On the growth of entire functions of $n$ complex variables

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Abstract. For entire functions of $n$ complex variables with gap series of homogeneous polynomials the rate of growth in the whole space is determined by the rate of growth on a real cone $K$ such that $C\bar{K}$ is non-pluripolar.

1. The main result. Let $f$ be an entire function of $n$ complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{v=1}^{\infty} f_v(z), \quad z \in C^n \quad (\deg f_v = v).$$

Suppose that $f_v = 0$ for $v \neq \lambda_k$ with $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$. Let $H : [0, \infty) \to (0, \infty)$ be a function with continuous, positive and increasing first derivative. Let $K$ denote a real cone in $C^n$ such that $C\bar{K} = \{z : z \in C, z \in \bar{K}\}$ is non-pluripolar. We write $\theta = \max \{r : \{||z|| \leq r\} \subset \bar{K}\}$, $\bar{K}$ being the convex hull of $\{||z|| = 1\} \cap C\bar{K}$ with respect to the family of all absolutely homogeneous plurisubharmonic functions on $C^n$. Since $C\bar{K}$ is non-pluripolar, $\theta$ is a positive number ([4], Corollary 11.2).

With these denotations we have

1.1. Theorem. (i) If $|f(z)| \leq H(||z||)$, $z \in K$ and $H(r)r^{-\gamma} \to \infty$, $r \to \infty$ for every positive integer $\gamma$, then for each $\sigma > 1$ there is a constant $A$ such that

$$M(r) = \sup \{||f(z)|| : ||z|| \leq r\} \leq AH\left(\frac{\sigma}{\theta}r\right), \quad r \in [0, \infty).$$

(ii) If $|f(z)| \leq H(||z||)$, $z \in K$ and $H(r)r^{-\mu} \to 0$, $r \to \infty$ for some positive integer $\mu$, then $f$ is a polynomial.

Let us note that for $n = 1$ Theorem 1.1 was proved by Anderson and Binmore ([1], Theorem 1, Theorem A).
2. Remarks and auxiliary theorems.

2.1. Remark. For a suitable $r_0 > 0$ a function

$$[r_0, \infty) \ni t \mapsto H^2(t) t^{-1} \in (0, \infty)$$

is increasing and so

$$\int_{r_0}^{r} H^2(t) t^{-1} \, dt \leq H^2(r), \quad r_0 \leq r < \infty.$$

2.2. Remark. $\int_{0}^{r} |f(tz)|^2 t^{-1} \, dt \leq M(r)^2$, $r > 0$, $\|z\| \leq 1$. Indeed, let $r > 0$ and $z \in \mathbb{C}^n$, $\|z\| \leq 1$. Since $f(0) = 0$ a function

$$\psi_z : C \ni \zeta \mapsto \zeta^{-1} f(\zeta z) \in \mathbb{C}$$

is analytic and $|\psi_z(\zeta)| \leq r^{-1} M(\zeta)$ for $|\zeta| = r$. By the maximum principle this inequality holds for $|\zeta| \leq r$. Hence

$$|f(tz)|^2 t^{-1} \leq M(r)^2 r^{-1}, \quad t \in [0, r],$$

from where the required inequality follows.

To prove Theorem 1.1 we need the following results due to Anderson, Binmore and Gaier.

2.3. Theorem ([2], Theorem 1). Let $\varphi : C \ni \zeta \mapsto \varphi(\zeta) = \sum_{v=1}^{\infty} a_v \zeta^v \in \mathbb{C}$ be an entire function with $a_v = 0$ for $v \neq \lambda_k$, where $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$. Then for every $r > 0$

$$|a_v r^v| \leq \sqrt{2v} \Pi(v) \left[ \int_{0}^{r} |\varphi(t)|^2 t^{-1} \, dt \right]^{1/2}, \quad v = 1, 2, \ldots,$$

where

$$\Pi(v) = \begin{cases} \prod_{j \neq k} \frac{\lambda_k + \lambda_j}{|\lambda_k - \lambda_j|} & \text{if } v = \lambda_k (k = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

2.4. Theorem ([3], Theorem 1). If $\{\lambda_k\}_{k \geq 1}$ is an increasing sequence of positive integers and $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$, then $\log \Pi(\lambda_k) = o(\lambda_k)$ ($k \to \infty$).

3. Proof of Theorem 1.1. Given $z \in \mathbb{C}^n$, we write $\varphi_z(\zeta) = f(\zeta z) = \sum_{v=1}^{\infty} f_v(z) \zeta^v$. Then the function $\varphi_z$ satisfies the hypothesis of Theorem 2.3.
and so

\[ 3(1) \quad |f_v(z)|r^v \leq \sqrt{2v} \Pi(v) \left[ \int_0^r |f(tv)|^2 t^{-1} dt \right]^{1/2}, \quad v \geq 1, \ r \geq 0. \]

Put \( K_1 = K \cap \{ \|z\| = 1 \} \) and take a number \( r_0 \) according to Remark 2.1. We claim that for every \( \alpha > 1 \) there exists a positive integer \( N \) such that

\[ 3(2) \quad |f_v(z)|r^v \leq \alpha^v H(r), \quad z \in K_1, \ r \geq r_0, \ v \geq N. \]

Indeed, by hypothesis we have

\[ \int_0^r |f(tv)|^2 t^{-1} dt \leq \int_0^{r_0} |f(tv)|^2 t^{-1} dt + \int_{r_0}^r H^2(t) t^{-1} dt, \quad z \in K_1, \ r \geq r_0. \]

So, by Remarks 2.2 and 2.1,

\[ \int_0^r |f(tv)|^2 t^{-1} dt \leq M(r_0)^2 + H^2(r), \quad z \in K_1, \ r \geq r_0. \]

Using Theorem 2.4, we can find a positive integer \( N \) such that

\[ \sqrt{2v} \Pi(v) \leq \alpha^{v/2}, \ v \geq N \]

and since \( H \) is increasing, the integer \( N \) can be enlarged so that

\[ [M(r_0)^2 + H^2(r)]^{1/2} \leq \alpha^{v/2} H(r), \quad v \geq N, \ r \geq r_0. \]

Hence by 3(1) \( \) the required inequality 3(2) holds.

Now, given \( \sigma > 1 \) take any \( \alpha \in (1, \sigma) \) and choose \( N \) for which 3(2) is satisfied. Let \( r \geq r_1 = \max \{ r_0, 1 \} \). Then for \( z \in K_1 \) and \( |z| \leq r \)

\[ |f(z)| \leq \sum_{v=1}^{N-1} |f_v(z)|r^v + \sum_{v=N}^{\infty} |f_v(z)|r^v \leq Br^N + \sum_{v=N}^{\infty} (|f_v(z)|r^v\sigma^v)\sigma^{-v} \]

\[ \leq Br^N + H(r\sigma) \sum_{v=N}^{\infty} \alpha^v \sigma^{-v} \leq C (r^N + H(r\sigma)), \]

where \( B, C \) are appropriate constants.

Finally,

\[ |f(z)| \leq C (r^N + H(r\sigma)), \quad z \in C \bar{K} \cap \{ \|z\| \leq r \}. \]

Since \( C \bar{K} \) is non-pluripolar, by the Sibony–Wong inequality ([4], Corollary 11.2)

\[ \sup \{ |f(z)|: \|z\| \leq r \} \leq \sup \{ |f(z)|: \|z\| \leq r/\theta, \ z \in C \bar{K} \}. \]

Hence

\[ M(r) \leq C [(r/\theta)^N + H(\sigma r/\theta)], \quad r \geq r_1. \]

Now, if \( H(r)r^{-N} \to \infty, \ r \to \infty \), we can find a number \( r_2 \geq r_1 \) and a
constant $A$ so that

$$M(r) \leq AH(\sigma r/\theta), \quad r \geq r_2.$$  

Taking $A$ greater than $M(r_2)/H(0)$, we obtain (i) of Theorem 1.1.

On the other hand, if $H(r)r^{-\mu} \to 0$, $r \to \infty$ for some positive integer $\mu$, then for $N \geq \mu$ we have

$$M(r) \leq Dr^N, \quad r \geq r_1$$  

with a suitable constant $D$. Hence $f$ is a polynomial.

References


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