COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

A CENTRAL LIMIT THEOREM FOR SUMS OF A RANDOM NUMBER OF INDEPENDENT RANDOM VARIABLES

BY

ZDZISŁAW RYCHLIK (LUBLIN)

1. Preliminary considerations. Following the classical work of Robbins [12], many authors (see, e.g., [1], [11], [2], [16], [9], [6], [14], [4]) have investigated the limit behaviour of the distribution of sums with random indices.

In the present paper we establish some theorems concerning the limit behaviour of sums of a random number of independent random variables. Introducing in Section 2 a so-called "random Lindeberg condition" we shall prove the random central limit theorem (Theorem 1) which is an extension of Lindeberg's result [7], and obtain some generalizations or extensions of results from [12], [14], [8] and [13] (Theorems 2 and 3). The proofs take advantage of the operator method introduced by Trotter [15].

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with mean value $\mathrm{E}X_k = a_k$ and finite variance $\sigma^2 X_k = \sigma_k^2$, and let

$$(1)\qquad S_n = \sum_{k=1}^{n} X_k, \quad A_n = \sum_{k=1}^{n} a_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2.$$

By N we denote a positive integer-valued random variable which has the distribution function dependent on a parameter λ ($\lambda > 0$), i.e.,

$$\mathrm{P}[N = n] = p_n, \quad \sum_{n=1}^{\infty} p_n = 1, \quad \text{where } p_n = p_n(\lambda).$$

We assume that random variables $N, X_1, X_2, \ldots$ are independent.

Let C_3 be the set of all uniformly continuous and bounded real-valued functions defined on the real number axis. These functions are three times differentiable while their first three derivatives are also uniformly continuous and bounded on the whole number axis. C_3 with the oridinary operations on functions and with the norm of f defined by the formula

$$\|f\| = \sup_x |f(x)|$$

is a normed linear space.

Let $F(y)$ be an arbitrary distribution function. A linear operator A_F defined by

$$A_F(f) = \int_{-\infty}^{\infty} f(x+y)\,dF(y)$$

is called the *operator associated with the distribution function F*. Operators associated with the probability distributions are commutative and are contractions ([10], p. 516).

Now, we put

$$Z_N = \frac{S_N - L}{\sqrt{M}}, \tag{2}$$

where $S_N = \sum_{k=1}^{N} X_k$, $L = \sum_{k=1}^{N} a_k$ and $M = \sum_{k=1}^{N} \sigma_k^2$.

The distribution function F_λ of the random variable Z_N is given by the formula

$$F_\lambda(x) = \sum_{n=1}^{\infty} p_n F_1 * F_2 * \ldots * F_n(s_n x), \tag{3}$$

where F_k is the distribution function of the random variable $X_k - a_k$, and $*$ denotes the operation of convolution. Hence, taking into account the Trotter rule [15] concerning the operator associated with the convolution of distribution functions for the operator A_{F_λ} associated with the distribution function $F_\lambda(x)$, we have the equality

$$A_{F_\lambda}(f) = \sum_{n=1}^{\infty} p_n A_{F_1}^{(n)} A_{F_2}^{(n)} \ldots A_{F_n}^{(n)}(f), \tag{4}$$

where $A_{F_k}^{(n)}$ denotes the operator associated with the distribution function $F_k(s_n x)$.

2. A random version of Lindeberg condition.

Definition. A sequence $\{X_n, n \geqslant 1\}$ of independent random variables is said to satisfy the *random Lindeberg condition* if, for every $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N} \int_{|x|\geqslant \varepsilon\sqrt{M}} x^2\,dF_k(x)\right\} = 0, \tag{5}$$

where F_k is the distribution function of the random variable $X_k - a_k$, while M is the random variable as in (2).

It is easy to see that in the special case where the parameter λ is a positive integer ($\lambda = n$) and, for every n, the random variable N takes the value n with probability one, condition (5) reduces to the classical Lindeberg condition, i.e., for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|\geqslant \varepsilon s_n} x^2 dF_k(x) = 0. \tag{6}$$

LEMMA 1. *If a sequence $\{X_n,\ n \geqslant 1\}$ of independent random variables satisfies* (6) *and if $N \xrightarrow{P} \infty$ as $\lambda \to \infty$* (P — *in probability*), *then* (5) *holds.*

Proof. By (6), for every $\delta > 0$ there exists a positive integer n_0 such that, for $n \geqslant n_0$,

$$\frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|\geqslant \varepsilon s_n} x^2 dF_k(x) < \frac{\delta}{2}.$$

Thus, for any given $n_1 \geqslant n_0$, we have

$$\mathrm{E}\left\{\frac{1}{M} \sum_{k=1}^{N} \int_{|x|\geqslant \varepsilon\sqrt{M}} x^2 dF_k(x)\right\} \leqslant \mathrm{P}[N \leqslant n_1] + \frac{\delta}{2}.$$

Since $N \xrightarrow{P} \infty$ as $\lambda \to \infty$, we can choose λ_0 such that $\mathrm{P}[N \leqslant n_1] < \delta/2$ for every $\lambda > \lambda_0$, and since $\delta > 0$ can be chosen arbitrarily small, Lemma 1 is proved.

LEMMA 2. *If* (5) *holds, then*

$$\lim_{\lambda\to\infty} \mathrm{E}\left\{\frac{1}{M} \sum_{k=1}^{N} \int_{|x|\geqslant \varepsilon\sqrt{M}} x^2 d\Phi\left(\frac{x}{\sigma_k}\right)\right\} = 0,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du.$$

Proof. First we show that

$$\lim_{\lambda\to\infty} \mathrm{E}\left(\max_{1\leqslant k\leqslant N} \frac{\sigma_k^2}{M}\right) = 0. \tag{7}$$

Indeed, it suffices to note that, for any given $\varepsilon > 0$, we have

$$\mathrm{E}\left\{\frac{1}{M} \sum_{k=1}^{N} \int_{|x|\geqslant \varepsilon\sqrt{M}} x^2 dF_k(x)\right\} \geqslant \mathrm{E}\left(\max_{1\leqslant k\leqslant N} \frac{\sigma_k^2}{M}\right) - \varepsilon^2$$

and to apply (5).

Now let $\delta > 0$ be given. Putting

$$A = \{n\colon \max_{1\leqslant k\leqslant n} \sigma_k^2 \geqslant \delta s_n^2\} \quad \text{and} \quad B_n^2 = s_n^2/(\max_{1\leqslant k\leqslant n} \sigma_k^2),$$

we get

$$\mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N} \int\limits_{|x|\geqslant \varepsilon\sqrt{M}} x^2 d\Phi\left(\frac{x}{\sigma_k}\right)\right\} \leqslant \sum_{n\in A} p_n + \sum_{n\in A^c} p_n \int\limits_{|x|\geqslant \varepsilon B_n} x^2 d\Phi(x),$$

where A^c denotes the set complementary to A.

Now, let η be any positive number. We can choose $\delta > 0$ such that

$$\sum_{n\in A^c} p_n \int\limits_{|x|\geqslant \varepsilon B_n} x^2 d\Phi(x) \leqslant \int\limits_{|x|\geqslant \varepsilon/\sqrt{\delta}} x^2 d\Phi(x) < \frac{\eta}{2}.$$

Furthermore, by (7), there exists λ_0 such that $\lambda > \lambda_0$ implies

$$\sum_{n\in A} p_n = \mathrm{P}[\max_{1\leqslant k\leqslant N} \sigma_k^2 \geqslant \delta M] < \frac{\eta}{2}.$$

Since $\eta > 0$ can be chosen arbitrarily small, the last two inequalities prove our assertion.

THEOREM 1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables, and N a positive integer-valued random variable independent of each X_n, $n = 1, 2, \ldots$ If condition* (5) *holds, then the random variable $Z_N = (S_N - L)/\sqrt{M}$ is asymptotically normal* (0, 1).

Proof. Let A_{F_λ} be the operator associated with the distribution function $F_\lambda(x)$ of the random variable Z_N. From (4) we have

$$A_{F_\lambda}(f) = \sum_{n=1}^{\infty} p_n A_{F_1}^{(n)} A_{F_2}^{(n)} \ldots A_{F_n}^{(n)}(f), \tag{8}$$

where $A_{F_k}^{(n)}$ denotes the operator associated with the distribution function $F_k(s_n x)$.

Now let $\{Y_n,\ n \geqslant 1\}$ be a sequence of independent normally distributed random variables with expectation 0 and variance σ_n^2. If $\Phi(x)$ is the normal distribution function with mean 0 and variance 1, then $\Phi(x/\sigma_n)$ is one of the random variables Y_n, $n = 1, 2, \ldots$ Furthermore, putting

$$V_N = \frac{Y_1 + Y_2 + \ldots + Y_N}{\sqrt{M}}, \tag{9}$$

we get $\mathrm{P}[V_N < x] = \Phi(x)$. Thus, for all λ, the random variable V_N is normally distributed with expectation 0 and variance 1. Hence, the operator A_Φ associated with the distribution function $\Phi(x)$ can be written in the form

$$A_{\Phi}(f) = \sum_{n=1}^{\infty} p_n A_1^{(n)} A_2^{(n)} \ldots A_n^{(n)}(f), \tag{10}$$

where $A_k^{(n)}$ is the operator associated with $\Phi(s_n x/\sigma_k)$.

From (8), (10) and Lemma 4 of [10], p. 517, it follows that, for every $f \epsilon C_3$,

$$\|A_{F_\lambda}(f) - A_{\Phi}(f)\| \leqslant \sum_{n=1}^{\infty} p_n \sum_{k=1}^{n} \|A_{F_k}^{(n)}(f) - A_k^{(n)}(f)\|. \tag{11}$$

Since $f \epsilon C_3$, $f(x+y)$ can be expanded into a finite Taylor series up to the second and third term, that is,

$$f(x+y) = f(x) + yf'(x) + \frac{1}{2} y^2 f''(x+\theta_1 y) \tag{12}$$

and

$$f(x+y) = f(x) + yf'(y) + \frac{1}{2} y^2 f''(x) + \frac{1}{6} y^3 f'''(x+\theta_2 y), \tag{13}$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ depend on x and y.

Now let $\varepsilon > 0$ be given. It is obvious that

$$A_{F_k}^{(n)}(f) = \int_{-\varepsilon}^{\varepsilon} f(x+y)\, dF_k(s_n y) + \int_{|y|>\varepsilon} f(x+y)\, dF_k(s_n y). \tag{14}$$

Using in the first integral on the right-hand side of (14), equality (13) and in the second integral of equality (12), we obtain

$$A_{F_k}^{(n)}(f) = f(x) + \frac{\sigma_k^2}{2s_n^2} f''(x) + \frac{1}{2} \int_{|y|>\varepsilon} y^2 \{f''(x+\theta_1 y) - f''(x)\}\, dF_k(s_n y) + \\ + \frac{1}{6} \int_{-\varepsilon}^{\varepsilon} y^3 f'''(x+\theta_2 y)\, dF_k(s_n y).$$

Hence, putting

$$M_1 = \sup_x |f''(x)| \quad \text{and} \quad M_2 = \sup_x |f'''(x)|,$$

we have

$$\left| A_{F_k}^{(n)}(f) - f(x) - \frac{\sigma_k^2}{2s_n^2} f''(x) \right| \leqslant \frac{M_1}{s_n^2} \int_{|y|\geqslant \varepsilon s_n} y^2\, dF_k(y) + \frac{\varepsilon M_2 \sigma_k^2}{6s_n^2}. \tag{15}$$

Analogously one can prove the following inequality:

$$\left| A_k^{(n)}(f) - f(x) - \frac{\sigma_k^2}{2s_n^2} f''(x) \right| \leqslant \frac{M_1}{s_n^2} \int_{|y|\geqslant \varepsilon s_n} y^2\, d\Phi\left(\frac{y}{\sigma_k}\right) + \frac{\varepsilon M_2 \sigma_k^2}{6s_n^2}. \tag{16}$$

Now, putting (15) and (16) into (11), we get

$$(17)\qquad \|A_{F_\lambda}(f)-A_\Phi(f)\| \leqslant M_1 \mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N}\int\limits_{|y|\geqslant\varepsilon\sqrt{M}} y^2 dF_k(y)\right\}+$$

$$+M_1\mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N}\int\limits_{|y|\geqslant\varepsilon\sqrt{M}} y^2 d\Phi\left(\frac{y}{\sigma_k}\right)\right\}+\frac{\varepsilon M_2}{3}.$$

In view of our assumption and Lemma 2 we have

$$\lim_{\lambda\to\infty}\|A_{F_\lambda}(f)-A_\Phi(f)\| = 0.$$

Thus we have proved that if $f\epsilon C_3$, then, for any value of x (and even uniformly in x),

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x+y)\,dF_\lambda(y) = \int_{-\infty}^{\infty} f(x+y)\,d\Phi(y).$$

The last equality and Criterion 1 of [5], p. 251, prove the assertion of Theorem 1.

From Theorem 1 and Lemma 1 one can deduce the following extension of Theorem 1 (cf. [10], p. 472):

COROLLARY 1. *If a sequence* $\{X_n,\ n\geqslant 1\}$ *of independent random variables satisfies* (6) *and if* $N\xrightarrow{\mathrm{P}}\infty$, *then the random variable* $(S_N-L)/\sqrt{M}$ *is asymptotically normal* $(0,1)$.

THEOREM 2. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of independent random variables, and* N *a positive integer-valued random variable independent of each* X_n, $n=1,2,\ldots$ *If* (5) *is satisfied and*

$$(18)\qquad \frac{M-\mathrm{E}M}{\sigma^2}\xrightarrow{\mathrm{P}} 0 \quad \textit{as } \lambda\to\infty,$$

then the random variable $(S_N-L)/\sigma$ *is asymptotically normal* $(0,\sqrt{1-d^2})$, *where* $d=\Delta/\sigma$.

Proof. We have

$$\frac{S_N-L}{\sigma}=\frac{S_N-L}{\sqrt{M}}\sqrt{\frac{M}{\sigma^2}}.$$

On the other hand, in view of Theorem 1, the random variable $(S_N-L)\sqrt{\mathrm{E}M/M\sigma^2}$ is asymptotically normal $(0,\sqrt{1-d^2})$, where $d=\Delta/\sigma$. Hence, taking into account Lemma 2 of [5], p. 247, it suffices to show that

$$\left(\frac{S_N-L}{\sqrt{M}}\right)\left(\frac{\sqrt{M}-\sqrt{\mathrm{E}M}}{\sigma}\right)\xrightarrow{\mathrm{P}} 0 \quad \text{as } \lambda\to\infty. \tag{19}$$

Since the random variable $(S_N-L)/\sqrt{M}$ is asymptotically normal $(0,1)$ and, for every $\varepsilon>0$,

$$\mathrm{P}[|\sqrt{M}-\sqrt{\mathrm{E}M}|\geqslant\varepsilon\sigma]\leqslant\mathrm{P}[|M-\mathrm{E}M|\geqslant\varepsilon^2\sigma^2],$$

condition (19) follows by Lemma 2 of [11], which implies the statement of Theorem 2.

3. Generalization of the central limit theorem of H. Robbins. Theorems and corollaries of this section are generalizations and extensions of the results given in [12], [14], [8] and [13].

Observing that if $A=\mathrm{E}L$, and $\Delta^2=\sigma^2L$, then $\mathrm{E}S_N=A$ and $\sigma^2=\sigma^2S_N=\mathrm{E}M+\Delta^2$, we have the following

THEOREM 3. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables, and N a positive integer-valued random variable independent of each X_n, $n=1,2,\ldots$ If* (5) *and* (18) *hold, then the random variable $(S_N-A)/\sigma$ has the limiting distribution function $H_\lambda(x/d)*\Phi(x/\sqrt{1-d^2})$, where $d=\Delta/\sigma$ and $H_\lambda(x)=\mathrm{P}[L-A<x\Delta]$.*

Proof. Let us consider the decomposition

$$\frac{S_N-A}{\sigma}=W_N+U_N, \tag{20}$$

where

$$W_N=\left(\frac{S_N-L}{\sqrt{M}}\right)\left(\frac{\sqrt{M}-\sqrt{\mathrm{E}M}}{\sigma}\right)\quad\text{and}\quad U_N=\frac{S_N-L}{\sqrt{M}}\sqrt{1-d^2}+\frac{L-A}{\sigma}.$$

It follows from (19) that $W_N\xrightarrow{\mathrm{P}}0$ as $\lambda\to\infty$. Thus, taking into account the Lemma of Cramér (see [3], p. 252), it suffices to prove that U_N has the limiting distribution function $H_\lambda(x/d)*\Phi(x/\sqrt{1-d^2})$. Therefore, the proof will be completed if we show (cf. Criterion 1 of [5], p. 251) that, for every element f of C_3,

$$\lim_{\lambda\to\infty}\|A_N(f)-A_\lambda(f)\|=0,$$

where A_N is the operator associated with the distribution function U_N, and A_λ is the operator associated with the distribution function $H_\lambda(x/d)*\Phi(x/\sqrt{1-d^2})$.

Since the random variables $N, X_1, X_2, \ldots$ are independent, we have

$$\mathrm{P}[U_N<x]=\sum_{n=1}^{\infty}p_nF_1*F_2*\ldots*F_n\left(\frac{xs_n}{\sqrt{1-d^2}}-\frac{s_n(A_n-A)}{\sigma\sqrt{1-d^2}}\right),$$

where, as in Section 2, $F_k(x)$ is the distribution function of the random variable $X_k - a_k$. Hence, by Trotter's rule [15], we obtain

$$A_N(f) = \sum_{n=1}^{\infty} p_n A_{F_1}^{(n)} A_{F_2}^{(n)} \ldots A_{F_n}^{(n)}(f), \tag{21}$$

where $A_{F_k}^{(n)}$ is the operator associated with the distribution function

$$F_k\left(\frac{xs_n}{\sqrt{1-d^2}} - \frac{\sigma_k^2(A_n - A)}{s_n\sigma\sqrt{1-d^2}}\right).$$

On the other hand, we have

$$H_\lambda\left(\frac{x}{d}\right) * \Phi\left(\frac{x}{\sqrt{1-d^2}}\right) = \sum_{n=1}^{\infty} p_n \Phi_1 * \Phi_2 * \ldots * \Phi_n\left(\frac{xs_n}{\sqrt{1-d^2}} - \frac{s_n(A_n - A)}{\sigma\sqrt{1-d^2}}\right),$$

where $\Phi_k(x) = \Phi(x/\sigma_k)$. Hence

$$A_\lambda(f) = \sum_{n=1}^{\infty} p_n A_1^{(n)} A_2^{(n)} \ldots A_n^{(n)}(f), \tag{22}$$

where $A_k^{(n)}$ is the operator associated with the distribution function

$$\Phi\left(\frac{xs_n}{\sigma_k\sqrt{1-d^2}} - \frac{\sigma_k(A_n - A)}{\sigma s_n\sqrt{1-d^2}}\right).$$

Therefore, it follows from (21), (22) and Lemma 4 of [10], p. 517, that, for any $f \in C_3$,

$$\|A_N(f) - A_\lambda(f)\| \leqslant \sum_{n=1}^{\infty} p_n \sum_{k=1}^{n} \|A_{F_k}^{(n)}(f) - A_k^{(n)}(f)\|. \tag{23}$$

It is obvious that

$$A_{F_k}^{(n)}(f)(x) = \int_{-\infty}^{\infty} f\left(x + \frac{(A_n - A)\sigma_k^2}{s_n^2\sigma} + z\sqrt{1-d^2}\right) dF_k(s_n z).$$

Thus we can apply (12) and (13) to $x + \sigma_k(A_n - A)/\sigma s_n^2$ and $z\sqrt{1-d^2}$ instead of x and y, respectively. The same calculations as those in the proof of Theorem 1 permit to prove that, for $f \in C_3$ and any $\varepsilon > 0$,

$$\left|A_{F_k}^{(n)}(f)(x) - f\left(x + \frac{(A_n - A)\sigma_k}{s_n^2\sigma}\right) - \frac{\sigma_k^2(1-d^2)}{2s_n^2} f''\left(x + \frac{(A_n - A)\sigma_k}{s_n^2\sigma}\right)\right| \tag{24}$$
$$\leqslant \frac{M_1(1-d^2)}{s_n^2} \int_{|z| \geqslant \varepsilon s_n} z^2 dF_k(z) + \frac{\varepsilon M_2 \sigma_k^2 (1-d^2)^{3/2}}{6s_n^2},$$

where

$$M_1 = \sup_x |f''(x)| \quad \text{and} \quad M_2 = \sup_x |f'''(x)|.$$

Similarly, we obtain the inequality

$$(25) \quad \left| A_k^{(n)}(f)(x) - f\left(x + \frac{(A_n - A)\sigma_k}{s_n^2 \sigma}\right) - \frac{\sigma_k^2(1-d^2)}{2s_n^2} f''\left(x + \frac{(A_n - A)\sigma_k}{s_n^2\sigma}\right)\right| \leqslant M_1(1-d^2) \int\limits_{|z|\geqslant \varepsilon s_n} z^2 d\Phi\left(\frac{z}{\sigma_k}\right) + \frac{\varepsilon M_2 \sigma_k^2 (1-d^2)^{3/2}}{6s_n^2}.$$

From (23), (24) and (25) we get

$$\|A_N(f) - A_\lambda(f)\| \leqslant M_1(1-d^2)\mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N} \int\limits_{|z|\geqslant \varepsilon\sqrt{M}} z^2 dF_k(z)\right\} + M_1(1-d^2)\mathrm{E}\left\{\frac{1}{M}\sum_{k=1}^{N} \int\limits_{|z|\geqslant\varepsilon\sqrt{M}} z^2 d\Phi\left(\frac{z}{\sigma_k}\right)\right\} + \frac{\varepsilon}{3} M_2(1-d^2)^{3/2}.$$

By (5) and Lemma 2,

$$\lim_{\lambda\to\infty} \|A_N(f) - A_\lambda(f)\| = 0 \quad \text{for every } f \epsilon C_3,$$

which was to be proved.

COROLLARY 2. *If the assumptions of Theorem* 3 *are satisfied and, moreover, the random variable* L *is asymptotically normal* (A, σ), *then the random variable* S_N *is asymptotically normal* (A, σ).

COROLLARY 3. *If the assumptions of Theorem* 3 *are satisfied, the random variable* $(L-A)/\Delta$ *has the asymptotic distribution function* $G(x)$, *and there exists*

$$\lim_{\lambda\to\infty} \frac{\mathrm{E}M}{\Delta^2} = s, \quad 0 \leqslant s < \infty,$$

then

$$G\left(x(1+s)^{1/2}\right) * \Phi\left(x\left(\frac{1+s}{s}\right)^{1/2}\right)$$

is the asymptotic distribution function of the random variable $(S_N - A)/\sigma$.

Now we are going to prove the following

LEMMA 3. *If the random variable* $(M - \mathrm{E}M)/\sigma M$ *has a limiting distribution function* $G(x)$ *such that* $G(x) > 0$ *for every finite* x, *then*

$$\frac{M - \mathrm{E}M}{\sigma^2} \xrightarrow{\mathrm{P}} 0 \quad as\ \lambda \to \infty.$$

Proof. First we shall show that $\sigma M = o(\mathrm{E}M)$ as $\lambda \to \infty$. Suppose it is not true. Then there exists a constant $\eta > 0$ such that, for every λ_0, there exists a $\lambda > \lambda_0$ such that

$$\frac{\mathrm{E}M}{\sigma M} < \eta. \tag{26}$$

Obviously, we can assume that $-\eta$ is a continuity point of $G(x)$.

Now choose λ_1 such that, for every $\lambda > \lambda_1$,

$$\mathrm{P}[M - \mathrm{E}M < -\eta\sigma M] > G(-\eta) - \frac{G(-\eta)}{2} > 0. \tag{27}$$

Thus, for some $\lambda > \lambda_1$ $(\lambda_1 > \lambda_0)$, we have both (26) and (27), whence

$$0 = \mathrm{P}[M < 0] = \mathrm{P}\left[\frac{M - \mathrm{E}M}{\sigma M} < -\frac{\mathrm{E}M}{\sigma M}\right] \geqslant \mathrm{P}[M - \mathrm{E}M < -\eta\sigma M] > 0,$$

a contradiction. It follows that $\sigma M = o(\mathrm{E}M)$ and $\sigma M = o(\sigma^2)$, since $\sigma^2 = \mathrm{E}M + \Delta^2$. Now Lemma 3 follows by Chebyshev's inequality.

From Theorem 3 and Lemma 3 one can deduce the following corollaries.

COROLLARY 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables, and N a positive integer-valued random variable independent of each X_n, $n = 1, 2, \ldots$ If (5) is satisfied and if the random variables L and M are asymptotically normal (A, Δ) and $(\mathrm{E}M, \sigma M)$, respectively, then the random variable S_N is asymptotically normal (A, σ).*

COROLLARY 5. *If the assumptions of Theorem 1 are satisfied, and the random variable M is asymptotically normal $(\mathrm{E}M, \sigma M)$, then the conclusion of Theorem 3 holds true.*

LEMMA 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $\sigma_k^2 \geqslant \sigma_0^2 > 0$ $(\sigma_0^2 = \mathrm{const})$. If the random variable $(M - \mathrm{E}M)/\sigma M$ has the limiting distribution function $G(x)$ such that $G(x) > 0$ for every finite x, then*

$$\frac{M - \mathrm{E}M}{\sigma^2} \xrightarrow{\mathrm{P}} 0, \quad \sigma^2 \to \infty, \quad \textit{as } \lambda \to \infty.$$

Proof. According to Lemma 3, $(M - \mathrm{E}M)/\sigma^2 \xrightarrow{\mathrm{P}} 0$ as $\lambda \to \infty$. Hence it suffices to prove that $\sigma^2 \to \infty$ as $\lambda \to \infty$.

First we shall show that $\mathrm{E}M \to \infty$ as $\lambda \to \infty$. If not, then, in view of $\sigma M = o(\mathrm{E}M)$ (see the proof of Lemma 3), $\sigma M \to 0$. Thus, by Chebyshev's inequality, for every $\varepsilon > 0$ we obtain

$$\mathrm{P}[|M - \mathrm{E}M| < \varepsilon] \to 1 \quad \text{as } \lambda \to \infty.$$

Without loss of generality we can take $0 < \varepsilon < \sigma_0^2/2$. Then there is at most one integer k satisfying $|s_k^2 - \mathrm{E}M| < \varepsilon$. Denoting this integer by k_λ, we have $\mathrm{P}[M = s_{k_\lambda}^2] \to 1$.

Let us put

$$I = \liminf_{\lambda\to\infty} \{(s_{k_\lambda}^2 - \mathrm{E}M)/\sigma M\}.$$

First we assume that $I > -\infty$ and let $x < I$ be a continuity point of $G(x)$. Hence, for sufficiently large λ,

$$s_{k_\lambda}^2 - \mathrm{E}M > x\sigma M \quad \text{and} \quad G(x) = \lim_{\lambda\to\infty} \mathrm{P}[M - \mathrm{E}M < x\sigma M].$$

Thus

$$\mathrm{P}[M - \mathrm{E}M < x\sigma M] \leqslant 1 - \mathrm{P}[M = s_{k_\lambda}^2],$$

which means that $G(x) = 0$; a contradiction. On the other hand, if $I = -\infty$, then $s_{k_\lambda}^2 - \mathrm{E}M < x\sigma M$ for every x and sufficiently large λ. This gives

$$\mathrm{P}[M - \mathrm{E}M < x\sigma M] \geqslant \mathrm{P}[M = s_{k_\lambda}^2],$$

whence $G(x) = 1$ for every x, which is a contradiction as well. Thus $\mathrm{E}M \to \infty$ as $\lambda \to \infty$, and hence $\sigma^2 \to \infty$ as $\sigma^2 = \mathrm{E}M + \Delta^2$.

From Lemma 4 and Theorem 1 of [13] we get immediately the following

COROLLARY 6. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $\sigma_k^2 \geqslant \sigma_0^2 > 0$, $k = 1, 2, \ldots$ If the sequence $\{X_n, n \geqslant 1\}$ satisfies the Lindeberg condition and the random variable $(M - \mathrm{E}M)/\sigma M$ is asymptotically normal, then the random variable $(S_N - A)/\sigma$ has the limiting distribution function $H_\lambda(x/d) * \Phi(x/\sqrt{1-d^2})$.*

REFERENCES

[1] F. J. Anscombe, *Large-sample theory of sequential estimation*, Proceedings of the Cambridge Philosophical Society 48 (1952), p. 600-607.

[2] J. R. Blum, D. L. Hanson and J. I. Rosenblatt, *On the central limit theorem for the sum of a random number of independent random variables*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 2 (1963), p. 389-393.

[3] H. Cramér, *Mathematical methods of statistics*, Princeton 1946.

[4] M. Csörgo and R. Fischler, *Some examples and results in the theory of mixing and random-sum central limit theorems*, Periodica Mathematica Hungarica 3 (1973), p. 41-57.

[5] W. Feller, *An introduction to probability theory and its applications*, Vol. II, New York 1966.

[6] S. Guiasu, *On the asymptotic distribution of the sequences of random variables with random indices*, The Annals of Mathematical Statistics 42 (1972), p. 2018-2028.

[7] J. W. Lindeberg, *Über das Exponentialgesetzes in der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift 15 (1922), p. 211-225.

[8] М. Маматов и Й. Нематов, *О предельной теореме для сумм случайного числа независимых случайных величин*, Известия Академии наук УзССР, серия физико-математических наук, 35 (1971), p. 18-24.

[9] J. Mogyoródi, *A remark on stable sequences of random variables and a limit distribution theorem for a random sum of independent random variables*, Acta Mathematica Academiae Scientiarum Hungaricae 17 (1966), p. 401-409.

[10] A. Rényi, *Probability theory*, Budapest 1970.

[11] — *On the central limit theorem for the sum of a random number of independent random variables*, Acta Mathematica Academiae Scientiarum Hungaricae 11 (1960), p. 97-102.

[12] H. Robbins, *The asymptotic distribution of the sum of a random number of random variables*, Bulletin of the American Mathematical Society 54 (1948), p. 1151-1161.

[13] Z. Rychlik and D. Szynal, *On the limit behaviour of sums of a random number of independent random variables*, Colloquium Mathematicum 28 (1973), p. 147-159.

[14] С. Х. Сираждинов и Г. Оразов, *Обобщение одной теоремы Г. Роббинса. Предельные теоремы и статистические выводы*, Ташкент 1966, p. 154-162.

[15] H. F. Trotter, *An elementary proof of the central limit theorem*, Archiv der Mathematik 9 (1959), p. 226-234.

Reçu par la Rédaction le 8. 1. 1974