\div$ have the meaning identical to that in ALGOL 60 (see [2]), i.e., $a \div b$ is equal to the integer part of $|a/b|$ multiplied by the sign of a/b.

We introduce the following notation:

$$A_i^0 = |l_i| \quad (i = 0, 1, \ldots, n), \tag{1}$$

$$c_{-1}^0 = 1, \quad c_i^0 = c_{i-1}^0 \operatorname{sgn} l_i \quad (i = 0, 1, \ldots, n), \tag{2}$$

$$E_{n+1}^j = 0 \quad (j = 1, 2, \ldots). \tag{3}$$

For $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots,$ we define

$$E_i^j = \{A_i^{j-1}(10c_i^{j-1} + E_{i+1}^j)\} \div m_i, \tag{4}$$

$$c_i^j = A_i^{j-1}(10c_i^{j-1} + E_{i+1}^j) - m_i E_i^j, \tag{5}$$

$$A_i^j = \begin{cases} 1 & (i = 0), \\ A_{i-1}^{j-1} & (i > 0). \end{cases} \tag{6}$$

Now, we can write

$$S_n^j = \frac{A_0^j}{m_0}\left(c_0^j + \frac{A_1^j}{m_1}\left(c_1^j + \ldots + \frac{A_{n-1}^j}{m_{n-1}}\left(c_{n-1}^j + \frac{A_n^j}{m_n}c_n^j\right)\ldots\right)\right) \tag{7}$$

$$(j = 0, 1, \ldots).$$

We prove the following

THEOREM 1. *For $j = 0, 1, \ldots$ we have*

$$S_n = \sum_{k=1}^{j} 10^{-k} E_0^k + 10^{-j} S_n^j. \tag{8}$$

Proof. It follows from the definitions of A_i^0 and c_i^0 that $S_n = S_n^0$, so formula (8) is true for $j = 0$.

We show that

(9) $$S_n^j = \frac{1}{10}(E_0^{j+1}+S_n^{j+1}) \quad (j = 0, 1, \ldots),$$

which implies that (8) is true for any $j = 1, 2, \ldots$

Let us transform (7) to the form

$$S_n^j = \frac{1}{10}\frac{A_0^j}{m_0}\left(10c_0^j+\frac{A_1^j}{m_1}\left(10c_1^j +\ldots+\frac{A_{n-1}^j}{m_{n-1}}\left(10c_{n-1}^j+\frac{A_n^j}{m_n}10c_n^j\right)\ldots\right)\right).$$

We see that S_n^j can be evaluated using the formulae

$$T_{n+1}^j = 0, \quad T_i^j = \frac{A_i^j}{m_i}(10c_i^j+T_{i+1}^j) \; (i = n, n-1, \ldots, 0), \quad S_n^j = \frac{1}{10}T_0^j.$$

Identity (9) can be now expressed in the equivalent form

$$T_0^j = E_0^{j+1}+\frac{1}{10}T_0^{j+1}.$$

The last equation is a particular case of the more general equality

(10) $$T_i^j = E_i^{j+1}+\frac{T_i^{j+1}}{10A_i^{j+1}}$$

which we prove to hold for $i = 0, 1, \ldots, n+1$.

In the proof of (10) we use induction on i. For $i = n+1$ this equation follows from the definitions of E_{n+1}^j and T_{n+1}^j. Assuming that (10) holds for a certain i $(0 < i \leqslant n+1)$ and using the equations

$$T_{i-1}^j = \frac{A_{i-1}^j}{m_{i-1}}(10c_{i-1}^j+T_i^j),$$

$$T_{i-1}^{j+1} = \frac{A_{i-1}^{j+1}}{m_{i-1}}(10c_{i-1}^{j+1}+T_i^{j+1}),$$

$$\frac{A_{i-1}^j(10c_{i-1}^j+E_i^{j+1})}{m_{i-1}} = E_{i-1}^{j+1}+\frac{c_{i-1}^{j+1}}{m_{i-1}}$$

(the last one follows from the definition of c_{i-1}^{j+1}), one can easily check that

$$T_{i-1}^j = E_{i-1}^{j+1}+\frac{T_{i-1}^{j+1}}{10A_{i-1}^{j+1}}.$$

Indeed, we have

$$\begin{aligned} T_{i-1}^{j} &= \frac{A_{i-1}^{j}(10c_{i-1}^{j}+E_{i}^{j+1}-E_{i}^{j+1}+T_{i}^{j})}{m_{i-1}} \\ &= E_{i-1}^{j+1}+\frac{c_{i-1}^{j+1}}{m_{i-1}}+\frac{A_{i-1}^{j}}{m_{i-1}}\,\frac{T_{i}^{j+1}}{10A_{i}^{j+1}} = E_{i-1}^{j+1}+\frac{c_{i-1}^{j+1}}{m_{i-1}}+\frac{T_{i}^{j+1}}{10m_{i-1}} \\ &= E_{i-1}^{j+1}+\frac{c_{i-1}^{j+1}}{m_{i-1}}+\frac{1}{10m_{i-1}}\left(\frac{m_{i-1}}{A_{i-1}^{j+1}}T_{i-1}^{j+1}-10c_{i-1}^{j+1}\right) = E_{i-1}^{j+1}+\frac{T_{i-1}^{j+1}}{10A_{i-1}^{j+1}}. \end{aligned}$$

LEMMA 1. *For* $j = 0, 1, \ldots$ *we have*

$$|S_n^j| < M, \quad \text{where } M = \sum_{i=0}^{n} \frac{\prod_{k=0}^{i} A_k^0}{\prod_{k=0}^{i-1} m_k}.$$

Proof. From (7) it follows immediately that

$$|S_n^j| = \left| \sum_{i=0}^{n} \frac{\prod_{k=0}^{i} A_k^j}{\prod_{k=0}^{i} m_k} c_i^j \right|,$$

which together with (5) implies

$$|S_n^j| \leqslant \sum_{i=0}^{n} \left| \frac{\prod_{k=0}^{i} A_k^j}{\prod_{k=0}^{i} m_k} c_i^j \right| < \sum_{i=0}^{n} \left| \frac{\prod_{k=0}^{i} A_k^j}{\prod_{k=0}^{i-1} m_k} \right| \leqslant \sum_{i=0}^{n} \frac{\prod_{k=0}^{i} A_k^0}{\prod_{k=0}^{i-1} m_k} = M.$$

Method Q consists in the evaluation of E_0^j for $j = 1, 2, \ldots$ by using formulae (1)-(6). By virtue of Theorem 1 and Lemma 1 the convergence of the method is guaranteed. Indeed, we have

$$\left| S_n - \sum_{k=0}^{j} 10^{-k} E_0^k \right| = 10^{-j} |S_n^j| < 10^{-j} M \quad (j = 1, 2, \ldots).$$

If the inequalities

$$0 \leqslant E_0^j < 10 \quad (j = 1, 2, \ldots) \tag{11}$$

hold, then in every step we obtain the next exact digit of the decimal expansion of S_n. In Section 2 we give some theorems which allow us to transform the bracket expression (7) in such a manner that method Q applied to the transformed form yields numbers E_0^j satisfying inequalities (11).

When one cannot guarantee (11) to hold, one has to store all the values of E_0^j and to use procedures correcting these values in order that E_0^j be the j-th exact digit of the expansion of S_n.

In Section 4 we give the declarations of ALGOL 60 procedures for method Q and for a modification of this method which we describe in Sections 2 and 3.

Below we give some examples of the application of method Q to certain series.

Example 1. Let us consider the series

$$\sum_{k=0}^{\infty} \frac{1}{2^k(k+1)} = 2\log\frac{1}{2}.$$

For $n = 8$ we have

$$\begin{aligned} S_8 &= \sum_{k=0}^{8} \frac{1}{2^k(k+1)} \\ &= 1+\frac{1}{4}+\frac{1}{12}+\frac{1}{32}+\frac{1}{80}+\frac{1}{192}+\frac{1}{448}+\frac{1}{1024}+\frac{1}{2304} \\ &= \frac{447047}{322560} = 1.385934399801\ldots \end{aligned}$$

In the bracket form we obtain

$$S_8 = 1\left(1+\frac{1}{4}\left(1+\frac{1}{3}\left(1+\frac{3}{8}\left(1+\frac{2}{5}\left(1+\frac{5}{12}\left(1+\frac{3}{7}\left(1+ +\frac{7}{16}\left(1+\frac{4}{9}\cdot 1\right)\right)\right)\right)\right)\right)\right)\right).$$

The action of method Q can be illustrated by means of Table 1.

In this example the number E_0^j is, for $j = 2, 3, 4, 5, 6$, the j-th exact decimal digit of S_8. Moreover, it follows from Theorem 4 (Section 2) that the inequalities $0 \leqslant E_0^j < 10$ hold also for $j > 6$, which means that E_0^j is the j-th exact digit of the expansion of S_8. Examples 2 and 3 (below) show that this is not true in general.

Example 1 shows that method Q is simple enough to be used successfully in paper-and-pencil calculations. Having this in mind, it is worth analyzing this example. After performing the first step we obtain

$$S_8^1 = 1\left(0+\frac{1}{4}\left(3+\frac{1}{3}\left(1+\frac{1}{8}\left(0+\frac{3}{5}\left(2+\frac{2}{12}\left(8+ +\frac{5}{7}\left(6+\frac{3}{16}\left(2+\frac{7}{9}\cdot 4\right)\right)\right)\right)\right)\right)\right)\right).$$

TABLE 1

i	0	1	2	3	4	5	6	7	8
m_i	1	4	3	8	5	12	7	16	9
c_i^0	1	1	1	1	1	1	1	1	1
A_i^0	1	1	1	3	2	5	3	7	4
E_i^1	13	3	5	6	6	6	6	6	4
c_i^1	0	3	1	0	2	8	6	2	4
A_i^1	1	1	1	1	3	2	5	3	7
E_i^2	8	8	4	3	24	21	49	9	31
c_i^2	0	2	1	0	3	6	2	9	1
A_i^2	1	1	1	1	1	3	2	5	3
E_i^3	5	5	3	1	9	18	14	29	3
c_i^3	0	3	2	1	3	6	0	1	3
A_i^3	1	1	1	1	1	1	3	2	5
E_i^4	9	9	7	2	7	5	1	3	16
c_i^4	0	1	1	1	0	1	2	4	6
A_i^4	1	1	1	1	1	1	1	3	2
E_i^5	3	3	3	1	0	1	4	8	13
c_i^5	0	1	2	2	1	2	0	1	3
A_i^5	1	1	1	1	1	1	1	1	3
E_i^6	4	4	7	2	2	1	0	1	10
c_i^6	0	1	1	6	1	8	1	4	0
A_i^6	1	1	1	1	1	1	1	1	1

One can demand that fractions A_i^1/m_i be irreducible, and to "contract" the bracket expression in order that all c_i^1 be different from zero:

$$S_8^1 = 1\cdot\frac{1}{4}\left(3+\frac{1}{3}\left(1+\frac{1}{8}\cdot\frac{3}{5}\left(2+\frac{2}{12}\left(8+\frac{5}{7}\left(6+\frac{3}{16}\left(2+\frac{7}{9}\cdot 4\right)\right)\right)\right)\right)\right)$$

$$= \frac{1}{4}\left(3+\frac{1}{3}\left(1+\frac{3}{40}\left(2+\frac{1}{6}\left(8+\frac{5}{7}\left(6+\frac{3}{16}\left(2+\frac{7}{9}\cdot 4\right)\right)\right)\right)\right)\right).$$

Such a procedure enables us to shorten the calculations in the next steps of method Q.

Assume that we apply method Q to the expression (¹)

$$S_n^0 = \frac{A_0^0}{m_0^0}\left(c_0^0+\frac{A_1^0}{m_1^0}\left(c_1^0+\ldots+\frac{A_{n-1}^0}{m_{n-1}^0}\left(c_{n-1}^0+\frac{A_n^0}{m_n^0}c_n^0\right)\ldots\right)\right).$$

The algorithm of "contraction" of the bracket expressions can be formulated as follows: Let us put

$$N_j = \{c_i^j\colon c_i^j \neq 0\}, \quad n_j = \overline{\overline{N}}_j - 1 \quad (j = 0, 1, \ldots),$$

where $\overline{\overline{N}}$ denotes the number of elements of the set N. It follows from the definition of c_i^0 that $n_0 = n$.

(¹) During "contraction" of a bracket expression the values of m_i are changed. This is why we have to write m_i^j instead of m_i in the expression S_n^j.

Consider the bracket expressions $S_n^0, \bar{S}_n^0, S_n^1, \bar{S}_n^1, S_n^2, \ldots$, where $\bar{S}_n^0 = S_n^0$ and for $j = 1, 2, \ldots$ the bracket expression

$$S_n^j = \frac{A_0^j}{m_0^{j-1}}\left(c_0^j + \frac{A_1^j}{m_1^{j-1}}\left(c_1^j + \ldots + \frac{A_{n_{j-1}-1}^j}{m_{n_{j-1}-1}^{j-1}}\left(c_{n_{j-1}-1}^j + \right.\right.\right.$$

$$\left.\left.\left. + \frac{A_{n_{j-1}}^j}{m_{n_{j-1}}^{j-1}} c_{n_{j-1}}^j\right)\ldots\right)\right)$$

is a result of applying method Q to $\bar{S}_n^{j-1}$, and the expression

$$\bar{S}_n^j = \frac{\bar{A}_0^j}{m_0^j}\left(\bar{c}_0^j + \frac{\bar{A}_1^j}{m_1^j}\left(\bar{c}_1^j + \ldots + \frac{\bar{A}_{n_j-1}^j}{m_{n_j-1}^j}\left(\bar{c}_{n_j-1}^j + \frac{\bar{A}_{n_j}^j}{m_{n_j}^j}\bar{c}_{n_j}^j\right)\ldots\right)\right)$$

is obtained from S_n^j by simplifying the fragments of the form

$$\ldots + c_l^j + \frac{A_{l+1}^j}{m_{l+1}^{j-1}}\left(0 + \ldots + \frac{A_{k-1}^j}{m_{k-1}^{j-1}}\left(0 + \frac{A_k^j}{m_k^{j-1}}(c_k^j + \ldots \qquad (c_l^j \neq 0,\ c_k^j \neq 0)\right.\right.$$

to the form

(12)
$$\ldots + c_l^j + \frac{A_{l+1}^j \ldots A_{k-1}^j A_k^j}{m_{l+1}^{j-1} \ldots m_{k-1}^{j-1} m_k^{j-1}}(c_k^j + \ldots,$$

and the fragment

$$\ldots + \frac{A_{s-1}^j}{m_{s-1}^{j-1}}\left(c_{s-1}^j + \frac{A_s^j}{m_s^{j-1}}\left(c_s^j + \frac{A_{s+1}^j}{m_{s+1}^{j-1}}\left(0 + \ldots + \frac{A_{n_{j-1}}^j}{n_{j-1}^{j-1}} \cdot 0 \ldots\right)\right)\right)\ldots\Big)\Big)$$

$$(c_s^j \neq 0)$$

to the form

$$\ldots + \frac{A_{s-1}^j}{m_{s-1}^{j-1}}\left(c_{s-1}^j + \frac{A_s^j}{m_s^{j-1}} c_s^j\right)\ldots\Big)\Big).$$

After such a simplification it suffices to numerate the obtained nominators $\bar{A}_i^j$, denominators m_i^j, and coefficients $\bar{c}_i^j$ by the numbers $0, 1, \ldots, n_j$ to obtain the expression $\bar{S}_n^j$.

The method Q for the "contracted" bracket form $\bar{S}_n^j$ $(j = 0, 1, \ldots)$ is performed according to the following formulae:

(1′)
$$\bar{A}_i^0 = A_i^0 = |l_i| \quad (i = 0, 1, \ldots, n_0),$$

(2′)
$$\bar{c}_{-1}^0 = 1, \quad \bar{c}_i^0 = c^0 = c_{i-1}^i \operatorname{sgn} l_i \quad (i = 0, 1, \ldots, n_0).$$

For $j = 1, 2, \ldots$ we have

$$E_{n+1}^{j} = 0, \tag{3}$$

$$E_i^j = \{\bar{\Lambda}_i^{j-1}(10\bar{c}_i^{j-1} + E_{i+1}^j)\} \div m_i^{j-1} \quad (i = 0, 1, \ldots, n_{j-1}), \tag{4'}$$

$$c_i^j = \bar{\Lambda}_i^{j-1}(10\bar{c}_i^{j-1} + E_{i+1}^j) - m_i^{j-1}E_i^j \quad (i = 0, 1, \ldots, n_{j-1}), \tag{5'}$$

$$\Lambda_i^j = \begin{cases} 1 & (i = 0), \\ \bar{\Lambda}_{i-1}^{j-1} & (i = 1, 2, \ldots, n_{j-1}). \end{cases} \tag{6'}$$

Repeat the calculations of Example 1 demanding that Λ_i^j/m_i^j be irreducible and using the contraction algorithm (Table 2).

Since $n_4 = n_5 = n_6 = n_7 = 4$ and $\bar{\Lambda}_i^4 = 1$ for $i = 0, 1, 2, 3, 4$, the values of Λ_i^j and m_i^j for $j = 5, 6, 7$ are equal to $\bar{\Lambda}_i^4$ and m_i^4, respectively.

TABLE 2

i	0	1	2	3	4	5	6	7	8	
c_i^0	1	1	1	1	1	1	1	1	1	
Λ_i^0	1	1	1	3	2	5	3	7	4	
m_i^0	1	4	3	8	5	12	7	16	9	
E_i^1	13	3	5	6	6	6	6	6	4	
c_i^1	0	3	1	0	2	8	6	2	4	$(n_1 = 6)$
Λ_i^1	1	1	1	1	3	2	4	3	7	
$\bar{\Lambda}_i^1$	1	1	3	1	5	3	7			
m_i^1	4	3	40	6	7	16	9			
$\bar{c}_i^1$	3	1	2	8	6	2	4			
E_i^2	8	4	3	21	49	9	31			
$\bar{c}_i^2 = c_i^2$	2	1	3	3	2	9	1			$(n_2 = 6)$
$\bar{\Lambda}_i^2 = \Lambda_i^2$	1	1	1	1	1	5	1			
m_i^2	4	3	40	2	7	16	3			
E_i^3	5	3	1	18	7	29	3			
c_i^3	3	2	8	1	0	1	1			$(n_3 = 5)$
Λ_i^3	1	1	1	1	1	1	5			
$\bar{\Lambda}_i^3$	1	1	1	1	1	5				
m_i^3	4	3	40	2	112	3				
$\bar{c}_i^3$	3	2	8	1	1	1				
E_i^4	9	7	2	5	0	16				
c_i^4	1	1	5	0	26	2				$(n_4 = 4)$
Λ_i^4	1	1	1	1	1	1				
$\bar{\Lambda}_i^4$	1	1	1	1	1					
m_i^4	4	3	40	224	3					
$\bar{c}_i^4$	1	1	5	26	2					
E_i^5	3	3	1	1	6					
c_i^5	1	2	11	42	2					$(n_5 = 4)$
E_i^6	4	7	2	1	6					
c_i^6	1	1	31	202	2					$(n_6 = 4)$
E_i^7	3	5	7	9	6					
c_i^7	3	2	39	100	2					$(n_7 = 4)$

Example 2. Let us consider the alternating series

$$\sum_{k=0}^{\infty}(-1)^k\frac{2^{k+1}}{3^{k+1}(k+1)}=\log\frac{5}{3}.$$

For $n = 7$ we have

$$S_7=\sum_{k=0}^{7}(-1)^k\frac{2^{k+1}}{3^{k+1}(k+1)}$$

$$=\frac{2}{3}-\frac{4}{18}+\frac{8}{81}-\frac{16}{324}+\frac{32}{1215}-\frac{64}{4374}+\frac{128}{15309}-\frac{256}{52488}$$

$$=\frac{116888}{229635}=.50901648267\ldots,$$

and in the bracket form we obtain

$$S_7=\frac{2}{3}\left(1+\frac{-1}{3}\left(1+\frac{-4}{9}\left(1+\frac{-1}{2}\left(1+\frac{-8}{15}\left(1+\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\left.+\frac{-5}{9}\left(1+\frac{-4}{7}\left(1+\frac{-7}{12}\cdot 1\right)\right)\right)\right)\right)\right)\right).$$

To this expression we apply method Q with contraction. As in the previous example the acting of the method is shown in a table (Table 3).

In this example the values of E_0^2 and E_0^3 differ from the second and the third digits of the decimal expansion of S_7 because $E_0^3 = -1 < 0$. However, observe that

$$10^{-1}E_0^1+10^{-2}E_0^2+10^{-3}E_0^3+10^{-4}E_0^4 = .5+.01-.001+.0000 = .5090;$$

thus we obtain exact values of four initial digits of the expansion of S_7.

Example 3. Let us take

$$S_3=\frac{3}{4}+\frac{3}{16}+\frac{3}{56}+\frac{1}{112}=1$$

or, in the bracket form,

$$S_3=\frac{3}{4}\left(1+\frac{1}{4}\left(1+\frac{2}{7}\left(1+\frac{1}{6}\cdot 1\right)\right)\right).$$

The results of application of method Q are shown in Table 4.

TABLE 3

i	0	1	2	3	4	5	6	7	
c_i^0	1	−1	1	−1	1	−1	1	−1	$(n_0 = 7)$
A_i^0	2	1	4	1	8	5	4	7	
m_i^0	3	3	9	2	15	9	7	12	
E_i^1	5	−2	3	−3	3	−4	2	−5	
$\bar{c}_i^1 = c_i^1$	1	1	1	−1	3	−4	6	−10	$(n_1 = 7)$
$\bar{A}_i^1 = A_i^1$	1	2	1	2	1	8	5	1	
m_i^1	3	3	9	1	15	9	7	3	
E_i^2	1	−7	−1	−20	0	−18	19	−33	
c_i^2	0	−1	−1	0	12	−6	2	−1	$(n_2 = 5)$
A_i^2	1	1	2	1	2	1	8	5	
$\bar{A}_i^2$	1	2	2	1	8	5			
$\bar{c}_i^2$	−1	−1	12	−6	2	−1			
m_i^2	9	9	15	9	7	3			
E_i^3	−1	1	15	6	4	−16			
c_i^3	0	1	3	−2	4	−2			$(n_3 = 4)$
A_i^3	1	1	2	2	1	8			
$\bar{A}_i^3$	1	2	2	1	8				
c_i^3	1	3	−2	4	−2				
m_i^3	81	15	9	7	3				
E_i^4	0	3	−4	−1	−53				
$\bar{c}_i^4 = c_i^4$	13	7	−6	−6	−1				
$\bar{A}_i^4 = A_i^4$	1	1	2	2	1				
m_i^4	81	15	9	7	3				

As in Example 2, the value of E_0^j is different from the j-th digit of the decimal expansion of S_3 ($j = 1, 2, 3, 4$) because $E_0^4 = 10$. However,

$$\sum_{j=0}^{3} 10^{-j} E_0^j = 1.0000.$$

The contraction algorithm, though very useful in paper-and-pencil calculations, is not applicable in automatic computation because the evaluation of $m_{l+1}^{j-1} m_{l+2}^{j-1} \ldots m_k^{j-1}$ (cf. (12)) may lead to fixed-point overflow.

By method Q one can also evaluate decimal expansions of finite products of rational factors. To this end one has to transform the product $P_n = \prod_{k=0}^{n} a_k/b_k$ to the form

$$P_n = \frac{a_0}{b_0}\left(0 + \frac{a_1}{b_1}\left(0 + \ldots + \frac{a_{n-1}}{b_{n-1}}\left(0 + \frac{a_n}{b_n} \cdot 1\right) \ldots\right)\right).$$

Now, method Q is obviously applicable. It suffices to set

$$A_i^0 = a_i, \quad m_i = b_i \quad (i = 0, 1, \ldots, n),$$

$$c_i^0 = \begin{cases} 0 & (i = 0, 1, \ldots, n-1), \\ 1 & (i = n). \end{cases}$$

TABLE 4

i	0	1	2	3
c_i^0	1	1	1	1
Δ_i^0	3	1	2	1
m_i^0	4	4	7	6
E_i^1	9	3	3	1
c_i^1	3	1	1	4
Δ_i^1	1	3	1	1
m_i^1	4	4	7	3
E_i^2	9	9	3	13
c_i^2	3	3	2	1
Δ_i^2	1	1	3	1
m_i^2	4	4	7	3
E_i^3	9	9	9	3
c_i^3	3	3	6	1
Δ_i^3	1	1	1	1
m_i^3	4	4	7	1
E_i^4	10	10	10	10
c_i^4	0	0	0	0

2. Properties of the bracket form of S_n. In this section we discuss some properties of the bracket form of S_n which enable us to modify method Q in such a way that every step of the method produces the succeeding digit of the decimal expansion of S_n.

THEOREM 2. *If the inequality*

$$S_n = \frac{l_0}{m_0}\left(1+\frac{l_1}{m_1}\left(1+\ldots+\frac{l_{n-1}}{m_{n-1}}\left(1+\frac{l_n}{m_n}\cdot 1\right)\ldots\right)\right) > 0$$

holds, then there exist integers $c_{-1}^0, c_0^0, c_1^0, \ldots, c_n^0$ satisfying the inequalities

$$c_{-1}^0 \geqslant 0, \quad |c_i^0| < m_i \quad (i = 0, 1, \ldots, n)$$

and such that

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0+\ldots+\frac{1}{m_{n-1}}\left(c_{n-1}^0+\frac{1}{m_n}c_n^0\right)\ldots\right)\right).$$

Proof. We have

$$S_n = \frac{l_0}{m_0}\left(1+\frac{l_1}{m_1}\left(1+\ldots+\frac{l_n}{m_n}\cdot 1\right)\ldots\right)$$

$$= \frac{1}{m_0}\left(l_0+\frac{l_0 l_1}{m_1}\left(1+\frac{l_2}{m_2}\left(1+\ldots+\frac{l_n}{m_n}\cdot 1\right)\ldots\right)\right)\ldots$$

$$= \frac{1}{m_0}\left(l_0+\frac{1}{m_1}\left(l_0 l_1+\ldots+\frac{1}{m_n}\prod_{i=0}^{n} l_i\right)\ldots\right).$$

Performing one step of a method analogical to method Q but containing no multiplication by 10 we obtain

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n} c_n^0\right) \ldots\right)\right).$$

Indeed, putting

$$E_i^0 = \left(\prod_{j=0}^{i} l_j + E_{i+1}^0\right) \div m_i \quad (i = n, n-1, \ldots, 0;\ E_{n+1}^0 = 0),$$

and

$$c_i^0 = \prod_{j=0}^{i} l_j + E_{i+1}^0 - m_i E_i^0 \quad (i = n,\ n-1, \ldots, 0),$$

we get

$$S_n = \frac{1}{m_0}\left(l_0 + \frac{1}{m_0}\left(l_0 l_1 + \ldots + \frac{1}{m_n} \prod_{j=0}^{n} l_j\right) \ldots\right)$$

$$= \frac{1}{m_0}\left(l_0 + \frac{1}{m_1}\left(l_0 l_1 + \ldots + \frac{1}{m_{n-1}}\left(\prod_{j=0}^{n-1} l_j + E_n^0 + \frac{1}{m_n} c_n^0\right) \ldots\right)\right)$$

$$= E_0^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n} c_n^0\right) \ldots\right)\right)$$

$$= c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n} c_n^0\right) \ldots\right)\right).$$

From the obvious inequality $|c_i^0| < m_i$ it follows successively that

$$\left|\frac{c_n^0}{m_n}\right| < 1,$$

$$\left|\frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{c_n^0}{m_n}\right)\right| < 1,$$

. . . .

$$\left|\frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{c_n^0}{m_n}\right) \ldots\right)\right)\right| < 1.$$

The last inequality implies $c_{-1}^0 \geqslant 0$.

THEOREM 3. *If*

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n} c_n^0\right) \ldots\right)\right) > 0,$$

where $c_{-1}^0, c_0^0, \ldots, c_n^0$ *are integers satisfying*

$$c_{-1}^0 \geqslant 0, \quad |c_i^0| < m_i \quad (i = 0, 1, \ldots, n),$$

then there exist constants $\bar{c}_{-1}^0, \bar{c}_0^0, \ldots, \bar{c}_n^0$ *satisfying the inequalities*

$$\bar{c}_{-1}^0 \geqslant 0, \quad 0 \leqslant \bar{c}_i^0 < m_i \quad (i = 0, 1, \ldots, n)$$

and such that

$$S_n = \bar{c}_{-1}^0 + \frac{1}{m_0}\left(\bar{c}_0^0 + \frac{1}{m_1}\left(\bar{c}_1^0 + \ldots + \frac{1}{m_{n-1}}\left(\bar{c}_{n-1}^0 + \frac{1}{m_n}\bar{c}_n^0\right)\ldots\right)\right).$$

Proof. Since $S_n > 0$, we have $c_s^0 > 0$, where $s = \min\{i\colon c_i^0 \neq 0\}$. We define

$$\tilde{c}_i^0 = \begin{cases} c_i^0 & (i < s), \\ c_i^0 - 1 & (i = s), \\ c_i^0 + m_i - 1 & (s < i < n), \\ c_i^0 + m_i & (i = n). \end{cases}$$

Obviously, $\tilde{c}_i^0 \geqslant 0$ $(i = -1, 0, 1, \ldots, n)$. Let us introduce the notation

$$\tilde{S}_n = \tilde{c}_{-1}^0 + \frac{1}{m_0}\left(\tilde{c}_0^0 + \frac{1}{m_1}\left(\tilde{c}_1^0 + \ldots + \frac{1}{m_{n-1}}\left(\tilde{c}_{n-1}^0 + \frac{1}{m_n}\tilde{c}_n^0\right)\ldots\right)\right).$$

We show that

(13) $$S_n = \tilde{S}_n.$$

S_n and $\tilde{S}_n$ can be evaluated by using the formulae

$$T_{n+1} = 0, \quad T_i = \frac{1}{m_i}(c_i^0 + T_{i+1}) \quad (i = n, n-1, \ldots, -1;\ m_{-1} = 1),$$

$$S_n = T_{-1}, \quad \tilde{T}_{n+1} = 0,$$

$$\tilde{T}_i = \frac{1}{m_i}(\tilde{c}_i^0 + \tilde{T}_{i+1}) \quad (i = n, n-1, \ldots, -1;\ m_{-1} = 1), \quad \tilde{S}_n = \tilde{T}_{-1}.$$

Identity (13) is equivalent to the identity $T_{-1} = \tilde{T}_{-1}$ following from the more general relation

(14) $$T_i = \tilde{T}_i,$$

which holds for $i = -1, 0, 1, \ldots, s$. Before proving (14) we check first that the following equation holds:

(15) $$\tilde{T}_i = T_i + 1 \quad (i = s+1, \ldots, n).$$

For $i = n$ we have

$$\tilde{T}_n = \frac{1}{m_n}\left(\tilde{c}_n^0 + \tilde{T}_{n+1}\right) = \frac{1}{m_n}\left(c_n^0 + m_n + T_{n+1}\right) = 1 + T_n.$$

Assuming that (15) is true for a certain i $(s+1 < i < n)$ we obtain

$$\tilde{T}_{i-1} = \frac{1}{m_{i-1}}(\tilde{c}_{i-1}^0 + T_i) = \frac{1}{m_{i-1}}(c_{i-1}^0 + m_{i-1} - 1 + 1 + T_i)$$

$$= 1 + \frac{1}{m_{i-1}}(c_{i-1}^0 + T_i) = 1 + T_{i-1}.$$

Equation (14) for $i = s$ follows from (15) and the definition of c_s^0, and for $i < s$ from (15) and the equality $\tilde{c}_i^0 = c_i^0$.

Now, all we have to do is to perform, as in the proof of Theorem 2, one step of method Q without multiplication by 10 and to obtain

$$S_n = \bar{S}_n = \bar{c}_{-1}^0 + \frac{1}{m_0}\left(\bar{c}_0^0 + \frac{1}{m_1}\left(\bar{c}_1^0 + \ldots + \frac{1}{m_{n-1}}\left(\bar{c}_{n-1}^0 + \frac{1}{m_n}\bar{c}_n^0\right)\ldots\right)\right),$$

where $\bar{c}_{-1}^0 = \tilde{c}_{-1}^0 + E_0^0 \geqslant 0$, $0 \leqslant \bar{c}_i^0 < m_i$ $(i = 0, 1, \ldots, n)$.

THEOREM 4. *If*

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n}c_n^0\right)\ldots\right)\right),$$

where $c_{-1}^0 \geqslant 0$ *and* $0 \leqslant c_i^0 < m_i$ $(i = 0, 1, \ldots, n)$, *then in every step of method* Q *the values of* E_0^j *satisfy the inequalities*

$$0 \leqslant E_0^j < 10 \quad (j = 1, 2, \ldots). \tag{16}$$

Proof. After performing the j-th step of method Q we obtain

$$S_n = c_{-1}^0 + \sum_{k=1}^{j} 10^{-k}E_0^k + 10^{-j}\left(\frac{1}{m_0}\left(c_0^j + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^j + \frac{1}{m_n}c_n^j\right)\ldots\right)\right)$$

$$= c_{-1}^0 + \sum_{k=1}^{j} 10^{-k}E_0^k +$$

$$+ 10^{-(j+1)}\left(E_0^{j+1} + \frac{1}{m_0}\left(c_0^{j+1} + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^{j+1} + \frac{1}{m_n}c_n^{j+1}\right)\ldots\right)\right).$$

It follows from the assumption on c_i^j that

$$S_n \leqslant c_{-1}^0 + \sum_{k=1}^{j} 10^{-k}E_0^k +$$

$$+10^{-j}\left(\frac{1}{m_0}\left(m_0-1+\ldots+\frac{1}{m_{n-1}}\left(m_{n-1}-1+\frac{1}{m_n}\left(m_n-1\right)\right)\ldots\right)\right)$$

$$=c_{-1}^0+\sum_{k=1}^{j}10^{-k}E_0^k+$$

$$+10^{-(j+1)}\left(\bar{E}_0^{j+1}+\frac{1}{m_0}\left(\bar{c}_0^{j+1}+\ldots+\frac{1}{m_{n-1}}\left(\bar{c}_{n-1}^{j+1}+\frac{1}{m_n}\bar{c}_n^{j+1}\right)\ldots\right)\right),$$

where

$$\bar{E}_{n+1}^{j+1}=0,\quad \bar{E}_i^{j+1}=\left(10(m_i-1)+\bar{E}_{i+1}^{j+1}\right)\div m_i\qquad (i=n,n-1,\ldots,0),$$

$$\bar{c}_i^{j+1}=10(m_i-1)+\bar{E}_{i+1}^{j+1}-m_i\bar{E}_i^{j+1}\qquad (i=n,\ n-1,\ldots,0).$$

It can be easily shown that $\bar{E}_0^{j+1}\geqslant E_0^{j+1}$. Indeed, it follows successively from the inequality $c_i^j\leqslant m_i-1$ that

$$\bar{E}_n^{j+1}=\left(10(m_n-1)+\bar{E}_{n+1}^{j+1}\right)\div m_n\geqslant(10c_n^j+E_{n+1}^{j+1})\div m_n=E_n^{j+1},$$

$$\bar{E}_{n-1}^{j+1}=\left(10(m_{n-1}-1)+\bar{E}_n^{j+1}\right)\div m_{n-1}\geqslant(10c_{n-1}^j+E_n^{j+1})\div m_{n-1}=E_{n-1}^{j+1},$$

. . . .

$$\bar{E}_0^{j+1}=\left(10(m_0-1)+\bar{E}_1^{j+1}\right)\div m_0\geqslant(10c_0^j+E_1^{j+1})\div m_0=E_0^{j+1}.$$

Now, it suffices to show that $\bar{E}_0^{j+1}<10$. If $\bar{E}_0^{j+1}>0$, then we would have

$$10\leqslant\bar{E}_0^{j+1}=\left(10(m_0-1)+\bar{E}_1^{j+1}\right)\div m_0,$$

hence $\bar{E}_1^{j+1}\geqslant 10$; analogously, $\bar{E}_2^{j+1}\geqslant 10,\ldots,\bar{E}_{n+1}^{j+1}\geqslant 0$. But the last inequality is contradictory to the definition of $\bar{E}_{n+1}^{j+1}$.

Example 4. Consider S_7 from Example 2. It follows from Theorem 2 that

$$S_7=\frac{1}{3}\left(2+\frac{1}{3}\left(-2+\frac{1}{9}\left(8+\frac{1}{2}\left(-8+\frac{1}{15}\left(64+\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+\frac{1}{9}\left(-320+\frac{1}{7}\left(1280+\frac{1}{12}\left(-8960\right)\right)\right)\right)\right)\right)\right)\right).$$

Performing one step of method Q without multiplication by 10 we obtain the results presented in Table 5.

TABLE 5

i	0	1	2	3	4	5	6	7
$\prod_{k=0}^{i} l_k$	2	−2	8	−8	64	−320	1286	−8960
m_i	3	3	9	2	15	9	7	12
E_i^0	0	0	0	−3	2	−26	76	−746
c_i^0	2	−2	5	0	8	0	2	−8

It follows from the last line that S_7 can be written in the form

$$S_7 = \frac{1}{3}\left(2+\frac{1}{3}\left(-2+\frac{1}{9}\left(5+\frac{1}{2}\left(0+\frac{1}{15}\left(8+\frac{1}{9}\left(0+\frac{1}{7}\left(2+\frac{1}{12}\left(-8\right)\right)\right)\right)\right)\right)\right)\right).$$

Using Theorem 3 we get

$$\begin{aligned} \tilde{c}_{-1}^0 &= c_{-1}^0 = 0, & s &= 0, \\ \tilde{c}_0^0 &= c_0^0 - 1 = 1, & \tilde{c}_1^0 &= c_1^0 + m_1 - 1 = 0, \\ \tilde{c}_2^0 &= c_2^0 + m_2 - 1 = 13, & \tilde{c}_3^0 &= c_3^0 + m_3 - 1 = 1, \\ \tilde{c}_4^0 &= c_4^0 + m_4 - 1 = 22, & \tilde{c}_5^0 &= c_5^0 + m_5 - 1 = 8, \\ \tilde{c}_6^0 &= c_6^0 + m_6 - 1 = 8, & \tilde{c}_7^0 &= c_7^0 + m_7 = 4. \end{aligned}$$

Performing again one step of method Q without multiplication by 10 we obtain the results presented in Table 6.

TABLE 6

i	0	1	2	3	4	5	6	7
$\tilde{c}_i^0$	1	0	13	1	22	8	8	4
m_i	3	3	9	2	15	9	7	12
E_i^0	0	0	1	1	1	1	1	0
$\bar{c}_i^0$	1	1	5	0	8	0	1	4

Finally, we obtain the expression

$$S_7 = 0+\frac{1}{3}\left(1+\frac{1}{3}\left(1+\frac{1}{9}\left(5+\frac{1}{2}\left(0+\frac{1}{15}\left(8+\frac{1}{9}\left(0+\frac{1}{7}\left(1+\frac{1}{12}\cdot 4\right)\right)\right)\right)\right)\right)\right)$$

satisfying the assumptions of Theorem 4. Obviously, the contracted form of the expression,

$$S_7 = \frac{1}{3}\left(1+\frac{1}{3}\left(1+\frac{1}{9}\left(5+\frac{1}{30}\left(8+\frac{1}{63}\left(1+\frac{1}{3}\cdot 1\right)\right)\right)\right)\right),$$

has also this property.

Performing three steps of method Q for S_7 we obtain $E_0^1 = 5$, $E_0^2 = 0$, $E_0^3 = 9$, i.e., the correct decimal digits of S_7.

Theorem 4 guarantees that method Q applied to the bracket expression

$$S_n - c_{-1}^0 = \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n}c_n^0\right)\ldots\right)\right)$$

gives in every step the next digit of the decimal expansion of S_n. However, transforming the expression S_n to the bracket form with all nominators equal to one is rather expensive. This follows from the fact that in practice one cannot use the method proposed in the proof of Theorem 2 because the product $\prod_{k=0}^{i} l_k$ cannot be stored in one machine word, even for small values of i.

In the next section we give another method of transforming S_n to the bracket form with nominators equal to one. The cost of the method is, however, remarkable, namely it is equal to the cost of evaluating $n/2$ digits of S_n by method Q.

3. Transformation of S_n to the bracket form with all nominators equal to one. In this section we give two algorithms for the transformation of S_n to the bracket form with all nominators equal to one. The first algorithm is a consequence of Theorem 5 which is formulated and proved below. The proof of the second algorithm is also based on this theorem. We give fragments of a program in pseudo-ALGOL realizing both algorithms. Also, we discuss in details a modification of method Q, which n every step evaluates the successive digit of the expansion of S_n.

THEOREM 5. *Let*

$$S_n = c_{-1}^0 + \frac{l_0}{m_0}\left(c^0 + \frac{l_1}{m_1}\left(c_1^0 + \ldots + \frac{l_{n-1}}{m_{n-1}}\left(c_{n-1}^0 + \frac{l_n}{m_n} c_n^0\right)\ldots\right)\right)$$

and

$$\bar{S}_n = \bar{d}_{-1}^0 + \frac{1}{m_0}\left(\bar{d}_0^0 + \frac{1}{m_1}\left(\bar{d}_1^0 + \ldots + \frac{1}{m_{n-1}}\left(\bar{d}_{n-1}^0 + \frac{1}{m_n}\bar{d}_n^0\right)\ldots\right)\right),$$

where $\bar{d}_{-1}^0, \bar{d}_0^0, \ldots, \bar{d}_n^0$ are defined recursively by the formulae

(17) $$\bar{d}_i^{n+1} = c_i^0 \quad (i = -1, 0, \ldots, n),$$

and for $j = n, n-1, \ldots, 0$

(18) $$d_i^j = \begin{cases} l_j \bar{d}_i^{j+1} & (n \geqslant i \geqslant j), \\ \bar{d}_i^{j+1} & (j > i \geqslant -1), \end{cases} \qquad E_i^j = d_i^j \div m_i \ (n \geqslant i \geqslant j),$$

$$\bar{d}_i^j = \begin{cases} d_i^j - E_i^j m_i & (n \geqslant i \geqslant j), \\ d_i^j + E_i^j & (i = j-1), \\ d_i^j & (j-1 \geqslant i \geqslant -1). \end{cases}$$

Then

(19) $$S_n = \bar{S}_n.$$

Proof. S_n can be evaluated by using the formulae

$$T_{n+1} = 0, \quad T_1 = \frac{l_i}{m_i}(c_i^0 + T_{i+1}) \quad (i = n, n-1, \ldots, 0),$$

$$S_n = c_{-1}^0 + T_0.$$

In view of (17) and (18) the following formulae describe an algorithm for the evaluation of $\bar{S}_n$:

$$D_{n+1}^{n+1} = 0, \quad E_{n+1}^{n+1} = 0,$$

$$D_j^{j+1} = \frac{1}{m_j}(c_j + E_{j+1}^{j+1} + D_{j+1}^{j+1}) \quad (j = n, n-1, \ldots, 0),$$

$$D_i^j = l_j D_i^{j+1} - E_i^j \quad (i = n, n-1, \ldots, j),$$

$$S_n = c_{-1}^0 + E_0^0 + D_0^0.$$

We show that

$$T_j = E_j^j + D_j^j \quad (j = n+1, n, \ldots, 0), \tag{20}$$

which implies (19). We use induction on j in the proof of (20). For $j = n+1$ this equation is obviously true. Assuming that it is true for a certain j $(0 < j \leqslant n+1)$ we obtain successively

$$T_{j-1} = \frac{l_{j-1}}{m_{j-1}}(c_{j-1}^0 + T_j) = \frac{l_{j-1}}{m_{j-1}}(c_{j-1}^0 + E_j + D_j^j) = l_{j-1} D_{j-1}^j = D_{j-1}^{j-1} + E_{j-1}^{j-1}.$$

Thus (20) is true for $j = 0, 1, \ldots, n+1$.

From Theorem 5 we obtain the following algorithm which for an expression $S_n > 0$ given in the form

$$S_n = c_{-1}^0 + \frac{l_0}{m_0}\left(c_0^0 + \ldots + \frac{l_{n-1}}{m_{n-1}}\left(c_{n-1}^0 + \frac{l_n}{m_n} c_n^0\right)\ldots\right),$$

where $c_{-1}^0 = 0$, $c_i^0 = 1$ $(i = 0, 1, \ldots, n)$, calculates new values of the coefficients c_i^0 in the transformed expression

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \frac{1}{m_1}\left(c_1^0 + \ldots + \frac{1}{m_{n-1}}\left(c_{n-1}^0 + \frac{1}{m_n} c_n^0\right)\ldots\right)\right)$$

satisfying the inequalities $c_{-1}^0 \geqslant 0$ and $|c_i^0| < m_i$ $(i = 0, 1, \ldots, n)$.

ALGORITHM I.

```
for i := n step -1 until 0 do
 begin
  for j := i step 1 until n do c_j^0 := c_j^0 × l_i;
  for j := n step -1 until i do
   begin
    E_j^0 := c_j^0 ÷ m_j;
    c_j^0 := c_j^0 - E_j^0 × m_j;
    c_{j-1}^0 := c_{j-1}^0 + E_j^0
   end j
 end i
```

It can be easily seen that the i-th step of the above algorithm is equivalent to one step of method Q without multiplication by 10, applied to the bracket expression

$$\frac{1}{m_i}\left(l_i c_i^0 + \frac{1}{m_{i+1}}\left(l_i c_{i+1}^0 + \ldots + \frac{1}{m_{n-1}}\left(l_i c_{n-1}^0 + \frac{1}{m_n} l_i c_n^0\right)\ldots\right)\right).$$

Algorithm I can be slightly modified. Let us assume that numbers $K, s_0, s_1, \ldots, s_K$ are given such that

$$K < n, \quad s_i + 2 < s_{i+1} \quad (i = 0, 1, \ldots, K-1),$$

$$s_0 = -2, \quad s_K = n$$

and such that the number M defined as

$$M = \max_{1 \leqslant i \leqslant K}\left\{m \prod_{j=s_{i-1}+2}^{s_i+1} l_j\right\},$$

where

$$m = \max_{0 \leqslant i \leqslant n}\{m_i - 1\}, \quad l_{n+1} = 1,$$

can be stored in one machine word. S_n can then be written in the form

$$S_n = c_{-1}^0 + \frac{1}{m_0}\left(c_0^0 + \ldots + \frac{1}{m_{s_1}}\left(c_{s_1}^0 + \frac{\prod_{j=s_0+2}^{s_1+1} l_j}{m_{s_1+1}}\left(c_{s_1+1}^0 + \right.\right.\right.$$

$$+ \frac{1}{m_{s_1+2}}\left(c_{s_1+2}^0 + \ldots + \frac{1}{m_{s_2}}\left(c_{s_2}^0 + \frac{\prod_{j=s_1+2}^{s_2+1} l_j}{m_{s_1+1}}\left(c_{s_2+1}^0 + \right.\right.\right.$$

$$\left.+ \frac{1}{m_{s_2+2}}\left(c_{s_2+2}^0 + \ldots + \frac{1}{m_{s_K}} c_{s_K}^0\right)\ldots\right)\Big)\ldots\Big)\Big)\Big)\ldots\Big),$$

where

$$c_i^0 = \begin{cases} 0 & (i = -1), \\ \prod_{j=s_k+2}^{i} l_j & (s_k + 2 \leqslant i \leqslant s_{k+1};\ k = 0, 1, \ldots, K-1), \\ 1 & (i = s_k + 1;\ k = 1, 2, \ldots, K-1). \end{cases}$$

The modified algorithm has the following pseudo-ALGOL form:

ALGORITHM II.

```
a := s_{K-1}+2;
for j := n step -1 until a do
begin
```

```
  E_j^0 := c_j^0 ÷ m_j;
  c_j^0 := c_j^0 − E_j^0 × m_j;
  c_{j−1}^0 := c_{j−1}^0 + E_j^0
 end j;
for i := K−1 step −1 until 1 do
 begin
 a := c_{s_i}^0 × l_i + 1;
 for j := s_i+1 step 1 until n do c_j^0 := c_j^0 × a;
 for j := n step −1 until s_{i−1}+2 do
  begin
   E_j^0 := c_j^0 ÷ m_j;
   c_j^0 := c_j^0 − E_j^0 × m_j;
   c_{j−1}^0 := c_{j−1}^0 + E_j^0
  end j
 end i
```

The cost of algorithm II is equal to the cost of evaluating $K/2$ digits of the decimal expansion of S_n by using method Q.

According to Theorem 4, for method Q to give in every step an exact decimal digit of S_n it suffices that

1° S_n is represented in the bracket form with all nominators equal to one,

2° c_i satisfy the inequalities $c_{-1}^0 \geqslant 0$, $0 \leqslant c_i^0 < m_i$ $(i = 1, 2, \ldots, n)$.

The first condition is fulfilled after application of either algorithm I or algorithm II to the given form of S_n. If one applies to the resulting form a procedure analogous to that used in the proof of Theorem 3, then the second condition is also fulfilled. Method Q is then realized according to the formulae

$$E_{n+1}^j = 0 \quad (j = 1, 2, \ldots), \tag{3}$$

$$E_i^j = (qc_i^{j-1} + E_{i+1}^j) \div m_i \quad (i = n, n-1, \ldots, 0), \tag{4''}$$

$$c_i^j = qc_i^{j-1} + E_{i+1}^j - m_i E_i^j \quad (i = n, n-1, \ldots, 0), \tag{5''}$$

where q denotes the base of the system in which we wish to expand S_n. The decimal expansion is obtained for $q = 10$, the binary one for $q = 2$, etc. The base $q = 10^k$, where k is a small natural number, is often used. In this case every step of method Q produces k succeeding exact decimal digits of S_n. This approach decreases k times the time of calculation. Since the integer part of S_n is always obtained in decimal form, the value of c_{-1}^0 should be represented in the system with base equal to q if q is not a multiplicity of 10. Though previous paragraphs have dealt with the case of $q = 10$, all obtained results can be readily reformulated for any arbitrary q.

4. Procedures Q and Qm. All the details of evaluating the expansion of the sum

$$S_n = \sum_{k=0}^{n} \frac{a_k}{b_k}$$

by method Q are explained below by means of declarations of ALGOL 60 procedures Q and Qm.

```
procedure Q(n, p, E0);
 value n, p;
 integer n, p;
 integer array E0;
 comment There are given natural numbers n and p and the bracket
         form of S_n, defined by integer procedures nom and den, each
         with one parameter i of type integer the values of which for
         given i (0 ≤ i ≤ n) are l_i and m_i, respectively. The results
         are numbers E_0^1, E_0^2, ..., E_0^p contained in the array E0. If
         n < p and S_n is the partial sum of a series with positive terms,
         then the expression

                        p
                        Σ 10^(-j) E0[j]
                       j=1

         gives the decimal expansion of S_n with p correct digits;
begin
 integer E, i, j, lambda, m, t;
 integer array c[0 : n];
 comment Declarations of procedures nom and den should be inserted
          here;
 for i := 0 step 1 until n do c[i] := 1;
 for j := 1 step 1 until p do
  begin
   E := 0;
   for i := n step -1 until 0 do
    begin
     lambda := if j > i then 1 else nom(i-j);
     m := den(i);
     t := lambda × (10 × c[i]+E);
     E := t ÷ m;
     c[i] := t - m × E
    end i;
   E0[j] := E
  end j
end Q
```

Procedure Q realizes method Q described in Section 1. In procedure Qm given below we use modifications discussed in Sections 2 and 3. Algorithm I (Section 3) and Theorems 3 and 4 are applied.

```
procedure Qm(B, B1, n, p, q, k);
 value n, p, q, k;
 integer n, p, q, k;
 Boolean B, B1;
 comment Procedure Qm evaluates and prints the expansion of the
         sum S_n in the system with base q, with p correct digits of
         the fractional part. S_n is given in the bracket form defined
         by integer procedures nom and den, each with one parameter
         i of type integer the values of which for given i (0 ≤ i ≤ n)
         are l_i and m_i, respectively. The Boolean variable B should
         be given the value false if |nom(i)| = 1 for i = 0, 1, ..., n,
         and the value true otherwise. The Boolean variable B1 should
         have the value false if S_n is the partial sum of a series with
         all positive terms. The actual parameter corresponding to k
         should contain an upper estimation of the number of digits
         of the integer part of S_n in the representation in the system
         with base q;

begin
 integer E, i, j, lambda, m, t;
 integer array c[-1 : n], d[0 : k];
 comment The declarations of procedures nom and den should be inserted
          here;
 if ¬B ∧ B1
  then begin
        c[0] := sign(nom(0));
        for i := 1 step 1 until n do c[i] := sign(nom(i)) × c[i-1]
       end
  else for i := 0 step 1 until n do c[i] := 1;
 c[-1] := 0;
 if B
  then begin
        c[n] := nom(n);
        for i := n-1 step -1 until 0 do
         begin
          lambda := nom(i);
          for j := i step 1 until n do c[j] := lambda × c[j];
          for j := step -1 until i do
           begin
            m := den(j);
```

```
            E := c[j]÷m;
            c[j] := c[j]−m×E;
            c[j−1] := c[j−1]+E
          end j
        end i
      end B;
if B1
 then begin
        i := −1;
        for t := c[i] while t = 0 do i := i+1;
        c[i] := c[i]−1;
        t := n−1;
        for j := i+1 step 1 until t do
        c[j] := c[j]+den(j)−1;
        c[n] := c[n]+den(n)
      end B1;
if ¬B∨B1
 then for j := n step −1 until 0 do
      begin
        m := den(j);
        E := c[j]÷m;
        c[j] := c[j]−m×E;
        c[j−1] := c[j−1]+E
      end j;
for j := 0 step 1 until k do d[j] := 0;
i := k+1;
E := c[−1];
for i := i−1 while t ≠ 0 do
 begin
  t := E÷q;
  d[i] := E−q×t;
  E := t
 end i;
print(d);
comment the output procedure print is taken from ALGOL 1204,
        a version of ALGOL 60 for the ODRA 1204 computer (cf. [1]);
for j := 1 step 1 until p do
 begin
 E := 0;
  for i := n step −1 until 0 do
   begin
    m := den(i);
    t := q×c[i]+E;
```

```
      E := t÷m;
      c[i] := c[i]−m×E
    end i;
    print(E)
  end j
end Qm
```

5. Results of tests. Procedure Qm given in the preceding section was tested on the ODRA 1204 computer installed at the Computing Centre of the Institute of Computer Science, University of Wrocław. The following values were calculated with 150 digits after the point:

e^x for $x = 1, -1, 2, -2$ (Table 7),

$\cos(1/m)$ for $m = 2, 3, \ldots, 10$ (Table 8),

$\arctan(1/m)$ for $m = 2, 3, \ldots, 10$ (Table 9).

In all the examples the time of calculations was proportional to the number $(p+b\times n/2+b1)(n+1)$ with the coefficient of about .01 sec, where

$$b = \begin{cases} 0 & (B \equiv \textbf{false}), \\ 1 & (B \equiv \textbf{true}), \end{cases} \qquad b1 = \begin{cases} 0 & (B1 \equiv \textbf{false}), \\ 1 & (B1 \equiv \textbf{true}). \end{cases}$$

1. e^x was calculated with the aid of the formula

$$S_n = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

We give the full results for the case $p = 150$ only. For the cases $p = 50$ and $p = 100$ we give the values of n and the time of calculations T.

TABLE 7

	$x = 1$,	$p = 150$, $n = 97$, $T = 135$,	$p = 100$, $n = 71$, $T = 67$,	$p = 50$, $n = 42$, $T = 20$,	
$e = 2.$	7 1 8 2 8	1 8 2 8 4	5 9 0 4 5	2 3 5 3 6	0 2 8 7 4
	7 1 3 5 2	6 6 2 4 9	7 7 5 7 2	4 7 0 9 3	6 9 9 9 5
	9 5 7 4 9	6 6 9 6 7	6 2 7 7 2	4 0 7 6 6	3 0 3 5 3
	5 4 7 4 9	4 5 7 1 3	8 2 1 7 8	5 2 5 1 6	6 4 2 7 4
	2 7 4 6 6	3 9 1 9 3	2 0 0 3 0	5 9 9 2 1	8 1 7 4 1
	3 5 9 6 6	2 9 0 4 3	5 7 2 9 0	0 3 3 4 2	9 5 2 6 0

$x = -1$,	$p = 150$,	$p = 100$,	$p = 50$,
	$n = 97$,	$n = 71$,	$n = 42$,
	$T = 137$,	$T = 69$,	$T = 21$,

$e^{-1} = 0.$	3 6 7 8 7	9 4 4 1 1	7 1 4 4 2	3 2 1 5 9	5 5 2 3 7
	7 0 1 6 1	4 6 0 8 6	7 4 4 5 8	1 1 1 3 1	0 3 1 7 6
	7 8 3 4 5	0 7 8 3 6	8 0 1 6 9	7 4 6 1 4	9 5 7 4 4
	8 9 9 8 0	3 3 5 7 1	4 7 2 7 4	3 4 5 9 1	9 6 4 3 7
	4 6 6 2 7	3 2 5 2 7	6 8 4 3 9	9 5 2 0 8	2 4 6 9 7
	5 7 9 2 7	9 0 1 2 9	0 0 8 6 2	6 6 5 3 5	8 9 4 9 4

$x = 2$,	$p = 150$,	$p = 100$,	$p = 50$,
	$n = 114$,	$n = 85$,	$n = 52$,
	$T = 243$,	$T = 127$,	$T = 43$,

$e^{2} = 7.$	3 8 9 0 5	6 0 9 8 9	3 0 6 5 0	2 2 7 2 3	0 4 2 7 4
	6 0 5 7 5	0 0 7 8 1	3 1 8 0 3	1 5 5 7 0	5 5 1 8 4
	7 3 2 4 0	8 7 1 2 7	8 2 2 5 2	2 5 7 3 7	9 6 0 7 9
	0 5 7 7 6	3 3 8 4 3	1 2 4 8 5	0 7 9 1 2	1 7 9 4 7
	7 3 7 5 3	1 6 1 2 6	5 4 7 8 8	6 6 1 2 3	8 8 4 6 0
	3 6 9 2 7	8 1 2 7 3	3 7 4 4 7	8 3 9 2 2	1 3 3 9 8

$x = -2$,	$p = 150$,	$p = 100$,	$p = 50$,
	$n = 114$,	$n = 85$,	$n = 52$,
	$T = 246$,	$T = 129$,	$T = 44$,

$e^{-2} = 0.$	1 3 5 3 3	5 2 8 3 2	3 6 6 1 2	6 9 1 8 9	3 9 9 9 4
	9 4 9 7 2	4 8 4 4 0	3 4 0 7 6	3 1 5 4 5	9 0 9 5 7
	5 8 8 1 4	6 8 1 5 8	8 7 2 6 5	4 0 7 3 3	7 4 1 0 1
	4 8 7 6 8	9 9 3 7 0	9 8 1 2 2	4 9 0 6 5	7 0 4 8 7
	5 5 0 7 7	2 8 7 1 8	9 6 3 3 5	5 2 2 1 2	4 4 9 3 4
	6 8 7 1 8	9 2 8 5 3	0 3 8 1 5	8 8 9 5 1	3 4 9 9 6

2. In the calculation of $\cos x$ the expression

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

was used.

TABLE 8

$x = 1/2$,	$p = 150$,	$n = 41$,	$T = 83$,

$\cos(1/2) = 0.$	8 7 7 5 8	2 5 6 1 8	9 0 3 7 2	7 1 6 1 1	6 2 8 1 5
	8 2 6 0 3	8 2 9 6 5	1 9 9 1 6	4 5 1 9 7	1 0 9 7 4
	4 0 5 2 9	9 7 6 1 0	8 6 8 3 1	5 9 5 0 7	6 3 2 7 4
	2 1 3 9 4	7 4 0 5 7	9 4 1 8 4	0 8 4 6 8	2 2 5 8 3
	5 5 4 7 8	4 0 0 5 9	3 1 0 9 0	5 3 9 9 3	4 1 3 8 2
	7 9 7 6 8	3 3 2 8 0	2 6 6 7 9	9 7 5 6 1	2 0 9 5 0

Table 8 (contd.)

$x = 1/3,\quad p = 150,\quad n = 39,\quad T = 79,$

cos(1/3) = 0.	9 4 4 9 5	6 9 4 6 3	1 4 7 3 7	6 6 4 3 8	8 2 8 4 0
	0 7 6 7 5	8 8 0 6 0	7 8 4 5 8	6 2 6 9 9	5 6 5 1 4
	0 7 3 7 6	7 7 6 4 5	7 3 3 7 5	0 0 9 9 5	6 2 1 9 6
	5 0 0 3 6	4 8 2 4 4	2 8 1 5 8	8 0 5 6 9	8 5 5 6 5
	9 2 7 0 5	5 0 4 9 0	4 2 8 8 6	4 6 2 1 6	7 2 5 2 8
	5 4 5 8 8	7 3 2 6 9	6 5 4 3 2	5 9 9 7 6	6 2 9 6 1

$x = 1/4,\quad p = 150,\quad n = 37,\quad T = 75,$

cos(1/4) = 0.	9 6 8 9 1	2 4 2 1 7	1 0 6 4 4	7 8 4 1 4	4 5 9 5 4
	4 9 4 9 4	1 8 9 1 9	9 8 0 4 1	3 4 1 9 0	2 8 7 4 4
	2 8 3 1 1	4 8 1 2 8	1 2 4 2 8	8 9 4 2 5	6 1 1 8 4
	5 2 3 3 2	7 2 6 4 6	5 5 2 0 2	7 9 9 6 8	5 0 2 5 5
	1 0 3 5 2	7 0 9 6 2	6 1 1 6 2	0 2 6 1 7	3 0 9 4 6
	4 4 8 4 0	5 0 1 4 9	9 6 3 5 9	1 8 3 1 7	4 6 8 0 9

$x = 1/5,\quad p = 150,\quad n = 35,\quad T = 71,$

cos(1/5) = 0.	9 8 0 0 6	6 5 7 7 8	4 1 2 4 1	6 3 1 1 2	4 1 9 6 5
	1 6 7 4 8	1 6 8 8 7	7 3 9 3 5	2 4 3 6 0	8 0 6 5 6
	7 9 9 4 0	5 2 5 4 8	2 9 0 1 2	6 1 8 7 4	0 7 8 8 2
	7 3 1 7 0	0 8 9 5 0	2 4 0 3 5	2 2 4 3 0	3 3 4 6 5
	0 5 2 9 9	3 9 1 7 6	7 5 9 8 5	0 9 4 3 4	5 0 9 0 2
	2 1 2 5 0	5 7 9 6 2	9 4 7 2 2	6 8 6 4 3	9 7 4 4 1

$x = 1/6,\quad p = 150,\quad n = 35,\quad T = 70,$

cos(1/6) = 0.	9 8 6 1 4	3 2 3 1 5	6 2 9 2 5	0 5 7 9 3	0 6 8 5 8
	2 2 9 5 6	1 1 4 9 3	8 9 3 8 0	2 5 5 3 9	0 7 6 3 7
	7 0 0 7 5	8 9 3 6 9	3 2 2 5 2	2 1 4 2 3	4 7 2 9 9
	3 5 6 5 2	1 5 0 8 8	9 2 8 4 6	2 0 9 5 7	4 7 9 9 8
	9 9 2 7 0	6 1 3 0 0	7 3 5 3 3	7 3 1 2 2	9 4 9 9 6
	3 2 9 1 5	9 6 0 0 4	6 3 3 7 4	7 0 1 9 9	5 9 4 6 6

$x = 1/7,\quad p = 150,\quad n = 33,\quad T = 67,$

cos(1/7) = 0.	9 8 9 8 1	3 2 6 0 4	4 6 6 1 5	0 8 2 6 9	5 7 2 6 1
	3 7 0 1 3	4 3 3 7 4	5 8 7 4 0	3 2 7 0 0	8 9 9 5 7
	1 4 1 7 6	3 1 8 1 0	3 4 3 6 5	1 6 1 2 6	3 0 8 0 8
	2 8 1 5 8	1 4 8 6 7	4 2 8 3 7	6 6 5 7 3	4 3 8 8 2
	1 3 2 9 1	1 3 9 9 5	7 9 1 0 8	1 5 0 0 8	5 1 4 3 7
	9 7 1 9 2	5 7 9 6 9	8 2 5 6 7	8 8 9 1 9	4 1 9 6 3

$x = 1/8,\quad p = 150,\quad n = 33,\quad T = 67,$

cos(1/8) = 0.	9 9 2 1 9	7 6 6 7 2	2 9 3 2 9	0 5 3 1 4	9 0 9 6 9
	0 7 7 8 8	2 5 0 8 6	9 5 4 3 3	2 7 3 0 4	7 3 6 6 0
	1 2 6 3 4	6 8 9 0 9	6 9 7 1 9	3 9 6 4 0	5 8 8 4 0
	0 6 1 4 8	1 7 2 1 2	3 6 2 9 0	9 3 5 8 4	4 3 5 7 2
	8 1 9 5 4	4 1 1 2 6	9 9 0 5 2	8 8 4 4 1	5 4 3 2 3
	4 2 4 5 8	6 8 5 4 2	2 2 5 4 1	2 9 7 7 7	6 9 5 3 7

	$x = 1/9$, $p = 150$, $n = 33$, $T = 67$,				
cos (1/9) = 0.	9 9 3 8 3	3 5 0 8 5	3 8 8 9 1	9 2 1 3 0	2 9 3 9 3
	0 3 0 7 0	4 7 2 9 9	9 9 2 4 6	7 8 7 1 3	7 8 2 7 5
	8 6 5 3 2	9 9 3 1 8	4 4 1 5 8	3 5 0 4 5	8 6 0 7 8
	3 7 2 4 9	2 6 2 5 9	0 7 3 2 8	7 2 3 0 7	6 8 4 6 3
	3 3 0 8 5	9 5 2 5 0	8 1 8 1 9	2 6 6 9 0	1 0 6 4 2
	7 1 1 8 4	9 1 3 3 6	9 0 0 0 5	5 6 4 7 4	0 7 4 0 0
	$x = 1/10$, $p = 150$, $n = 31$, $T = 63$,				
cos (1/10) = 0.	9 9 5 0 0	4 1 6 5 2	7 8 0 2 5	7 6 6 0 9	5 5 6 1 9
	8 7 8 0 3	8 7 0 2 9	4 8 3 8 5	7 6 2 2 5	4 1 5 0 8
	4 0 3 5 9	5 9 3 5 2	7 4 4 6 8	5 2 6 5 9	1 0 2 1 8
	2 4 0 4 6	6 5 2 9 6	6 3 6 1 8	5 2 8 2 6	2 9 2 7 9
	1 0 7 2 3	6 8 5 8 8	0 8 3 6 8	7 1 8 6 0	3 9 4 1 2
	9 7 0 3 5	2 2 1 2 0	4 5 9 9 3	2 5 8 2 2	7 7 7 6 5

3. In the evaluation of arctan$(1/m)$ the expression

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k m}{(2k+1)m^{2k+2}}$$

was applied.

TABLE 9

	$m = 2$, $p = 150$, $n = 245$, $T = 849$,				
arctan (1/2) = 0.	4 6 3 6 4	7 6 0 9 0	0 0 8 0 6	1 1 6 2 1	4 2 5 6 2
	3 1 4 6 1	2 1 4 4 0	2 0 2 8 5	3 7 0 5 4	2 8 6 1 2
	0 2 6 3 8	1 0 9 3 3	0 8 8 7 2	0 1 9 7 8	6 4 1 6 5
	7 4 1 7 0	5 3 0 0 6	0 0 2 8 3	9 8 4 8 8	7 8 9 2 5
	5 6 5 2 9	8 5 2 2 5	1 1 9 0 8	3 7 5 1 3	5 0 5 8 1
	8 1 8 1 6	2 5 0 1 1	1 5 5 4 7	1 5 3 0 5	6 9 9 4 4
	$m = 3$, $p = 150$, $n = 155$, $T = 434$,				
arctan (1/3) = 0.	3 2 1 7 5	0 5 5 4 3	9 6 6 4 2	1 9 3 4 0	1 4 0 4 6
	1 4 3 5 8	6 6 1 3 1	9 0 2 0 7	5 5 2 9 5	5 5 7 6 5
	6 1 9 1 4	3 2 8 0 3	0 5 9 3 5	6 7 5 6 2	3 7 4 0 5
	8 1 0 5 4	4 3 5 6 4	0 8 4 2 2	3 5 0 6 4	1 3 7 4 4
	3 9 0 0 7	1 6 9 3 7	7 1 2 9 7	3 9 1 4 8	2 6 7 6 4
	2 9 7 0 7	6 2 6 3 4	4 0 2 4 5	9 8 0 9 2	8 2 0 8 8
	$m = 4$, $p = 150$, $n = 123$, $T = 316$,				
arctan (1/4) = 0.	2 4 4 9 7	8 6 6 3 1	2 6 8 6 4	1 5 4 1 7	2 0 8 2 4
	8 1 2 1 1	2 7 5 8 1	0 9 1 4 1	4 4 0 9 8	3 8 1 1 8
	4 0 6 7 1	2 7 3 7 5	9 1 4 6 6	7 3 5 5 1	1 9 5 8 7
	6 4 2 0 9	6 5 7 4 5	3 4 1 5 7	6 6 8 7 0	1 9 9 1 3
	6 3 8 3 4	8 0 4 4 9	0 0 3 7 1	1 8 3 7 4	2 9 5 4 8
	5 4 2 0 9	9 5 0 5 9	9 7 6 9 5	8 9 8 6 9	6 0 6 1 4

Table 9 (contd.)

$m = 5,\quad p = 150,\quad n = 105,\quad T = 259,$

arctan (1/5) = 0.	19739	55598	49880	75837	00497
	65194	79029	34475	85103	78785
	21015	17688	94024	10339	69978
	24378	57326	97828	03728	80441
	12628	11807	36913	60104	45647
	98867	94239	35574	75654	95216

$m = 6,\quad p = 150,\quad n = 95,\quad T = 225,$

arctan (1/6) = 0.	16514	86774	14626	83827	91282
	89643	94345	39983	86660	46502
	78190	18034	43001	11456	62179
	80514	14748	87737	89870	01078
	89996	56405	37600	00222	86898
	67512	19851	92205	73322	52536

$m = 7,\quad p = 150,\quad n = 87,\quad T = 202,$

arctan (1/7) = 0.	14189	70546	04163	92281	28516
	17102	55308	30077	81758	72846
	40723	78130	02936	34416	26759
	93116	09441	91861	63424	65181
	17522	68287	40610	98365	23817
	52108	62376	75301	17212	87856

$m = 8,\quad p = 150,\quad n = 81,\quad T = 186,$

arctan (1/8) = 0.	12435	49945	46761	43503	13548
	49163	87102	55731	70191	76980
	40899	15114	11911	57222	67427
	56675	86237	10594	31335	33303
	26379	05130	34383	79043	81116
	30839	68395	04671	22437	86871

$m = 9,\quad p = 150,\quad n = 77,\quad T = 173,$

arctan (1/9) = 0.	11065	72211	73895	64655	91398
	72210	06210	59752	86095	00306
	40321	22814	44310	76452	05740
	49871	37652	28889	43348	79731
	11511	27179	75831	51030	92257
	44680	78125	15316	28801	38149

$m = 10,\quad p = 150,\quad n = 73,\quad T = 162,$

arctan (1/10) = 0.	09966	86524	91162	02737	84461
	19878	02059	02432	78322	50431
	46480	15508	77681	00277	47447
	55065	44206	12624	43428	63715
	79558	38640	88273	98969	56792
	70665	63156	91279	03020	72085

Some of the results given in Tables 7-9 were compared with the appropriate values tabulated in [3]. In this way the values of e and e^{-1} were found to be correct. The correctness of the expansions of arctan (1/2), arctan (1/5), and arctan (1/8) was ascertained by the evaluation of π from the formula

$$\pi = 4(\arctan\tfrac{1}{2}+\arctan\tfrac{1}{5}-\arctan\tfrac{1}{8})$$

and by comparing the result obtained with the expansion given in [3].

References

[1] K. Jerzykiewicz and J. Szczepkowicz, *ALGOL 1204*, Warszawa 1973.
[2] S. Paszkowski, *Język ALGOL 60*, Warszawa 1978.
[3] I. Peters and I. Šteĭn (И. Петерс и И. Штейн), *Математические таблицы*, Выч. центр АН СССР, Москва 1965.
[4] H. J. Sale, *The calculation of e to many significant digits*, Comput. J. 11 (1968), p. 229.

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