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PRODUCTS AND SUMS OF ABSOLUTE PROPER RETRACTS

BY

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1. Introduction. In [12] we introduced the notion of absolute proper retract (APR). This notion is more restrictive than that of absolute retract (AR) and plays a role in the geometric theory of non-compact spaces and proper maps. In this paper* we consider products and sums of APR's, obtaining results which (while differing somewhat in detail) are analogous to the well-known results of Borsuk on products and sums of AR's (see [2], Chapter IV, Theorem 6.1 (i) and (iii), and Theorem 7.1).

2. Definitions, notations, and preliminary results. We begin by recalling some definitions, establishing our notations, and stating a few results which shall be needed for the sequel. At the risk of being overly verbose, we shall try to make the current paper as self-contained as possible.

We use I to denote the interval $[0, 1]$, id_X to denote the identity function on the set X, and $\approx$ to mean "is homeomorphic to". We use E^n, S^n, E^n_+, and E^n_- to denote Euclidean n-space, the n-sphere, the set of points in E^n with non-negative n-th coordinate, and the set of points in E^n with non-positive n-th coordinate, respectively. $X \in AR$ (ANR) means that X is an absolute retract (absolute neighborhood retract) for metric spaces. If X is a space and $A \subset X$, we use $\mathrm{Cl}_X A$ to denote the closure of A in X.

Suppose X and Y are topological spaces. Let $f\colon X \to Y$ be a map (i.e., a continuous function). Then f is said to be *proper* if $f^{-1}(C)$ is compact whenever C is a compact subset of Y.

Remark. Call a set U in the space Z *unbounded* if U lies in no compact subset of Z. Then, for X and Y Hausdorff, $f\colon X \to Y$ is proper if and only if $f(U)$ is unbounded whenever U is unbounded. This preservation of unboundedness helps explain why proper maps make good sense as a tool for the study of the geometry of non-compact spaces.

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We shall say that $f\colon X \to Y$ is *perfect* if f is closed and $f^{-1}(p)$ is compact for every $p \in Y$.

Remark. The reader is warned that the terminology used for these notions is varied in the literature. Every perfect map is proper, but not conversely. However, the two concepts are equivalent if X and Y are Hausdorff and Y is locally compact. Later on our attention will be restricted to spaces having these properties, and we shall adopt the term "proper" for exclusive use at that time.

Maps $f\colon X \to Y$ and $g\colon X \to Y$ are *properly homotopic* if there exists a proper map $H\colon X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. (This is not the same as declaring that there is a homotopy joining f and g each of whose level maps is proper.) If f and g are properly homotopic, we write $f \underset{p}{\simeq} g$. The relation $\underset{p}{\simeq}$ is easily seen to be an equivalence relation on the set of proper maps from X to Y. If there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $gf \underset{p}{\simeq} \mathrm{id}_X$ and $fg \underset{p}{\simeq} \mathrm{id}_Y$, then X and Y are of the same *proper homotopy type* and we write $X \underset{p}{\simeq} Y$, the relation $\underset{p}{\simeq}$ being an equivalence relation as expected. If only $gf \underset{p}{\simeq} \mathrm{id}_X$ is known to hold, then we say that Y *properly homotopically dominates* X, and we write $X \underset{p}{\leqslant} Y$ or $Y \underset{p}{\geqslant} X$. The following lemma is easily proved and will be useful in the next section:

LEMMA 2.1. *Suppose X, X', Y and Y' are spaces.*

(i) *If* $X \underset{p}{\simeq} X'$ *and* $Y \underset{p}{\simeq} Y'$, *then* $X \times Y \underset{p}{\simeq} X' \times Y'$.

(ii) *If* $X \underset{p}{\leqslant} X'$ *and* $Y \underset{p}{\leqslant} Y'$, *then* $X \times Y \underset{p}{\leqslant} X' \times Y'$.

A *compactification* of the Hausdorff space X is a compact Hausdorff space containing X as a dense subspace. We say that the compactifications Z and W of X are *equivalent*, $Z \equiv W$, if there exists a homeomorphism $h\colon Z \to W$ such that $h(x) = x$ for all $x \in X$. The space X is *rim compact* if each point of X has a local basis of open neighborhoods with compact frontiers. Let Λ denote the class of rim compact Hausdorff spaces and, for $X \in \Lambda$, let FX denote the Freudenthal compactification of X (see [8] or [10]; or, for separable metric spaces, [6]). Let $EX = FX - X$. The points of EX are the *ends* of X. The following result is established in [10] (as Theorem 1) and is restated here for convenience:

THEOREM 2.1. *Suppose $X \in \Lambda$. Then*

(a) *each point of FX has a local basis of open neighborhoods whose frontiers lie in X, and*

(b) *if Z is a compactification of X such that each point of Z has a local basis of open neighborhoods whose frontiers lie in X, then there exists a map $g\colon FX \to Z$ such that $g(x) = x$ for all $x \in X$.*

Furthermore,

(c) *if* $\tilde{F}X$ *is a compactification of* X *having properties* (a) *and* (b), *then* $\tilde{F}X \equiv FX$.

An important fact [14] is that if X, $Y \in \Lambda$ and $f\colon X \to Y$ is perfect, then f extends to a unique map of pairs $Ff\colon (FX, EX) \to (FY, EY)$. (There are several versions of this result in the literature.) Note that, by uniqueness, F is functorial; that is, F preserves compositions and identities. We shall say that the perfect map $f\colon X \to Y$ is *end-preserving* if $Ff \mid EX$ is injective. We write $(X, X_0) \in \Lambda'$ if $X \in \Lambda$, X_0 is a closed subset of X, and the inclusion of X_0 into X is end-preserving. It is easily verified that $(X, X_0) \in \Lambda'$ if and only if $\mathrm{Cl}_{FX} X_0 \equiv FX_0$ whenever $X \in \Lambda$ and X_0 is closed in X.

LEMMA 2.2. *Suppose* X_1, X_2, *and* $X_0 = X_1 \cap X_2$ *are closed subsets of* $X_1 \cup X_2 = X \in \Lambda$. *Suppose further that* $(X_1, X_0) \in \Lambda'$ *and* $(X_2, X_0) \in \Lambda'$. *Then* $(X, X_i) \in \Lambda'$ *for* $i = 0, 1$ *or* 2.

Proof. Since $(X_1, X_0) \in \Lambda'$ and $(X_2, X_0) \in \Lambda'$, we may regard FX_0 as being a closed subspace of each of FX_1 and FX_2. Pasting FX_1 and FX_2 together *via* the identity on FX_0 yields a compactification K of X such that $\mathrm{Cl}_K X_i \equiv FX_i$ for $i = 0, 1, 2$ and $\mathrm{Cl}_K X_1 \cap \mathrm{Cl}_K X_2 = \mathrm{Cl}_K X_0$. It is easily verified that each point of K has a local basis of open neighborhoods whose frontiers lie in X. For $i = 1$ or 2, let j_i denote the inclusion of X_i into X and let $f_i\colon \mathrm{Cl}_K X_i \to FX_i$ be a homeomorphism such that $f_i(x) = x$ for all $x \in X_i$. Let

$$(Fj_i)f_i = G_i\colon \mathrm{Cl}_K X_i \to FX.$$

Then

$$G_1 \mid (\mathrm{Cl}_K X_1) \cap (\mathrm{Cl}_K X_2) = G_1 \mid \mathrm{Cl}_K X_0 = G_2 \mid \mathrm{Cl}_K X_0 = G_2 \mid (\mathrm{Cl}_K X_1) \cap (\mathrm{Cl}_K X_2),$$

and hence the union of G_1 and G_2 yields a map $G\colon K \to FX$ such that $G(x) = x$ for all $x \in X$. It follows immediately that K satisfies condition (b) of Theorem 2.1, and hence, by condition (c) of Theorem 2.1, $K \equiv FX$. Since $\mathrm{Cl}_K X_i \equiv FX_i$ for $i = 0, 1, 2$, it follows that $\mathrm{Cl}_{FX} X_i \equiv FX_i$, and hence $(X, X_i) \in \Lambda'$.

Given X_0, X_1, X_2 and X as in the statement of Lemma 2.2, let $i = 1$ or 2 and let $j\colon X_0 \to X_i$, $k\colon X_i \to X$, and $h\colon X_0 \to X$ be inclusions. Then $h = kj$, so $Fh = F(kj) = (Fk)(Fj)$. Thus F_j is injective if Fh is. Hence, if $(X, X_0) \in \Lambda'$, then $(X_1, X_0) \in \Lambda'$ and $(X_2, X_0) \in \Lambda'$. However, it is possible to have $(X, X_1) \in \Lambda'$ and $(X, X_2) \in \Lambda'$ while $(X, X_0), (X_1, X_0), (X_2, X_0) \notin \Lambda'$. This is the case, for example, if $X_1 = E^2_+$ and $X_2 = E^2_-$. Combining these observations with Lemma 2.2, we obtain the following

THEOREM 2.2. *Suppose X_1, X_2, and $X_0 = X_1 \cap X_2$ are closed subsets of $X_1 \cup X_2 = X \in \Lambda$. Then*

$$\begin{array}{c} (X_1, X_0) \in \Lambda' \text{ and } (X_2, X_0) \in \Lambda' \\ \Uparrow \quad \Downarrow \\ (X, X_0) \in \Lambda' \\ \Downarrow \quad \not\Uparrow \\ (X, X_1) \in \Lambda' \text{ and } (X, X_2) \in \Lambda'. \end{array}$$

We shall be particularly interested in the situation in which FX is metrizable, and this will occur if X is a locally compact separable metric space for which QX, the quasi-component space of X, is compact (see [8], Chapter VI, Theorem 42). (Local compactness is not a necessary condition for the metrizability of FX, but shall be required in Sections 3 and 4.) We shall let Σ denote the class of such spaces and make the *standing hypothesis* that *all spaces henceforth considered in this paper shall be of the class Σ*. If $(X, X_0) \in \Lambda'$ and $X, X_0 \in \Sigma$, we write $(X, X_0) \in \Sigma'$.

LEMMA 2.3. *Suppose X is a connected, locally compact and locally connected metric space having more than one end. Then there exists an end-preserving embedding $h\colon E^1 \to X$.*

Proof. Since X is connected, FX is a continuum. Moreover, FX is locally connected by Theorem 4.1 of [4]. Hence FX is arcwise connected and, since EX contains at least two points, there exists an arc pq in FX such that $p \in EX$ and $q \in EX$. Since EX is closed in FX and has dimension zero, we may assume, by a standard "first point - last point" argument, that the interior of pq lies in X. Now we need only map E^1 homeomorphically onto the interior of pq to obtain h.

We say that Z, $Z \subset W$, is a *proper retract* of W if there exists a proper map $r\colon W \to Z$ such that $r(z) = z$ for all $z \in Z$; such a map r is a *proper retraction* of W onto Z. Similarly, we define a *proper deformation retraction*, etc. Suppose $X \in \Sigma$. Then X is an *absolute proper retract*, for which we shall write $X \in APR$, if, for each $Y \in \Sigma$ and end-preserving embedding $h\colon X \to Y$, $h(X)$ is a proper retract of Y. This notion was introduced and studied in [12], and we wish now to restate the primary results of that paper; however, before doing so it will be convenient to first recall some definitions. By a *tree* we shall mean a locally finite connected and simply connected simplicial 1-complex. In [7], a geometric property was introduced called *property SUV^∞*, and this was further studied in [11]. We need not recall the original definition of property SUV^∞ as given in [7] or [11], but shall only require the fact (which follows from [11], Corollary 3.5, and [1], Theorem 3.12) that the space $X \in ANR$ has property SUV^∞ if and only if there exists a tree T such that $X \underset{p}{\leqslant} T$. If X has property SUV^∞, we write $X \in SUV^\infty$. Finally, $Z' \subset Z$ is said to be an *unstable* subset [9] of a space Z if there exists a homotopy $H: Z\times I\to Z$ such that $H(z, 0)=z$ for all $z\in Z$ and $H(z, t)\notin Z'$ for all $z\in Z$ and $0<t\leqslant 1$. Now, the main object of [12] was to state identifying criteria for the APR's, and this identification is given by the following theorem (cf. [12], Theorems 4.1 and 4.2):

THEOREM 2.3. *Suppose X is a locally compact metric space. Then the following are equivalent:*

(i) $X\in APR$.

(ii) *X is non-compact, $X\in ANR$, and $X\in SUV^\infty$.*

(iii) *X is non-compact, $FX\in AR$, and EX is an unstable subset of FX.*

Now, every proper retract of a non-compact absolute neighborhood retract is a non-compact absolute neighborhood retract. Also, if X is a proper retract of Y and $Y\underset{p}{\leqslant} Z$, then $X\underset{p}{\leqslant} Y\underset{p}{\leqslant} Z$, and hence $X\underset{p}{\leqslant} Z$. Thus, applying Theorem 2.3 (in particular, the equivalence of (i) and (ii)) and the characterization of property SUV^∞ mentioned before, we obtain the following result:

THEOREM 2.4. *If X is a proper retract of Y and $Y\in APR$, then $X\in APR$.*

Now, $F(E^n)\approx S^n$ if $n>1$, while $F(E^1)\approx I$. Hence, by Theorem 2.3 (in particular, the equivalence of (i) and (iii)) we have the following

THEOREM 2.5. *$E^n\in APR$ if and only if $n=1$.*

3. Cartesian products. Let X and Y be two spaces. Then $X\times Y\in AR$ if and only if $X\in AR$ and $Y\in AR$ (see [2], Chapter IV, Theorem 7.1). Such is not the case for APR's and, indeed, neither of the analogous implications holds. For example, using Theorem 2.5, $E^1\in APR$, but

$$E^1\times E^1=E^2\notin APR.$$

Also, $E^2\notin APR$, but

$$E^2\times[0, \infty)\approx E^3_+\underset{p}{\simeq}[0, \infty),$$

so that $E^2\times[0, \infty)\in APR$. This latter example can be generalized as follows:

THEOREM 3.1. *Suppose X is a locally compact metric absolute retract. Then $\{p\}\times[0, \infty)$ is a strong proper deformation retract of $X\times[0, \infty)$ if $p\in X$.*

Proof. We may suppose X is non-compact, for otherwise the proof follows in the obvious manner from the fact that $\{p\}$ is a strong deformation retract of X. Let $f: X\to[0, \infty)$ be a proper map such that $f(p)=0$. (The existence of f follows easily from [11], Lemma 2.1.) Let

$$A=\{(x, t)\in X\times[0, \infty)\mid t\leqslant f(x) \text{ or } x=p\},$$

$$B=\{(x, t)\in X\times[0, \infty)\mid t\geqslant f(x)+1 \text{ or } x=p\}.$$

Then A and B are each closed in $X\times[0,\infty)$. Let us define a map $H_0\colon A\cup B\to X$ by

$$H_0(x,t)=\begin{cases} x & \text{if } (x,t)\in A,\\ p & \text{if } (x,t)\in B.\end{cases}$$

Then, since $A\cap B=\{p\}\times[0,\infty)$, H_0 is continuous. Since X is an absolute retract, there exists an extension $H\colon X\times[0,\infty)\to X$ of H_0. Now define

$$K\colon \big(X\times[0,\infty)\big)\times I\to X\times[0,\infty)$$

by

$$K\big((x,t),s\big)=\Big(H\big(x,s(f(x)+1)\big),\,t+sf(x)\Big)\quad \text{for } (x,t)\in X\times[0,\infty),\ s\in I.$$

Then K is a proper map. To verify this, it will suffice to show that $K^{-1}(C\times[0,n])$ is compact for all compact $C\subset X$ and $n>0$. Letting C be a compact subset of X and $n>0$, assume that D is a compact subset of X such that

$$C\cup f^{-1}([0,n+1])\subset D.$$

We shall show in (1) and (2) below that

$$K\big[\big(X\times[0,\infty)\big)\times I-(D\times[0,n])\times I\big]\subset\big(X\times[0,\infty)\big)-(C\times[0,n]),$$

thus proving that $K^{-1}(C\times[0,n])$ lies in the compact set $(D\times[0,n])\times I$. Let

$$(x,t)\in\big[\big(X\times[0,\infty)\big)-(D\times[0,n])\big].$$

Then $(x,t)\in X\times(n,\infty)$ or $(x,t)\in(X-D)\times[0,\infty)$.

(1) If $(x,t)\in X\times(n,\infty)$ and $s\in I$, then

$$K\big((x,t),s\big)=\Big(H\big(x,s(f(x)+1)\big),\,t+sf(x)\Big)\notin C\times[0,n],$$

since $t+sf(x)>n+sf(x)\geqslant n$.

(2) If $(x,t)\in(X-D)\times[0,\infty)$ and $s\in I$, then $f(x)>n+1$, so if $t+sf(x)\leqslant n$, then

$$s\big(f(x)+1\big)=sf(x)+s\leqslant n-t+s\leqslant n-t+1\leqslant n+1<f(x)$$

and, therefore,

$$H\big(x,s(f(x)+1)\big)=x\in X-D\subset X-C.$$

Hence, either $t+sf(x)>n$ or $H\big(x,s(f(x)+1)\big)\in X-C$. In either case

$$K\big((x,t),s\big)=\Big(H\big(x,s(f(x)+1)\big),\,t+sf(x)\Big)\notin C\times[0,n].$$

The proof is now completed by noting that if $x \in X$, $t \in [0, \infty)$, and $s \in I$, then

$$\text{(i)} \qquad K\big((x, t), 0\big) = \Big(H\big(x, 0(f(x)+1)\big), t + 0f(x)\Big) = \big(H(x, 0), t\big) = (x, t),$$

$$\text{(ii)} \qquad K\big((x, t), 1\big) = \Big(H\big(x, 1(f(x)+1)\big), t + 1f(x)\Big) = \big(H(x, f(x)+1), t+f(x)\big) = \big(p, t+f(x)\big) \in \{p\} \times [0, \infty),$$

and

$$\text{(iii)} \quad K\big((p, t), s\big) = \Big(H\big(p, s(f(p)+1)\big), t + sf(p)\Big) = \big(H(p, s), t\big) = (p, t).$$

COROLLARY 3.1. *Suppose X is a locally compact metric absolute retract. Then $X \times [0, \infty] \in APR$.*

Proof. By Theorem 3.1, $X \times [0, \infty) \underset{p}{\simeq} [0, \infty)$. Hence $X \times [0, \infty)$ is non-compact, $X \times [0, \infty) \in ANR$, and $X \times [0, \infty) \in SUV^{\infty}$, which implies, by Theorem 2.3, that $X \times [0, \infty) \in APR$.

By the above, if

$$\prod_{i=1}^{n} X_i \in APR,$$

it need not be the case that each $X_i \in APR$. In fact, it may happen that no $X_i \in APR$ (see Example 3.1 below). We can, however, obtain some information about the X_i's as given by the following result:

THEOREM 3.2. *Let $X_1, X_2, \ldots, X_n$ be locally compact metric spaces. Then $\prod_{i=1}^{n} X_i \in APR$ only if*

(1) *exactly one of the spaces $X_1, X_2, \ldots, X_n$ is non-compact, or*

(2) *one of the spaces $X_1, X_2, \ldots, X_n$ has precisely one end.*

Proof. Suppose that

$$\prod_{i=1}^{n} X_i \in APR.$$

Then $\prod_{i=1}^{n} X_i$ is non-compact, and hence so is at least one of the spaces $X_1, X_2, \ldots, X_n$. Supposing condition (2) fails to hold, we may assume (relabeling if necessary) that, for some $k \geqslant 1$, each of $X_1, X_2, \ldots, X_k$ is non-compact and has more than one end and that, for $k < n$, each of $X_{k+1}, X_{k+2}, \ldots, X_n$ is compact.

Since $\prod_{i=k+1}^{n} X_i$ is compact, $\prod_{i=1}^{k} X_i$ is a proper retract of $\prod_{i=1}^{n} X_i$. It follows from Theorem 2.4 that

$$\prod_{i=1}^{k} X_i \in APR.$$

By Lemma 2.3, if $i = 1, 2, \ldots, k$, there exists a topological line $L_i \subset X_i$ such that $(X_i, L_i) \in \Sigma'$. Since $L_i \in APR$ for $i = 1, 2, \ldots, k$, there exists a proper retraction r_i of X_i onto L_i. Then

$$\prod_{i=1}^{k} r_i\colon \prod_{i=1}^{k} X_i \to \prod_{i=1}^{k} L_i$$

is a proper retraction. But then, by Theorem 2.4, $\prod_{i=1}^{k} L_i \in APR$. However, $\prod_{i=1}^{k} L_i \approx E^k$, and hence, by Theorem 2.5, $k = 1$. Thus condition (1) holds.

We are now ready to combine our results to obtain our main theorem on products of APR's.

THEOREM 3.3. *Suppose $X_1, X_2, \ldots, X_n \in APR$, where $n > 1$. Then $\prod_{i=1}^{n} X_i \in APR$ if and only if $X_j \underset{p}{\simeq} [0, \infty)$ for some j, $1 \leqslant j \leqslant n$.*

Proof. Suppose first that $\prod_{i=1}^{n} X_i \in APR$. Then, by Theorem 3.2, there exists a j, $1 \leqslant j \leqslant n$, such that X_j has precisely one end. Now, since $X_j \in APR$, $X_j \in SUV^\infty$, and hence there exists a tree T such that $X_j \underset{p}{\simeq} T$ (see [1], Theorem 3.12, and [11], Corollary 3.5). Since the number of ends of a space is an invariant of proper homotopy type, T has precisely one end, and hence $T \underset{p}{\simeq} [0, \infty)$ by [11], Theorem 2.4. Thus $X_j \underset{p}{\simeq} [0, \infty)$.

On the other hand, suppose that $X_j \underset{p}{\simeq} [0, \infty)$ for some j, $1 \leqslant j \leqslant n$. Relabeling, if necessary, we may assume that $j = n$. Since $X_1, X_2, \ldots, X_{n-1} \in APR$, it follows that $X_1, X_2, \ldots, X_{n-1} \in AR$, and hence $\prod_{i=1}^{n-1} X_i \in AR$ (see [2], Chapter IV, Theorem 7.1). Now, using Lemma 2.1 and Theorem 3.1, we have

$$\prod_{i=1}^{n} X_i \approx \Big(\prod_{i=1}^{n-1} X_i\Big) \times X_n \underset{p}{\simeq} \Big(\prod_{i=1}^{n-1} X_i\Big) \times [0, \infty) \underset{p}{\simeq} [0, \infty).$$

It follows that $\prod_{i=1}^{n} X_i$ is a non-compact ANR having property SUV^∞, and hence, by Theorem 2.3, $\prod_{i=1}^{n} X_i \in APR$.

Example 3.1. *There exist spaces X and Y such that $X \notin APR$ and $Y \notin APR$, but $X \times Y \in APR$.*

Construction. Let B denote a polyhedral 3-cell and let A be an arc in B such that

(1) A intersects ∂B (the boundary of B) in a single point, that being an endpoint of A;

(2) A is a locally polyhedral modulo p, where p is the endpoint of A lying in $\mathrm{Int}B$ (the interior of B); and

(3) A fails to be locally tame in B at p.

Let $X = B - A$. Then X is 3-pseudo-half space [13]; i.e. X is a 3-manifold with boundary such that $\mathrm{Int}X \approx E^3$ and $\partial X \approx E^2$. Now X has precisely one end, which we denote by $[a]$, and it is easy to show that $\pi_1(X, a) \neq 0$, where π_1 is the "proper fundamental group" functor of [3]. Hence, by [11], Theorem 4.3, $X \notin SUV^\infty$ and thus, by Theorem 2.3, $X \notin APR$. Let $Y = E^2$. Then $Y \notin APR$ by Theorem 2.5. But $M = X \times Y$ is a 5-manifold with boundary and

$$\partial M = (\partial X) \times Y \approx E^4 \quad \text{while } \mathrm{Int}M = (\mathrm{Int}X) \times Y \approx E^5.$$

Thus, by [5], Theorem 1, $M \approx E^5_+ \approx E^4 \times [0, \infty)$ from which it follows, by Corollary 3.1, that $M \in APR$.

Remark. We have only considered finite products thus far. Of course, $\prod_{i=1}^{\infty} X_i$ is locally compact if and only if each X_i is locally compact and almost all are compact. Thus if, for example, $\prod_{i=1}^{\infty} X_i \in APR$, then there exists an integer n such that X_i is a compact absolute retract for $i > n$. Then

$$\prod_{i=1}^{\infty} X_i \underset{p}{\simeq} \prod_{i=1}^{n} X_i,$$

and, by previously stated results,

$$\prod_{i=1}^{n} X_i \in APR.$$

In this manner we may reduce the study of infinite products to the finite cases already considered.

4. Sums of spaces. If X_1 and X_2 are closed in $X = X_1 \cup X_2$ and $X_1 \cap X_2 = X_0$, $X_1, X_2 \in AR$, then $X \in AR$ (see [2], Chapter IV, Theorem 6.1 (i)). The corresponding statement for absolute proper retracts is not true. For example, E^2_+, E^2_-, and $E^2_+ \cap E^2_- \in APR$, but $E^2_+ \cup E^2_- = E^2 \notin APR$. However, the desired result does hold if $(X, X_0) \in \Sigma'$ (or, equivalently, by Theorem 2.2, if $(X_1, X_0) \in \Sigma'$ and $(X_2, X_0) \in \Sigma'$).

THEOREM 4.1. *Suppose that X_1 and X_2 are closed subsets of $X = X_1 \cup X_2$, that X_1, X_2, and $X_0 = X_1 \cap X_2$ are absolute proper retracts, and* *that* $(X, X_0) \in \Sigma'$ *(or, equivalently, that* $(X_1, X_0) \in \Sigma'$ *and* $(X_2, X_0) \in \Sigma'$*). Then* X *is an absolute proper retract.*

Proof. By Theorem 2.2, $(X, X_1) \in \Sigma'$ and $(X, X_2) \in \Sigma'$. Hence we may regard FX as $FX_1 \cup FX_2$, where $FX_1 \cap FX_2 = FX_0$ (cf. the proof of Lemma 2.2). By [2] (see Chapter IV, Theorem 6.1 (i)), $FX \in AR$ and our proof will be complete, by Theorem 2.3, if we can show that EX is an unstable subset of FX.

Now, since $X_0 \in APR$, EX_0 is an unstable subset of FX_0. Hence there exists a homotopy $H\colon FX_0 \times I \to FX_0$ such that $H(x, 0) = x$ for all $x \in FX_0$ and $H(x, t) \notin EX_0$ for all $x \in FX_0$ and $0 < t \leqslant 1$. Define

$$H'\colon (FX_1 \times \{0\}) \cup (FX_0 \times I) \to FX_1$$

by

$$H'(x, 0) = x \quad \text{if } (x, 0) \in FX_1 \times \{0\}$$

and by

$$H'(x, t) = H(x, t) \quad \text{if } (x, t) \in FX_0 \times I.$$

Since $X_1 \in APR$, EX_1 is an unstable subset of FX_1, and since $(FX_1 \times \{0\}) \cup (FX_0 \times I)$ is a closed subset of $FX_1 \times I$, we apply Lemma 2.1 of [11] to obtain an extension of H' to a map $H_1\colon FX_1 \times I \to FX_1$ such that $H(x, t) \notin EX_1$ for all $x \in FX_1$ and $0 < t \leqslant 1$. Similarly, there exists a map

$$H_2\colon FX_2 \times I \to FX_2$$

such that $H_2(x, 0) = x$ for all $x \in FX_2$, $H_2(x, t) \notin EX_2$ for all $x \in FX_2$ and $0 < t \leqslant 1$, and $H_2(x, t) = H(x, t)$ for all $x \in FX_0$ and $0 \leqslant t \leqslant 1$. Define

$$K\colon FX \times I \to FX$$

by

$$K(x, t) = H_1(x, t) \quad \text{if } (x, t) \in FX_1 \times I$$

and by

$$K(x, t) = H_2(x, t) \quad \text{if } (x, t) \in FX_2 \times I.$$

Noting that $EX = EX_1 \cup EX_2$, we see that K is a homotopy showing that EX is an unstable subset of FX.

A slight modification of the argument of the preceding proof also establishes the following two statements. The second of these includes the first as a special case, and its proof uses [2], Chapter V, Theorem 9.1 and Remark 9.17, rather than [2], Chapter IV, Theorem 6.1 (i).

COROLLARY 4.1. *Suppose that* X_1 *and* X_2 *are closed subsets of* $X = X_1 \cup X_2$, X_1 *and* X_2 *are absolute proper retracts, and* $X_0 = X_1 \cap X_2$ *is a compact absolute retract. Then* X *is an absolute proper retract. (The conclusion also holds if one of* X_1, X_2 *is a compact absolute retract rather than an absolute proper retract.)*

COROLLARY 4.2. *Suppose that* X_1 *and* X_2 *are absolute proper retracts such that* $X_1 \cap X_2 = \emptyset$, *that* X_0 *is a compact absolute retract such that* $X_0 \subset X_1$, *and that* $f\colon X_0 \to X_2$ *is a map. Then* $X = X_1 \underset{f}{\cup} X_2$ *is an absolute proper retract.* (*The conclusion also holds if one of* X_1, X_2 *is a compact absolute retract rather than an absolute proper retract.*)

Borsuk has also shown ([2], Chapter IV, Theorem 6.1 (iii)) that if X_1 and X_2 are closed in $X = X_1 \cup X_2$, $X \in AR$ and $X_1 \cap X_2 = X_0 \in AR$, then $X_1, X_2 \in AR$. In the sequel we establish a version of this result for absolute proper retracts.

THEOREM 4.2. *Suppose that* X_1 *and* X_2 *are closed subsets of* $X = X_1 \cup X_2$, *that* X *and* $X_0 = X_1 \cap X_2$ *are absolute proper retracts, and that* $(X, X_0) \in \Sigma'$ (*or, equivalently, that* $(X_1, X_0) \in \Sigma'$ *and* $(X_2, X_0) \in \Sigma'$). *Then* X_1 *and* X_2 *are absolute proper retracts.*

Proof. Since $(X_1, X_0) \in \Sigma'$ and $X_0 \in APR$, there exists a proper retraction $r_1\colon X_1 \to X_0$. Define $r\colon X \to X_2$ by $r(x) = x$ if $x \in X_2$ and by $r(x) = r_1(x)$ if $x \in X_1$. Then r is a proper retraction of X onto X_2 and so, by Theorem 2.4, $X_2 \in APR$. Similarly, $X_1 \in APR$.

Remark. Actually, the hypothesis $(X, X_0) \in \Sigma'$ of Theorem 4.2 can be weakened. However, the proof of this uses some work on homotopy theoretical characterizations of APR's (to appear later) which is beyond the scope of the current paper.

REFERENCES

[1] B. J. Ball and R. B. Sher, *A theory of proper shape for locally compact metric spaces*, Fundamenta Mathematicae 86 (1974), p. 163-192. (Research announcement, Bulletin of the American Mathematical Society 79 (1973), p. 1023-1026.)

[2] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1967.

[3] E. M. Brown, *Proper homotopy theory in simplicial complexes*, in Topology Conference 1973, Lecture Notes in Mathematics 375 (1974), p. 41-46.

[4] J. DeGroot and R. H. McDowell, *Locally connected spaces and their compactifications*, Illinois Journal of Mathematics 11 (1967), p. 353-364.

[5] P. H. Doyle, *Certain manifolds with boundary that are products*, Michigan Mathematical Journal 11 (1964), p. 177-181.

[6] H. Freudenthal, *Neuaufbau der Endentheorie*, Annals of Mathematics 43 (1942), p. 261-279.

[7] D. Hartley, *Quasi-cellularity in manifolds*, Thesis, University of Georgia, 1973 (condensed version, to appear).

[8] J. R. Isbell, *Uniform spaces*, Mathematical Surveys 12, Providence 1964.

[9] W. Kuperberg, *Homotopically labile points of locally compact metric spaces*, Fundamenta Mathematicae 73 (1971-1972), p. 133-136.

[10] K. Morita, *On bicompactifications of semibicompact spaces*, Science Reports of Tokyo Bunrika Daigaku, Section A.4, 92 (1952), p. 222-229.

[11] R. B. Sher, *Property SUV^{∞} and proper shape theory*, Transactions of the American Mathematical Society 190 (1974), p. 345-356.
[12] — *A theory of absolute proper retracts*, Fundamenta Mathematicae (to appear).
[13] C. D. Sikkema, *A duality between certain spheres and arcs in S^3*, Transactions of the American Mathematical Society 122 (1966), p. 399-415.
[14] E. G. Skljarenko, *Perfect bicompact extensions*, Soviet Mathematics Doklady 3 (1962), p. 1455-1458.

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