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ON ABSOLUTE FIXED-POINT SETS

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1. Introduction. A subset A of a topological space X is said to be a *fixed-point set* of X if there is a map (continuous function) $f\colon X \to X$ such that $f(x) = x$ iff $x \in A$. Clearly, a fixed-point set of a Hausdorff space X must be a closed subset of X. In [8] Ward, Jr., defines a space X to have the *complete invariance property* if each of its non-empty closed subsets is a fixed-point set of X.

The problem as to when a topological space X has the complete invariance property has been investigated by Robbins [4], Schirmer [5]-[7], and Ward, Jr. [8]. In particular, spaces known to have the complete invariance property include n-cells [4], dendrites [5], compact manifolds without boundary [6], arcwise connected subspaces of locally smooth dendroids [8], a class of Peano continua containing the local dendrites and the continua containing no continuum of condensation [8], and compact triangulable manifolds with or without boundary [7]. However, the general question as to what properties a space must possess in order to have the complete invariance property is far from being resolved. Indeed, in [8] Ward, Jr., asks if every Peano continuum has this property.

The purpose of this paper* is to pose and study the following related question:

If Q denotes a class of topological spaces, then what properties must a Q-space have to insure that, whenever it is embedded as a closed subset of an arbitrary Q-space Z, it is a fixed-point set of Z?

We shall call a Q-space which satisfies this condition an *absolute fixed-point set relative to* Q (abbreviated AFPS(Q)). It is shown that, for many classes Q, the class of AFPS(Q)-spaces contains the class of absolute retracts relative to Q and lies in the class of connected, locally connected Q-spaces. In the case where Q denotes the class of compacta in the plane,

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it is shown that every locally connected continuum in Q is an AFPS(Q)-space. However, for the case where Q denotes the class of compact Hausdorff spaces, more restrictive conditions would be required. Indeed, we give an example of an ordered continuum which is not an AFPS(Q)-space.

2. Absolute fixed-point sets. All spaces will be Hausdorff and, unless stated otherwise, Q shall denote one of the following classes of spaces: compact metric, separable metric, metric, compact, Lindelöf, paracompact, collectionwise normal, perfectly normal, normal. Definitions for these spaces may be found in [2] and we shall adopt the notation used in [2]. In particular, we shall let AR(Q) denote the class of absolute retracts relative to the class Q.

Definition. A space X is an *absolute fixed-point set relative to the class* Q (abbreviated AFPS(Q)) if $X \in Q$ and whenever X is embedded as a closed subset of a Q-space Z, then X is a fixed-point set of Z.

From this definition we obtain the following result:

THEOREM 1. *Every* AR(Q)-*space is an* AFPS(Q)-*space.*

Proof. Let $X \in \mathrm{AR}(Q)$. Suppose X is embedded as a closed subset of a Q-space Z. Then there is a retraction $r\colon Z \to X$, and hence X is a fixed-point set of Z. Thus $X \in \mathrm{AFPS}(Q)$.

Example 1. Consider the plane continuum $X = A \cup B \cup C$, where A denotes the closed interval $[-1, 0]$ on the x-axis, B – the closed interval $[-1, 1]$ on the y-axis, and C – the set whose equation is $y = \sin(\pi/x)$, $0 < x \leqslant 1$. Then $S = \{(-1, 0), (1, 0)\}$ is not a fixed-point set of X. For suppose $f\colon X \to X$ is a map whose fixed-point set is S. Then $f(X)$ must be a continuum containing S. Since arc components must be preserved under f, it follows that $f(C) = C$. Hence $f(B) = B$. But since B has the fixed-point property, S is not the fixed-point set of f.

THEOREM 2. *Every* AFPS(Q)-*space is connected.*

Proof. Suppose Y is an AFPS(Q)-space which is not connected. Then Y is a Q-space which has at least two components, say H and K. Let X denote the space defined in Example 1. Let Z denote the Q-space obtained by taking the disjoint union of Y and X, and then identifying the point $(-1, 0)$ in X with a point p in H, and identifying the point $(1, 0)$ in X with a point q in K. Now Y is a closed subspace of Z, and thus there is a map $f\colon Z \to Z$ such that $f(z) = z$ iff $z \in Y$. Let T denote the component in Z containing the sets H, X and K. Since $p = (-1, 0)$ and $q = (1, 0)$ must remain fixed under f, it follows that $f(T) \subset T$. Let $r\colon T \to X$ denote the retraction defined by

$$r(z) = \begin{cases} p & \text{if } z \in H, \\ q & \text{if } z \in K, \\ z & \text{if } z \in X. \end{cases}$$

Let $g = f|X$ be the restriction of f to X. Then rg: $X \to X$ is a mapping from X into itself whose fixed-point set is $\{(-1, 0), (1, 0)\}$. This contradiction shows that Y must be connected as required.

THEOREM 3. *Every first countable* AFPS(Q)-*space is locally connected.*

Proof. Suppose X is a first countable AFPS(Q)-space which is not locally connected at p. Then there exists a neighborhood V of p in X and points $p_1, p_2, \ldots$ such that $\lim p_i = p$ and each of the points $p, p_1, p_2, \ldots$ lies in a different component of V. To complete the proof of the theorem it suffices to construct a Q-space Z containing X as a closed subset such that X is not a fixed-point set of Z.

Let q_i denote the point $(1/i, 0)$ in the plane R^2 and let 0 denote the origin in R^2. For each integer $i = 1, 2, \ldots$, we construct a plane continuum A_i, homeomorphic to the continuum in Example 1, joining q_i to q_{i+1}. Let a denote the midpoint of the closed interval $[q_{i+1}, q_i]$. Then A_i is the union of a triod consisting of the horizontal interval $[q_{i+1}, a]$ and the closed vertical interval with midpoint a and length $2/i$, and the curve in the plane whose equation is given by

$$y = \frac{1}{i} \sin \frac{\pi(1-ai)}{i(x-a)} \qquad \text{for } a < x \leqslant \frac{1}{i}.$$

Let Y be the plane continuum defined by

$$Y = \bigcup_{i=1}^{\infty} A_i \cup \{0\}.$$

Define Z to be the space obtained by taking the disjoint union of X and Y, and then identifying p_i with q_i for $i = 1, 2, \ldots$, and p with 0. Then Z is a Q-space and X is a closed subset of Z.

Suppose f: $Z \to Z$ is a map whose fixed-point set is X. Let U be a neighborhood of p in Z such that $U \cap X \subset V$. Since f is continuous at p, there is a neighborhood W of p in Z such that $W \subset U$ and $f(W) \subset U$. Now W contains infinitely many sets of the form A_i. Thus, for some j, $f(A_j)$ is a continuum in U containing p_j and p_{j+1}. Moreover, no subcontinuum of $f(A_j)$ containing p_j and p_{j+1} can lie in V, for otherwise p_j and p_{j+1} would belong to the same component in V. Thus $f(A_j) \supset A_j$ and, as in Example 1, it is easy to show that some point of A_j other than p_j or p_{j+1} must remain fixed under f. This contradiction shows that X must be locally connected as required.

The preceding theorems show that, for many classes Q, the class of AFPS(Q)-spaces contains the class of AR(Q)-spaces and lies in the class of connected, locally connected Q-spaces. The following theorem shows that if Q denotes the class of compacta in the plane, then every connected, locally connected Q-space is an AFPS(Q)-space.

THEOREM 4. *If Q is the class of planar compacta, then every Peano continuum in Q is an* AFPS(Q)*-space.*

Proof. Suppose X is a Peano continuum which is embedded in a compact subset Z of the plane R^2. Let $\{A_n\}$, $n = 0, 1, 2, \ldots$, be the set of components in $R^2 - X$, with A_0 denoting the unbounded component. Let $B_n = Z \cap A_n$ for $n = 0, 1, 2, \ldots$ If we add to X all the bounded components of the set $R^2 - X$, we obtain a locally connected continuum $X_0 = R^2 - A_0$ which does not separate the plane R^2. Then X_0 is an AR--space and, by Corollary (13.5) in [1], p. 137, there exists a retraction f_0 of the closure $\mathrm{Cl}\, A_0$ of the set A_0 to its boundary $\mathrm{Bd}\, A_0$.

Now consider a bounded component A_i of the set $R^2 - X$. Then either (1) $B_i \neq A_i$ or (2) $B_i = A_i$. For $B_i \neq A_i$, let a_i be a point in $A_i - B_i$. Now if we add to X all the bounded components of $R^2 - X$ different from A_i, we obtain a locally connected continuum X_i whose complement in R^2 consists of the two components A_0 and A_i. Then X_i is an ANR-space and there exists a retraction f_i (see [1], p. 139) of the set $(\mathrm{Cl} A_i) - \{a_i\}$ to $\mathrm{Bd} A_i$.

Suppose $B_i = A_i$. We may assume that the metric d on R^2 is such that $d(x, y) \leqslant 1$ for all x, y in B_i. Let p be a point in $\mathrm{Bd} B$ such that $C = (\mathrm{Int} B_i) \cup \{p\}$ is homeomorphic to the convex subset of R^2 defined by $\{(x, y) \mid x^2 + y^2 < 1\} \cup \{(1, 0)\}$. It then follows that there is a homotopy $H\colon C \times I \to C$ such that $H(x, 0) = x$ for all x in C, and if $x \neq p$ and $t > 0$, then $H(x, t) \neq x$ ([8], p. 554). Define $f_i\colon \mathrm{Cl} B_i \to \mathrm{Cl} B_i$ by

$$f_i(x) = \begin{cases} H\big(x, d(x, \mathrm{Bd} B_i)\big) & \text{if } x \in C, \\ x & \text{if } x \in \mathrm{Bd} B_i. \end{cases}$$

Then f_i is a map whose fixed-point set is precisely $\mathrm{Bd} B_i$.

Now set

$$f(z) = \begin{cases} f_i(z) & \text{if } z \in \mathrm{Cl} B_i,\ i = 0, 1, 2, \ldots, \\ z & \text{if } z \in X. \end{cases}$$

Then, clearly, $f\colon Z \to Z$ is a mapping from Z into itself whose fixed--point set is X. Therefore, X is an AFPS(Q)-space as required.

We note that the proof of Theorem 4 actually shows that every planar Peano continuum is an AFPS(Q)-space, where Q denotes the class of planar spaces (compact or not).

Example 2. Let Q denote the class of compact Hausdorff spaces, and let Y be the one-point compactification of the long line. Then Y is an ordered continuum (i.e., a tree without branch points) with two endpoints, say a and b ([3], p. 56). However, Y is not an AFPS(Q)-space. For suppose otherwise. Let X denote the space defined in Example 1. Let Z be the Q-space obtained by taking the disjoint union of Y and X, and then identifying the point $(-1, 0)$ in X with a, and identifying the point $(1, 0)$ in X with b. Since Y is a closed subset of Z, there is a map $f\colon Z \to Z$ such that $f(z) = z$ iff $z \in Y$. Now $f(X)$ is a continuum containing a and b. Moreover, we cannot have $f(X) \supset X$, for then some point in X different from a and b would remain fixed under f. Since every point in $Y - \{a, b\}$ is a cut point, it follows that $f(X) \supset Y$. But X is separable, and hence $f(X)$ is separable, while Y is not separable. This contradiction shows that Y is not an AFPS(Q)-space.

The preceding example shows that, for the class Q of compact Hausdorff spaces, only a very restricted subclass of the class of connected, locally connected Q-spaces will be AFPS(Q)-spaces.

Our initial study leaves the following general problem to be considered:

Problem. Characterize the AFPS(Q)-spaces for the classes Q listed at the beginning of Section 2. (**P 971**)

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