ZASTOSOWANIA MATEMATYKI
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S. TRYBUŁA (Wrocław)

MINIMAX CONTROL OF A LINEAR SYSTEM WITH MULTINOMIAL DISTURBANCES

In the paper we consider the problem of minimax control of system (1) with disturbances v_{ij} distributed according to the multinomial laws with different parameters for different i. It is assumed that the horizon of the control is random and bounded and that the loss function depends also on the parameters of the disturbances. This is defined in (3).

1. Preliminary remarks and definitions. Statement of the problem. The system considered in the paper is defined by the equation

(1) $$x_{n+1} = A_n x_n + B_n u_n + C_n \sum_{i=1}^{s} v_{in}, \qquad n = 0, 1, \ldots, N, \; u_n \in R^l,$$

where

x_n is the state variable,

u_n is the control,

v_{in} $(i = 1, \ldots, s)$ are independent random variables distributed according to the laws given by (2),

N is a random variable independent of v_{in}, $1 \leqslant N \leqslant M$,

A_n, B_n, C_n are $(k \times k)$-, $(k \times l)$-, $(k \times m)$-matrices, respectively.

Obviously, x_n, u_n, v_{in} are k-, l-, m-dimensional column vectors, respectively; x_0 is given.

Let V be the set of values of the random variables

$$v_n = \sum_{i=1}^{s} v_{in}.$$

The matrices C_n satisfy the following assumptions: the linear equations $C_n y_n = z$ for $z \in \{z \in R^k \colon \exists (y \in V)(C_n y = z)\}$ have exactly one solution y_n for $n = 0, 1, \ldots, M-1$.

It is assumed that the data available at time n are

$$X_n = (x_0, x_1, \ldots, x_n) \quad \text{and} \quad U_{n-1} = (u_0, u_1, \ldots, u_{n-1}).$$

It follows then from the above that at time n the value of the random variable v_n is known.

The control u_n is a Borel function of (X_n, U_{n-1}).

It is assumed that the random variables v_{in} have multinomial distributions with parameters $q, \lambda_{i1}, \ldots, \lambda_{im}$, i.e., their probability function is

$$(2)\qquad p(v^{(1)}, \ldots, v^{(m+1)}; \lambda_{i1}, \ldots, \lambda_{im+1}) = \frac{q!}{v^{(1)}! \ldots v^{(m+1)}!} \lambda_{i1}^{v^{(1)}} \ldots \lambda_{im+1}^{v^{(m+1)}},$$

where $\{v^{(1)}, \ldots, v^{(m)}\}$ is the value of the random variable v_{in},

$$v^{(m+1)} = q - \sum_{j=1}^{m} v^{(j)}, \qquad \lambda_{im+1} = 1 - \sum_{j=1}^{m} \lambda_{ij}.$$

The parameter q is known; the parameters λ_{ij} are unknown.

The case $q = 1$ is of special interest.

It is supposed that the horizon $1 \leqslant N \leqslant M$ of the control has a given distribution $P(N = n) = p_n$, $p_M > 0$.

We put

$$\bar{\lambda} = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{sm}), \qquad \lambda_j = \frac{1}{s} \sum_{i=1}^{s} \lambda_{ij}, \; j = 1, \ldots, m+1,$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \ldots \\ \lambda_m \end{bmatrix}.$$

In the paper we denote by A' the matrix transposed to the matrix A.

Let us define the risk function for the control strategy $U = (u_0, u_1, \ldots, u_M)$ as

$$(3)\qquad R(\bar{\lambda}, U) = \mathrm{E}_p \{\mathrm{E}_{\bar{\lambda}} [\sum_{i=0}^{N} ((x_i', \lambda') T_i (x', \lambda')' + u_i' K_i u_i)]\},$$

where

$\mathrm{E}_p(\cdot)$ denotes the expectation with respect to the distribution of the random variable N,

$\mathrm{E}_{\bar{\lambda}}(\cdot)$ denotes the expectation with respect to the distribution of the random variables v_{in} for fixed $\bar{\lambda}$,

(x_i', λ') is the vector x_i' with added coordinates $\lambda_1, \ldots, \lambda_m$,

T_i and K_i are $(k+m) \times (k+m)$ and $l \times l$ symmetric matrices, respectively, nonnegative definite.

We consider only control strategies U for which the risk function $R(\bar{\lambda}, U)$ exists for each $\bar{\lambda} \in \bar{\Lambda} = \Lambda^s$, where

$$\Lambda = \{(y_1, \ldots, y_m) \in R^m \colon \; y_i \geqslant 0, \; \sum_{i=1}^{m} y_i \leqslant 1\}.$$

The set of all these strategies is denoted by Δ.

A strategy $U^{(0)} \in \Delta$ such that

$$\sup_{\bar{\lambda} \in \bar{\Lambda}} R(\bar{\lambda}, U^{(0)}) = \inf_{U \in \Delta} \sup_{\bar{\lambda} \in \bar{\Lambda}} R(\bar{\lambda}, U)$$

is called a *minimax control strategy.*

The problem is to determine the minimax control strategy for the risk function (3).

2. A filtration problem. Suppose that the random variables v_{in} have the distribution defined in (2) with $\lambda_{ij} = \lambda_j$. Then the random variable v_n has the multinomial distribution with the parameters qs and λ, i.e., its probability function $\hat{p}(v, \lambda)$ is

$$\hat{p}(v, \lambda) = \begin{cases} \dfrac{(qs)!}{v^{(1)}! \ldots v^{(m+1)}!} \lambda_1^{v^{(1)}} \ldots \lambda_{m+1}^{v^{(m+1)}} & \text{if } \lambda \in \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$v = \{v^{(1)}, \ldots, v^{(m+1)}\}, \qquad \sum_{i=1}^{m+1} v^{(i)} = qs.$$

Let us assume in this section that the parameter λ is a random variable which has the (a priori) distribution $\pi_{\beta,r}$, $r = (r_1, \ldots, r_m)'$, with the density

$$g(\lambda; \beta, r) = \begin{cases} \dfrac{\Gamma(\beta)}{\Gamma(r_1) \ldots \Gamma(r_{m+1})} \lambda_1^{r_1 - 1} \ldots \lambda_{m+1}^{r_{m+1} - 1} & \text{if } \lambda \in \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$r_{m+1} = \beta - \sum_{k=1}^{m} r_k.$$

Having observed x_n, the a posteriori density $g(\lambda | X_n, U_{n-1})$ of the parameter λ given (X_n, U_{n-1}) is equal to the density of the parameter λ given $\sum_{i=0}^{n-1} v_i$ and can be computed according to the Bayes rule to obtain

$$g(\lambda | X_n, U_{n-1}) = g(\lambda; \beta_n, \hat{r}_n), \tag{6}$$

where

$$\beta_n = \beta + nqs, \qquad r_n^{(k)} = r_k + \sum_{i=1}^{s} \sum_{j=1}^{n-1} v_{ij}^{(k)}, \; k = 1, \ldots, m,$$

$$v_{ij} = \begin{bmatrix} v_{ij}^{(1)} \\ \ldots \\ v_{ij}^{(m)} \end{bmatrix}, \qquad \hat{r}_n = \begin{bmatrix} r_n^{(1)} \\ \ldots \\ r_n^{(m)} \end{bmatrix}, \qquad r_n^{(m+1)} = \beta_n - \sum_{k=1}^{m} r_n^{(k)}.$$

Moreover, put

$$v_n = \begin{bmatrix} v_n^{(1)} \\ \dots \\ v_n^{(m)} \end{bmatrix}, \qquad v_n^{(m+1)} = qs - \sum_{k=1}^{m} v_n^{(k)}.$$

Given (X_n, U_{n-1}), the conditional distribution of the random variable v_n is

$$\begin{aligned} p(v_n|X_n, U_{n-1}) &= \int_\Lambda \hat{p}(v_n, \lambda) g(\lambda; \beta_n, \hat{r}_n) d\lambda \\ &= \frac{\hat{p}(v_n, \lambda) g(\lambda; \beta_n, \hat{r}_n)}{g(\lambda; \beta_{n+1}, \hat{r}_{n+1})} \\ &= \frac{(qs)!}{v_n^{(1)}! \dots v_n^{(m+1)}!} \frac{\Gamma(\beta+nqs)}{\Gamma(r_n^{(1)}) \dots \Gamma(r_n^{(m+1)})} \\ &\quad \times \frac{\Gamma(r_n^{(1)}+v_n^{(1)}) \dots \Gamma(r_n^{(m+1)}+v_n^{(m+1)})}{\Gamma(\beta+(n+1)qs)}. \end{aligned}$$

Then

$$\begin{aligned} &\mathrm{E}(v_n^{(1)}|X_n, U_{n-1}) \\ &= \sum{}' v_n^{(1)} p(v_n|X_n, U_{n-1}) \\ &= \frac{\Gamma(\beta+nqs)qs}{\Gamma(r_n^{(1)}) \dots \Gamma(r_n^{(m+1)})} \int_\Lambda x_1^{r_n^{(1)}} x_2^{r_n^{(2)}-1} \dots x_m^{r_n^{(m)}-1} \Big(1-\sum_{j=1}^{m} x_j\Big)^{r_n^{(m+1)}-1} \\ &\quad \times \Bigg(\sum{}'' \frac{(qs-1)!}{(v_n^{(1)}-1)! \, v_n^{(2)}! \dots v_n^{(m+1)}!} \\ &\quad \times x_1^{v_n^{(1)}-1} x_2^{v_n^{(2)}} \dots x_m^{v_n^{(m)}} \Big(1-\sum_{j=1}^{m} x_j\Big)^{v_n^{(m+1)}}\Bigg) dx_1 \dots dx_m \\ &= \frac{\Gamma(\beta+nqs)qs}{\Gamma(r_n^{(1)}) \dots \Gamma(r_n^{(m+1)})} \int_\Lambda x_1^{r_n^{(1)}} x_2^{r_n^{(2)}-1} \dots x_m^{r_n^{(m)}-1} \\ &\quad \times \Big(1-\sum_{j=1}^{m} x_j\Big)^{r_n^{(m+1)}-1} dx_1 \dots dx_m = qs \frac{r_n^{(1)}}{\beta_n}, \end{aligned}$$

where in $\sum{}'$ the summation runs over the set

$$v_n^{(1)} \geqslant 0, \dots, v_n^{(m+1)} \geqslant 0, \qquad v_n^{(1)} + \dots + v_n^{(m+1)} = qs,$$

and in $\sum{}''$ over the set

$$v_n^{(1)} \geqslant 1, \, v_n^{(2)} \geqslant 0, \dots, v_n^{(m+1)} \geqslant 0, \qquad v_n^{(1)} + \dots + v_n^{(m+1)} = qs.$$

In general, we have

$$\mathrm{E}(v_n^{(k)}|X_n, U_{n-1}) = qs\frac{r_n^{(k)}}{\beta_n} \quad (k = 1, \ldots, m+1). \tag{7}$$

In a similar way we prove that

$$\mathrm{E}\big(v_n^{(k)}(v_n^{(k)}-1)|X_n, U_{n-1}\big) = qs(qs-1)\frac{r_n^{(k)}(r_n^{(k)}+1)}{\beta_n(\beta_n+1)} \tag{8}$$

and

$$\mathrm{E}(v_n^{(k)}v_n^{(l)}|X_n, U_{n-1}) = qs(qs-1)\frac{r_n^{(k)}r_n^{(l)}}{\beta_n(\beta_n+1)}, \quad k \neq l \tag{9}$$

$$(k, l = 1, \ldots, m+1).$$

Moreover, taking into account equation (6), we have

$$\mathrm{E}(\lambda_k|X_n, U_{n-1}) = \frac{r_n^{(k)}}{\beta_n}, \tag{10}$$

$$\mathrm{E}(\lambda_k^2|X_n, U_{n-1}) = \frac{r_n^{(k)}(r_n^{(k)}+1)}{\beta_n(\beta_n+1)} \tag{11}$$

and

$$\mathrm{E}(\lambda_k\lambda_l|X_n, U_{n-1}) = \frac{r_n^{(k)}r_n^{(l)}}{\beta_n(\beta_n+1)}, \quad k \neq l \tag{12}$$

$$(k, l = 1, \ldots, m+1).$$

Consequently, for example, we have

$$\begin{aligned} \mathrm{E}(x_{n+1}|X_n, U_{n-1}) &= \mathrm{E}(A_n x_n + B_n u_n + C_n v_n|X_n, U_{n-1}) \\ &= A_n x_n + B_n u_n + C_n qs\frac{\hat{r}_n}{\beta_n}, \\ \mathrm{E}(\hat{r}_{n+1}|X_n, U_{n-1}) &= \mathrm{E}(\hat{r}_n + v_n|X_n, U_{n-1}) = \left(1+\frac{qs}{\beta_n}\right)\hat{r}_n. \end{aligned} \tag{13}$$

3. Bayes strategies. Let π be the a priori distribution of the parameter λ and let U be a control strategy. The function

$$r(\pi, U) = \int_{\bar{\Lambda}} R(\bar{\lambda}, U)\pi(d\bar{\lambda}) = \mathrm{E}\big(R(\bar{\lambda}, U)\big)$$

is called the *Bayes risk*.

A control strategy $U_\pi \in \Delta$ such that

$$r(\pi, U_\pi) = \inf_{U \in \Delta} r(\pi, U)$$

is called the *Bayes control strategy* with respect to π.

Suppose that the disturbances v_n have the distribution given by (4) and the a priori distribution $\pi = \pi_{\beta,r}$ of the parameter λ is given by (5), where $r_i > 0$, $i = 1, \ldots, m$, $\sum_{j=1}^{m} r_j < \beta$. Consider the problem of determining the control strategy

$$U^*_{\beta,r} = (u^*_0, u^*_1, \ldots, u^*_M)$$

which minimizes the Bayes risk $r(\pi_{\beta,r}, U)$. It is sufficient to minimize the following function with respect to $U^{(n)} = (u_n, \ldots, u_M)$ for given (X_n, U_{n-1}) consequently for $n = M, \ldots, 1, 0$ and for $\pi = \pi_{\beta,r}$:

$$r_n(\pi, U^{(n)}) = \mathrm{E}_p\left\{\mathrm{E}\left[\sum_{i=n}^{N}\left((x_i', \lambda')\, T_i(x_i', \lambda')' + u_i' K_i u_i\right)\Big| X_n, U_{n-1}\right]\Big|\ N \geqslant n\right\}$$
$$= \mathrm{E}\left\{\sum_{i=n}^{M}\frac{\pi_i}{\pi_n}\left((x_i', \lambda')\, T_i(x_i', \lambda')' + u_i' K_i u_i\right)\Big| X_n, U_{n-1}\right\},$$

where

$$\pi_k = \sum_{i=k}^{M} p_i.$$

Write

$$W_n = \min_{U^{(n)}} r_n(\pi, U^{(n)}).$$

Similarly as in [4], applying the Bellman's dynamic programming optimality principle, we can prove that W_n satisfy the recurrence equation

$$W_n = \min_{u_n}\left\{x_n' T_n^{(1)} x_n + 2\frac{\hat{r}_n'}{\beta_n} T_n^{(2)} x_n + \frac{1}{\beta_n(\beta_n+1)}\hat{r}_n' T_n^{(3)} \hat{r}_n\right.$$
$$\left. + \frac{1}{\beta_n(\beta_n+1)}\mathrm{Row}(T_n^{(3)})\hat{r}_n + u_n' K_n u_n + \frac{\pi_{n+1}}{\pi_n}\mathrm{E}\left[W_{n+1} | X_n, U_{n-1}\right]\right\}, \tag{14}$$

where the matrix T_n is divided into submatrices

$$T_n = \begin{bmatrix} T_n^{(1)} & T_n^{(2)\prime} \\ T_n^{(2)} & T_n^{(3)} \end{bmatrix}$$

such that

$$(x_n', \lambda')\, T_n(x_n', \lambda')' = x_n' T_n^{(1)} x_n + 2\lambda' T_n^{(2)} x_n + \lambda' T_n^{(3)} \lambda$$

and

$$\mathrm{Row}(A) = [a_{11}, \ldots, a_{mm}]$$

if $A = [a_{ij}]_1^m$.

We show that W_n can be expressed in the form

$$W_n = x_n' D_n x_n + 2\frac{\hat{r}_n'}{\beta_n} F_n x_n + \hat{r}_n' G_n \hat{r}_n + H_n \hat{r}_n, \tag{15}$$

where D_n is a nonnegative definite matrix.

For $n = M$ equation (15) holds with

$$\begin{gathered} D_M = T_M^{(1)}, \quad F_M = T_M^{(2)}, \quad G_M = \frac{1}{\beta_n(\beta_n+1)} T_M^{(3)}, \\ H_M = \frac{1}{\beta_n(\beta_n+1)} \operatorname{Row}(T_M^{(3)}). \end{gathered} \tag{16}$$

Assume that equation (15) holds for $n+1$. Since

$$x_{n+1} = A_n x_n + B_n u_n + C_n v_n,$$

W_n exists in (14) and for determining the Bayes control it is sufficient to solve the equation

$$2K_n u_n + \frac{\pi_{n+1}}{\pi_n} \operatorname{grad}_{u_n} \mathrm{E}[W_{n+1} | X_n, U_{n-1}] = 0, \tag{17}$$

where $\operatorname{grad}_{u_n} \mathrm{E}[W_{n+1} | X_n, U_{n-1}]$ is the column vector defined as usual. But

$$\begin{aligned} \mathrm{E}[W_{n+1} | X_n, U_{n-1}] = \mathrm{E}\Big[&(A_n x_n + B_n u_n + C_n v_n)' D_{n+1} (A_n x_n + B_n u_n + C_n v_n) \\ &+ \frac{2}{\beta_{n+1}} (\hat{r}_n + v_n)' F_{n+1} (A_n x_n + B_n u_n + C_n v_n) + (\hat{r}_n + v_n)' G_{n+1} (\hat{r}_n + v_n) \\ &+ H_{n+1} (\hat{r}_n + v_n) | X_n, U_{n-1} \Big]. \end{aligned} \tag{18}$$

Then from (17), using (7)–(9) and (13), we obtain for the Bayes control u_n^* the equation

$$\begin{aligned} &\left[K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right] u_n^* \\ &\quad + \frac{\pi_{n+1}}{\pi_n} [B_n' D_{n+1} A_n x_n + B_n'(qs D_{n+1} C_n + F_{n+1}')] \frac{\hat{r}_n}{\beta_n} = 0. \end{aligned} \tag{19}$$

Assume that equation (19) has a solution u_n^*. Then the Bayes control is

$$u_n^* = -P_n x_n - Q_n \frac{\hat{r}_n}{\beta_n}, \tag{20}$$

where

$$P_n = \frac{\pi_{n+1}}{\pi_n} \left(K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right)^+ B_n' D_{n+1} A_n, \tag{21}$$

$$(22)\qquad Q_n = \frac{\pi_{n+1}}{\pi_n}\left(K_n + \frac{\pi_{n+1}}{\pi_n} B_n' D_{n+1} B_n\right)^{+} B_n'(qsD_{n+1} C_n + F_{n+1}'),$$

and A^+ is the Moore–Penrose pseudoinverse matrix to the matrix A (see [2], p. 407).

Moreover, by (14) using (7)–(13) and (18) we prove (15) for n with D_n and F_n satisfying the equations

$$(23)\qquad D_n = T_n^{(1)} + \frac{\pi_{n+1}}{\pi_n} A_n' D_{n+1}(A_n - B_n P_n),$$

$$(24)\qquad F_n = T_n^{(2)} + \frac{\pi_{n+1}}{\pi_n}(qsC_n' D_{n+1} + F_{n+1})(A_n - B_n P_n)$$

and the boundary conditions (16).

From (21) and (23) we obtain also another form of the equation for D_n:

$$(25)\qquad D_n = T_n^{(1)} + P_n' K_n P_n + \frac{\pi_{n+1}}{\pi_n}(A_n - B_n P_n)' D_{n+1}(A_n - B_n P_n).$$

From (16) and (25) it follows (by inductive argument) that D_n is a symmetric nonnegative definite matrix.

In the paper we assume that for $n = 0, 1, \ldots, M-1$ and for each $x_n' \in R^k$, $\tilde{r}_n'/\beta_n \in R^m$, equation (19) has a solution u_n^*.

For determining Bayes control strategies see [1], [3] and [5].

4. Determining the risk. We say that (β, r) *belongs to* S and write $(\beta, r) \in S$, $r = (r_1, \ldots, r_m)$, if $r_i > 0$, $i = 1, \ldots, m$, $\sum_{i=1}^{m} r_i < \beta$.

Let u_n^* be the control defined in (20)–(24) and let

$$U_{\beta,r}^* = (u_0^*, u_1^*, \ldots, u_M^*).$$

For $(\beta, r) \in S$ we put

$$R_n(\bar{\lambda}, U_{\beta,r}^*) = \mathrm{E}_{\bar{\lambda}}\left[\sum_{i=n}^{M} \frac{\pi_i}{\pi_n}[(x_i', \lambda')\, T_i(x_i', \lambda')' + u_i^{*\prime} K_i u_i^*] \,|\, X_n, U_{n-1}^*\right],$$

where $U_{n-1}^* = (u_0^*, u_1^*, \ldots, u_{n-1}^*)$. We have

$$(26)\qquad R(\bar{\lambda}, U_{\beta,r}^*) = R_0(\bar{\lambda}, U_{\beta,r}^*)$$

and the functions R_n satisfy the equations

$$(27)\qquad R_n(\bar{\lambda}, U_{\beta,r}^*) = (x_n', \lambda')\, T_n(x_n', \lambda')' + u_n^{*\prime} K_n u_n^*$$

$$+ \frac{\pi_{n+1}}{\pi_n} \mathrm{E}_{\bar{\lambda}}\left[\mathrm{E}_{\bar{\lambda}}\left[\sum_{i=n+1}^{M} \frac{\pi_i}{\pi_{n+1}}[(x_i', \lambda')\, T_i(x_i', \lambda')' + u_i^{*\prime} K_i u_i^*] \,|\, X_{n+1}, U_n^*\right] \Bigg|\, X_n, U_{n-1}^*\right]$$

$$= (x_n', \lambda')\, T_n (x_n', \lambda')' + u_n^{*\prime} K_n u_n^* + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}_{\bar{\lambda}} [R_{n+1}(\bar{\lambda}, U_{\beta,r}^*) | X_n, U_{n-1}^*].$$

The risk $R(\bar{\lambda}, U_{\beta,r}^*)$ can be determined by (26) and (27). Similarly as in the previous section we prove that $R_n(\bar{\lambda}, U_{\beta,r}^*)$ is of the form

$$R_n(\bar{\lambda}, U_{\beta,r}^*) = x_n' a_n x_n + 2x_n' b_n \lambda + \hat{r}_n' c_n \hat{r}_n + \lambda' e_n \lambda + 2\hat{r}_n' f_n \lambda + i_n \lambda + \sum_{i=1}^{s} \hat{\lambda}_i' l_n \hat{\lambda}_i, \tag{28}$$

where

$$\hat{\lambda}_i = \begin{bmatrix} \lambda_{i1} \\ \cdots \\ \lambda_{im} \end{bmatrix} \tag{29}$$

and, for $n = 0, 1, \ldots, M$,

$$a_n = D_n, \quad b_n = F_n', \quad c_n = \frac{1}{\pi_n} \sum_{i=n}^{M-1} \frac{1}{\beta_i^2} S_i,$$

$$\begin{aligned} e_n &= \frac{1}{\pi_n}\left[Z_n + qsY_n - 2qs \sum_{i=n+1}^{M-1} \frac{i-n}{\beta_i} S_i + q^2 s^2 \sum_{i=n+1}^{M-1} \frac{(i-n)^2}{\beta_i^2} S_i \right], \\ f_n &= \frac{1}{\pi_n}\left[-\sum_{i=n}^{M-1} \frac{1}{\beta_i} S_i + qs \sum_{i=n+1}^{M-1} \frac{i-n}{\beta_i^2} S_i \right], \end{aligned} \tag{30}$$

$$i_n = \frac{qs}{\pi_n} \mathrm{Row}\left[Y_n + \sum_{i=n+1}^{M-1} \frac{i-n}{\beta_i^2} S_i \right], \quad l_n = -\frac{q}{\pi_n}\left[Y_n + \sum_{i=n+1}^{M-1} \frac{i-n}{\beta_i^2} S_i \right],$$

where

$$\begin{aligned} S_n &= Q_n' (\pi_n K_n + \pi_{n+1} B_n' D_{n+1} B_n) Q_n, \\ Y_n &= \sum_{i=n}^{M-1} \pi_{i+1} C_i' D_{i+1} C_i, \end{aligned} \tag{31}$$

$$Z_n = \pi_n T_n^{(3)} + \sum_{i=n}^{M-1} \pi_{i+1} \{ qs(qs-1) C' D_{i+1} C_i + qs(F_{i+1} C_i + C_i' F_{i+1}') + T_{i+1}^{(3)} \}.$$

From (28)–(31) we obtain

$$\begin{aligned} R(\bar{\lambda}, U_{\beta,r}^*) = \lambda' &\left[Z_0 + qsY_0 - 2qs \sum_{i=1}^{M-1} \frac{i}{\beta_i} S_i + q^2 s^2 \sum_{i=1}^{M-1} \frac{i^2}{\beta_i^2} S_i \right] \lambda \\ &+ \left[2x_0' F_0' - 2r' \sum_{i=0}^{M-1} \frac{\beta}{\beta_i^2} S_i + qs\, \mathrm{Row}\left(Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i \right) \right] \lambda \end{aligned} \tag{32}$$

$$+x_0' D_0 x_0 + r' \sum_{i=0}^{M-1} \frac{1}{\beta_i^2} S_i r - q \sum_{j=1}^{s} \hat{\lambda}_j' \left(Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i \right) \hat{\lambda}_j.$$

5. Extended Bayes strategies. Denote by $U_m^+ = (u_0^+, u_1^+, \ldots, u_M^+)$, $m \in \Lambda$, the control strategy for which

$$u_M^+ = 0_l, \quad u_n^+ = -P_n x_n - Q_n m \;\; (n = 0, 1, \ldots, M-1), \tag{33}$$

where 0_l is an l-dimensional zero column vector.

Moreover, let $U_m^- = (u_0^-, u_1^-, \ldots, u_M^-)$, $m \in \Lambda$, be the control strategy for which

$$\begin{aligned} u_M^- = 0_l, \quad u_0^- &= -P_0 x_0 - Q_0 m, \\ u_n^- &= -P_n x_n - Q_n \frac{\sum_{j=0}^{n-1} v_j}{nqs} \quad (n = 1, \ldots, M-1). \end{aligned} \tag{34}$$

From (20) and (31)–(34) we obtain

$$R(\bar{\lambda}, U_m^+) = \lim_{\substack{t \to \infty \\ t/\gamma \to m}} R(\bar{\lambda}, U_{\gamma,t}^*), \quad R(\bar{\lambda}, U_m^-) = \lim_{\substack{t \to 0 \\ t/\gamma \to m}} R(\bar{\lambda}, U_{\gamma,t}^*) \tag{35}$$

for $(t, \gamma) \in S$, $m \in \Lambda$.

6. Some lemmae. In the next section we need the following lemmae:

LEMMA 1 (Sion Theorem). *Let f be a function mapping $X \times Y$ into R^1. Suppose that*

(a) *X and Y are convex and compact subsets of R^m, $m \in \{1, 2, \ldots\}$;*

(b) *for each $y \in Y$, $x \to f(x, y)$ is convex and continuous;*

(c) *for each $x \in X$, $y \to f(x, y)$ is concave and continuous.*

Then there exists a saddle point $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} f(\bar{x}, y) = f(\bar{x}, \bar{y}) = \inf_{x \in X} f(x, \bar{y})$$
$$= \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

LEMMA 2. *Let $\{\pi_k\}_1^\infty$ be a sequence of a priori distributions on $\bar{\Lambda}$ and let $\{U_k\}_1^\infty$ and $\{r(\pi_k, U_k)\}_1^\infty$ be the corresponding sequences of Bayes strategies and Bayes risks. If $U^{(0)}$ is a strategy for which the risk function R satisfies the condition*

$$\sup_{\bar{\lambda} \in \bar{\Lambda}} R(\bar{\lambda}, U^{(0)}) \leqslant \limsup_{k \to \infty} r(\pi_k, U_k),$$

then $U^{(0)}$ is a minimax strategy.

LEMMA 3. *If the matrix A is nonnegative definite and $\lambda \in \Lambda$, then*

$$\lambda' A \lambda \leqslant \mathrm{Row}(A) \lambda.$$

LEMMA 4. *For $\bar{\lambda} \in \bar{\Lambda}$ and $\hat{\lambda}_j$ defined by* (29), *we have*

$$q \sum_{j=1}^{s} \hat{\lambda}_j' \left[Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i \right] \hat{\lambda}_j \geqslant qs\lambda' \left[Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i \right] \lambda. \tag{36}$$

Lemma 1 is well known in game theory. Lemma 2 is well known in decision theory (see, e.g., Theorem 6.5.2 in [6]). Lemma 3 follows from the theorem that a convex function considered in a convex polyhedron takes its maximum at a vertex of this polyhedron and that a nonnegative definite matrix A can be presented as a limit of positive definite matrices. To prove inequality (36) it is sufficient to notice that the matrix in the square brackets is nonnegative definite.

7. Minimax theorem. Put $\varrho = r/\beta$. From Lemma 4 and equation (32) it follows that

$$\begin{aligned} R(\bar{\lambda}, U^*_{\beta,\varrho\beta}) &\leqslant \lambda' \left[Z_0 + \sum_{i=1}^{M-1} \left(-1 + \frac{\beta^2 - qsi}{\beta_i^2} \right) S_i \right] \lambda \\ &+ \left[2x_0' F_0' - 2\varrho' \sum_{i=0}^{M-1} \frac{\beta^2}{\beta_i^2} S_i + qs \operatorname{Row} \left(Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i \right) \right] \lambda \\ &+ x_0' D_0 x_0 + \varrho' \sum_{i=0}^{M-1} \frac{\beta^2}{\beta_i^2} S_i \varrho \stackrel{\mathrm{df}}{=} \lambda' Z^{(1)}(\beta) \lambda + Z^{(2)}(\beta, \varrho) \lambda + Z^{(3)}(\beta, \varrho) \end{aligned} \tag{37}$$

with equality if $\lambda_{ij} = \lambda_j$ ($i = 1, \ldots, s$; $j = 1, \ldots, m$). Then

$$\begin{aligned} & r(\pi_{\beta,\varrho\beta}, U^*_{\beta,\varrho\beta}) \\ &= \frac{\beta}{\beta+1} \varrho' Z^{(1)}(\beta) \varrho + \frac{1}{\beta+1} \operatorname{Row}(Z^{(1)}(\beta)) \varrho + Z^{(2)}(\beta, \varrho) \varrho + Z^{(3)}(\beta, \varrho). \end{aligned} \tag{38}$$

Notice that matrices S_n, Y_n, Z_n are symmetric and S_n, Y_n are nonnegative definite.

THEOREM. I. *If the matrix $-Z_0$ is nonnegative definite, then the minimax control strategy is $U^+_{m_0}$, where*

$$\begin{aligned} & m_0' \left[Z_0 - \sum_{i=0}^{M-1} S_i \right] m_0 + [2x_0' F_0' + qs \operatorname{Row}(Y_0)] m_0 \\ &= \max_{m \in \Lambda} \left\{ m' \left[Z_0 - \sum_{i=0}^{M-1} S_i \right] m + [2x_0' F_0' + qs \operatorname{Row}(Y_0)] m \right\}. \end{aligned} \tag{39}$$

II. *If the matrix*

$$Z_0 - \sum_{i=1}^{M-1} \frac{qsi+1}{qsi} S_i$$

is nonnegative definite, then the minimax control strategy is $U^-_{m_0}$, *where*

$$(40)\qquad -m_0' S_0 m_0 + [2x_0' F_0' + \mathrm{Row}(Z_0 + qsY_0 - \sum_{i=1}^{M-1} S_i)]\, m_0$$
$$= \max_{m\in\Lambda} \{-m' S_0 m + [2x_0' F_0' + \mathrm{Row}(Z_0 + qsY_0 - \sum_{i=1}^{M-1} S_i)]\, m\}.$$

III. *If there is* β $(0 < \beta < \infty)$ *such that*

$$(41)\qquad Z_0 - \sum_{i=1}^{M-1} \left(-1 + \frac{\beta^2 - qsi}{\beta_i^2}\right) S_i = 0,$$

where 0 *is the zero matrix, then the minimax control strategy is*

$$U^{**}_{\beta, m_0\beta} = \lim_{m\to m_0} U^*_{\beta, m\beta},$$

where

$$-m_0' \sum_{i=0}^{M-1} \frac{\beta^2}{\beta_i^2} S_i m_0 + \left[2x_0' F_0' + qs\,\mathrm{Row}\left(Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i\right)\right] m_0$$
$$= \max_{m\in\Lambda} \left\{-m' \sum_{i=0}^{M-1} \frac{\beta^2}{\beta_i^2} S_i m + \left[2x_0' F_0' + qs\,\mathrm{Row}\left(Y_0 + \sum_{i=1}^{M-1} \frac{i}{\beta_i^2} S_i\right)\right] m\right\}.$$

Proof. Let us notice that the function $(\beta^2 - qsi)/\beta_i^2$ of the variable $\beta > 0$ is increasing and the matrices S_i are nonnegative definite. This implies that if the matrix

$$-Z_0 = \lim_{\beta\to\infty} [-Z^{(1)}(\beta)]$$

is nonnegative definite, then $-Z^{(1)}(\beta)$ is nonnegative definite for all β $(0 < \beta < \infty)$.

Suppose that case I occurs. By (35) and (37) for $m \in \Lambda$ and $\varrho \in \mathrm{Int}\,\Lambda$ we have

$$(42)\qquad R(\bar{\lambda}, U_m^+) = \lim_{\substack{\beta\to\infty \\ \varrho\to m}} R(\bar{\lambda}, U^*_{\beta, m\beta})$$
$$\leqslant (\lambda - m)' Z_0 (\lambda - m) + \left[2m'(Z_0 - \sum_{i=0}^{M-1} S_i) + 2x_0' F_0' + qs\,\mathrm{Row}(Y_0)\right] \lambda$$

$$+x_0' D_0 x_0 + m'\Big(\sum_{i=0}^{M-1} S_i - Z_0\Big) m$$

$$\leqslant \Big[2m'\Big(Z_0 - \sum_{i=0}^{M-1} S_i\Big) + 2x_0' F_0' + qs \operatorname{Row}(Y_0)\Big]\lambda + x_0' D_0 x_0 + m'\Big(\sum_{i=0}^{M-1} S_i - Z_0\Big) m$$

$$\stackrel{\mathrm{df}}{=} R_1(\lambda, m).$$

The function $R_1(\lambda, m)$ is a convex function of m for fixed λ and a concave function of λ for fixed m. Then from Lemma 1 it follows that there exists a point $(\lambda_0, m_0) \in \Lambda \times \Lambda$ such that

$$\text{(43)} \quad \inf_{m \in \Lambda} \sup_{\lambda \in \Lambda} R_1(\lambda, m) = \sup_{\lambda \in \Lambda} R_1(\lambda, m_0) = R_1(\lambda_0, m_0) = \inf_{m \in \Lambda} R_1(\lambda_0, m) = \sup_{\lambda \in \Lambda} \inf_{m \in \Lambda} R_1(\lambda, m).$$

It is well known that λ_0 can be chosen – independently – from the equation

$$\text{(44)} \quad \max_{\lambda \in \Lambda} \inf_{m \in \Lambda} R_1(\lambda, m) = \inf_{m \in \Lambda} R_1(\lambda_0, m).$$

Assuming that the matrix $\sum_{i=0}^{M-1} S_i - Z_0$ is positive definite, we infer that the only infimum $\inf_{m \in \Lambda} R_1(\lambda, m)$ is attained at $m = \lambda$. Then by (44) we have

$$\text{(45)} \quad \max_{\lambda \in \Lambda} R_1(\lambda, \lambda) = R_1(\lambda_0, \lambda_0).$$

Moreover, $R_1(\lambda_0, m)$ is a strictly convex function of variable m and at $m_0 = \lambda_0$ is its unique minimum. Then from (43) it follows that (λ_0, λ_0) is the only saddle point of the function $R_1(\lambda, m)$.

If the matrix $\sum_{i=0}^{M-1} S_i - Z_0$ is nonnegative definite, then each (λ_0, λ_0) satisfying (45) is a saddle point.

Then we have

$$\text{(46)} \quad \sup_{\bar{\lambda} \in \Lambda} R(\bar{\lambda}, U_{m_0}^{+}) \leqslant \sup_{\lambda \in \Lambda} R_1(\lambda, m_0) = R_1(m_0, m_0) = \lim_{\substack{\beta \to \infty \\ \varrho \to m_0}} r(\pi_{\beta,\varrho\beta}, U_{\beta,\varrho\beta}^{*}),$$

where the last equality follows directly from the equations for the functions $r(\pi_{\beta,\varrho\beta}, U_{\beta,\varrho\beta}^{*})$ and $R_1(\lambda, m)$ given in (38) and (42), respectively.

As was proved, the strategy $U_{\beta,\varrho\beta}^{*}$ is Bayes with respect to the a priori distribution $\pi_{\beta,\varrho\beta}$ for $\varrho \in \operatorname{Int} \Lambda$. Then from Lemma 2 we infer that the control strategy $U_{m_0}^{+}$ is minimax. Formula (39) for m_0 follows from (45).

Suppose that the matrix

$$Z_0 - \sum_{i=1}^{M-1} \frac{qsi+1}{qsi} S_i = \lim_{\beta \to 0+} Z^{(1)}(\beta)$$

is nonnegative definite. Then $Z^{(1)}(\beta)$ is nonnegative definite for all β $(0 < \beta < \infty)$. In this case we have

$$\begin{aligned} R(\bar{\lambda}, U_m^-) &= \lim_{\substack{\beta \to 0+ \\ \varrho \to m}} R(\bar{\lambda}, U^*_{\beta,\varrho\beta}) \\ &\leqslant \lambda' \left[Z_0 - \sum_{i=1}^{M-1} \frac{qsi+1}{qsi} S_i \right] \lambda \\ &\quad + \left[2x_0' F_0' - 2m' S_0 + \mathrm{Row}\left(qsY_0 + \sum_{i=1}^{M-1} \frac{1}{qsi} S_i \right) \right] \lambda + x_0' D_0 x_0 + m' S_0 m \\ &\leqslant [2x_0' F_0' - 2m' S_0 + \mathrm{Row}(Z_0 + qsY_0 - \sum_{i=1}^{M-1} S_i)] \lambda + x_0' D_0 x_0 + m' S_0 m \\ &\stackrel{\mathrm{df}}{=} R_2(\lambda, m), \end{aligned}$$

where the last inequality follows from Lemma 3.

The function $R_2(\lambda, m)$ is convex with respect to m for fixed λ and is concave with respect to λ for fixed m. Moreover,

$$\inf_{m \in \Lambda} R_2(\lambda, m) = R_2(\lambda, \lambda).$$

Then there exists a saddle point (λ_0, m_0), $\lambda_0 = m_0$, which is determined from the condition

$$R_2(\lambda_0, \lambda_0) = \max_{\lambda \in \Lambda} R_2(\lambda, \lambda).$$

Moreover,

$$R_2(m_0, m_0) = \lim_{\substack{\beta \to 0+ \\ \varrho \to m_0}} r(\pi_{\beta,\varrho\beta}, U^*_{\beta,\varrho\beta}).$$

Then, similarly as in case I, we prove that the control strategy $U_{m_0}^-$, when m_0 is determined from (40), is minimax.

Assume that the case III occurs. Then from (37) for β satisfying (41) we have

$$R(\bar{\lambda}, U^*_{\beta,\varrho\beta}) \leqslant Z^{(2)}(\beta, \varrho)\lambda + Z^{(3)}(\beta, \varrho) \stackrel{\mathrm{df}}{=} R_3(\lambda, \varrho),$$

and it follows from (37) that this function is convex with respect to ϱ for fixed λ, is concave with respect to λ for fixed ϱ, and the infimum $\inf_{\varrho \in \Lambda} R_3(\lambda, \varrho)$

is attained at $\lambda = \varrho$. Moreover, we get

$$R_3(\varrho_0, \varrho_0) = \lim_{\varrho \to \varrho_0} r(\pi_{\beta,\varrho\beta}, U^*_{\beta,\varrho\beta}),$$

which can be verified directly from the equations for the functions R_3 and r. Then the control strategy $U^*_{\beta,\varrho_0\beta}$, where

$$R_3(\varrho_0, \varrho_0) = \max_{\varrho \in \Lambda} R_3(\varrho, \varrho),$$

is minimax. This proves part III of the Theorem.

Notice that a minimax control strategy exists for all natural k, l and s in (1) when $m = 1$.

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW

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