

ON THE TOTAL POSITIVITY
OF THE TRUNCATED POWER KERNEL

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A kernel \( K(x, t) \) is said to be totally positive on \( X \times T \) if there is an \( \varepsilon = +1 \) or \( \varepsilon = -1 \) such that

(1) \[ \varepsilon \det \{ K(x_i, t_j) \}_{i=1}^{N} \geq 0 \]

for each choice of the points \( x_1 < \ldots < x_N \) in \( X \) and \( t_1 < \ldots < t_N \) in \( T \).

The truncated power kernel

\[ (x - t)^{r - 1}_{+} := \begin{cases} (x - t)^{r - 1} & \text{if } x - t \geq 0, \\ 0 & \text{if } x - t < 0, \end{cases} \]

is totally positive on \( X \times T \) for each \( X, T \subset \mathbb{R} \) (see Karlin [4]). This fact plays a fundamental role in the theory of spline functions. The purpose of this note is to show that the relation (1) remains true in a more general setting involving Birkhoff type matrices \( \{ K(x_i, t_j) \} \).

1. Preliminaries. B-splines with Birkhoff knots. Consider a pair \((x, E)\) with \( x = (x_i)_{i=1}^m, x_1 < \ldots < x_m \), and with an incidence matrix \( E = (e_{ij})_{i=1}^m, j=0 \). Denote by \( |E| \) the number of 1-entries in \( E \).

We shall say that the pair \((x, E)\) is regular (respectively, s-regular) if:

(i) \( E \) is conservative;
(ii) \( E \) satisfies the Pólya condition (respectively, the strong Pólya condition).

All notions used above are well known in the theory of Birkhoff interpolation (see [5] for details).

Let \((x, E)\) be a regular pair with \( |E| = r + 1 \). Then, by the Atkinson–Sharma theorem [1], the Birkhoff interpolation problem

(2) \[ p^{(j)}(x_i) = f^{(j)}(x_i) \quad \text{if} \quad e_{ij} = 1 \]

has a unique solution \( p_f \in \pi_r \) (\( \pi_n \) denotes the set of all algebraic polynomials of degree \( n \)). Equivalently, there exists a unique linear functional

\[ D[(x, E); f] = \sum_{e_{ii} = 1} c_{ij} f^{(j)}(x_i) \]
satisfying the conditions:

\[ D[(x, E); f] = 0 \quad \text{for} \quad f(x) = x^k, \quad k = 0, \ldots, r - 1, \]
\[ D[(x, E); f] = 1 \quad \text{for} \quad f(x) = x^r. \]

\( D[(x, E); f] \) is called the divided difference of \( f \) at \((x, E)\). Note that \( D[(x, E); f] \) coincides with the coefficient of \( x^r \) in the polynomial \( p_f \) which interpolates \( f \) at \((x, E)\), i.e., which satisfies (2).

**Lemma 1.** Suppose that the pair \((x, E)\) is regular, \( x = (x_1, \ldots, x_m) \), \( E = (e_{ij})_{i=1, i=j=0}^m \) and \( |E| = r + 1 \). Then \( c_{m, \lambda} > 0 \), where \( \lambda \) is the order of the highest derivative of \( f \) at \( x_m \), appearing in the expression \( D[(x, E); f] \).

**Proof.** Let \( \varphi \) be the polynomial from \( \pi_r \) that satisfies the interpolation conditions

\[ \varphi^{(j)}(x_i) = \delta_{im} \delta_{j \lambda} \quad \text{if} \quad e_{ij} = 1. \]

Then

\[ c_{m, \lambda} = D[(x, E); \varphi]. \]

On the other hand, \( D[(x, E); \varphi] \) is the coefficient \( C \) of \( x^r \) in the polynomial \( \varphi \). Thus

\begin{equation}
\text{sign} \ c_{m, \lambda} = \text{sign} \ C = \text{sign} \ \varphi^{(r)}(x_m).
\end{equation}

Now a very careful study of the behaviour of the sign changes in the sequence \( \varphi(x), \varphi'(x), \ldots, \varphi^{(r)}(x) \) when \( x \) runs from \( a := x_1 \) to \( b := x_m \) shows that \( \varphi^{(r)}(b), \ldots, \varphi^{(r)}(b) \) does not contain a sign change. Therefore \( \text{sign} \ \varphi^{(r)}(b) = \text{sign} \ \varphi^{(r)}(b) = 1 \), which, in view of (3), completes the proof.

For regular \((x, E)\) with \( |E| = r + 1 \), the function

\[ B[(x, E); t] := D[(x, E); ( \cdot, -t)^{r-1}] \]

is said to be a \( B\)-spline of degree \( r - 1 \) with knots \((x, E)\). This natural extension of the original Curry–Schoenberg \( B\)-splines was introduced and studied in [2]. Many of the crucial properties of the extended \( B\)-splines \( B[(x, E); t] \) were proved there. It is known, for instance, that \( B[(x, E); t] \) has a finite support and does not change sign on \( \mathbb{R} \). This could be derived from a general theorem about the number of zeros of polynomial splines with Birkhoff knots (see Theorem 7.13 in [5]). We give here a new, simple direct proof of this fact.

**Proposition 1.** Let a pair \((x, E)\) be \( s\)-regular and \( |E| = r + 1 \). Then

\begin{align}
(5) & \quad B[(x, E); t] = 0 \quad \text{for} \quad t \notin [x_1, x_m], \\
(6) & \quad B[(x, E); t] > 0 \quad \text{for} \quad t \in (x_1, x_m). 
\end{align}
Proof. The equality (5) is clear since the function \( g(x) := (x - t)^{r-1}_t \) vanishes on \((x_1, x_m)\) for \( t > x_m \) and \( g \) coincides on \((x_1, x_m)\) with the polynomial \((x - t)^{r-1}\) for each fixed \( t < x_1 \).

Let us prove (6). According to the remark after the definition of the divided difference, \( B(t) := B[(x, E); t] \) is the coefficient of \( x^r \) in the polynomial \( p \in \pi_r \) which interpolates \( g \) at \((x, E)\). Note that \( p \not\equiv 0 \) and \( p \not\equiv g \) in \((x_1, x_m)\), and consequently, in any subinterval of \((x_1, x_m)\). Therefore \( p(x) - g(x) \) has only isolated zeros in \((x_1, x_m)\). Since \( p(x) - g(x) \) vanishes at \((x, E)\), we see by Rolle’s theorem and the \( s \)-regularity assumption that \( p^{(r-1)}(x) - g^{(r-1)}(x) \) must have at least two sign changes in \((x_1, x_m)\). This is possible only if \( p^{(r-1)}(x) \) is an increasing linear function, i.e., if \( p \) has positive leading coefficient \( B(t) \). This completes the proof.

Our further considerations are based on the total positivity of a certain matrix of the form \( \{B[(x_i, E_i); t_j]\} \). In order to formulate the result we need some definitions.

Given an integer \( r > 0 \) and a pair \((x, E)\) such that \( x_1 < \ldots < x_m \), \( E = (e_{ij})_{i=1, j=0}^{m, r-1}, |E| = r + N \), we defined in [2] the \((r + 1)\)-partition of \((x, E)\) to be a sequence of pairs \( \{(x_i, E_i)\}_{i=1}^N \) obtained from \((x, E)\) in the following way. Order the elements of \( E \) row by row, i.e., in the manner \( e_{10}, \ldots, e_{1,r-1}, \ldots, e_{m0}, \ldots, e_{mr-1} \) and number the 1-entries in this sequence from 1 to \( r + N \). Let \( e_p, e_{p+1}, \ldots, e_q \) be the rows of \( E \) which contain \( r + 1 \) consecutive 1-entries starting from the \( i \)-th one. Suppose that the first row \( e_p \) (respectively, the last row \( e_q \)) contains \( n_1 \) (respectively, \( n_2 \)) \( 1 \)-entries of this \((r + 1)\)-sample. We denote by \( E_i \) the matrix \( \{e_p, \ldots, e_q\} \) in which all 1’s in the sequence \( e_{p0}, \ldots, e_{pr-1} \) (respectively, in \( e_{q0}, \ldots, e_{qr-1} \)) except the first \( n_1 \) (respectively, \( n_2 \)) are replaced by 0. Finally, define \( x_i := (x_p, \ldots, x_q) \).

We say that the \((r + 1)\)-partition \( \{(x_i, E_i)\} \) of \((x, E)\) is \( s \)-regular if each \((x_i, E_i)\) is \( s \)-regular.

Proposition 2. Let \( x = (x_0, \ldots, x_{m+1}) \), \( x_0 < x_1 < \ldots < x_{m+1} \), \( E = (e_{ij})_{i=0, j=0}^{m+1, r-1} \) and \(|E| = r + N\). Suppose that the \((r + 1)\)-partition \( \{(x_k, E_k)\}_{k=1}^N \) of \((x, E)\) is \( s \)-regular. Then

\[
\Delta := \det\{B_k(\tau_j)\}_{k=1}^N, j=1^N \geq 0
\]

for any \( \tau_1 \leq \ldots \leq \tau_N \) satisfying \( \tau_j < \tau_{j+r}, j = 1, \ldots, N - r \), where \( B_k := B[(x_k, E_k); \cdot] \). Moreover, \( \Delta \) is positive if and only if

\[
\tau_k \in \text{supp } B_k, \quad k = 1, \ldots, N.
\]

This extension of the Schoenberg–Whitney theorem (see [6]) was proved in [2].
2. **Main result.** A spline function of degree \( r - 1 \) with knots \( \xi_1 < \ldots < \xi_n \) of respective multiplicities \( \nu_1, \ldots, \nu_n \) is any expression of the form

\[
s(x) = p(x) + \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} a_{k\lambda} (x - \xi_k)^{r-\lambda-1}
\]

where \( \{a_{k\lambda}\} \) are real constants and \( p \in \pi_{r-1} \).

Let \((x, E)\) be a given pair with \( x = (x_0, \ldots, x_{m+1}) \), \( a = x_0 < x_1 < \ldots < x_{m+1} = b \), and with an incidence matrix \( E = (e_{ij})_{i=0}^{m+1} {r-1}_{j=0} \) such that \( |E| = r + N \). Consider the Birkhoff interpolation problem

\[
s^{(j)}(x_i) = f_{ij} \quad \text{if} \quad e_{ij} = 1,
\]

where \( \{f_{ij}\} \) are given values and \( p(x) \) is written in the form

\[
p(x) = a_0 + a_1(x-a) + \ldots + a_{r-1}(x-a)^{r-1}.
\]

In what follows we define \( s^{(j)}(x) \) as \( s^{(j)}(x+0) \) in case \( s^{(j)} \) is discontinuous at \( x \). Denote by \( V = V[(x, E), (\xi, \nu)] \) the matrix of the system (8) with respect to the unknowns

\[
a_0, \ldots, a_{r-1}, a_{10}, \ldots, a_1, \nu_1-1, \ldots, a_{n0}, \ldots, a_n, \nu_n-1.
\]

We shall show that

\[
\varepsilon \det V[(x, E), (\xi, \nu)] \geq 0
\]

for each \( x \) and \( \xi \) with some \( \varepsilon = (-1)^{\sigma} \) where \( \sigma \) depends only on the structure of \( E \). In a fairly general situation, including quasi-Hermitian \( E \), we find the explicit value of \( \sigma \) and thus provide a new proof of a fundamental result of S. Karlin [4].

We start with an auxiliary lemma.

Denote, for simplicity, by

\[
\begin{bmatrix}
\{u_1(t), \ldots, u_n(t)\}^{(j)}_{t=t_i} \\
e_{ij} = 1, \quad e_{ij} \in E
\end{bmatrix}
\]

the matrix consisting of the rows

\[
u_1^{(j)}(t_i), \ldots, u_n^{(j)}(t_i)
\]

ordered according to the position of the 1-entries \( e_{ij} \) in the sequence of consecutive rows of the incidence matrix \( E = (e_{ij}) \).

**Lemma 2.** Let \((y, G)\) be a given pair with \( y = (y_1, \ldots, y_k) \) and with an incidence matrix \( G = (g_{ij})^r_{i=1, j=0} \) such that \( |G| = r \). Let

\[
A = \begin{bmatrix}
\{1, x-a, \ldots, (x-a)^{r-1}\}^{(j)}_{x=y_i} \\
g_{ij} = 1, \quad g_{ij} \in G
\end{bmatrix}.
\]
Suppose that \((y, E)\) is a regular pair. Then there is a positive integer \(\sigma\) depending only on \(G\) such that
\[
(-1)^{\sigma} \det A > 0
\]
for each \(a \leq y_1 < \ldots < y_k\). Moreover, if the 1-entries of \(g_i := (g_{i0}, \ldots, g_{i,r-1})\) remain in the lowest \(|g_i|\) positions of the 1-entries in the coalescence of \(g_i\) and \(g_{i+1}\), for \(i = 1, \ldots, k - 1\), then \(\sigma = 0\), and if this holds for \(i = 2, \ldots, k - 1\), then
\[
\sigma = \frac{p(p+1)}{2} + i_1 + \ldots + i_p,
\]
where \(i_1, \ldots, i_p\) are the positions of 1's in \(g_i\).

\textbf{Proof.} Since \((y, G)\) is a regular pair, by the Atkinson-Sharma theorem [1], the interpolation problem
\[
\{a_0 + a_1(x - a) + \ldots + a_{r-1}(x - a)^{r-1}\}_{|x=y_i} = 0 \quad \text{if} \quad g_{ij} = 1
\]
has a unique solution. Thus \(\det A \neq 0\) for each \(a \leq y_1 < \ldots < y_k\). One can even find the sign of \(\det A\). In order to do this, note that for fixed \(y_1, \ldots, y_{k-1}\), \(\det A\) is a polynomial function of \(x := y_k - y_{k-1}\). Denote this function by \(A_k(x)\). By Taylor's formula,
\[
A_k(x) = \sum_{j=0} A_k^{(j)}(0)x^j/j!.
\]
Let \(A_k^{(\lambda)}(0)\) be the first nonzero coefficient in (9). It is not difficult to see that \(A_k^{(\lambda)}(0)\) is equal (up to a positive integer factor) to a determinant \(A_{k-1}\) that is obtained from \(A_k(x)\) by replacing its last \(n := |g_k|\) rows with rows of the form
\[
\{1, x - a, \ldots, (x - a)^{r-1}\}_{|x=y_{k-1}}^{(j)}
\]
for \(j = j_1, \ldots, j_n\), where \(j_1, \ldots, j_n\) are the positions of the first \(n\) 0-entries in the sequence \((g_{k-1,\mu}, \ldots, g_{k-1,r-1})\), \(\mu\) being the position of the first 1-entry in \(g_k\). Clearly,
\[
\text{sign } A_k(x) = \text{sign } A_{k-1}
\]
for sufficiently small \(x > 0\).

Now \(A_{k-1}\) is a determinant corresponding to \(y_1 < \ldots < y_{k-1}\) and an incidence matrix \(G_{k-1}\) which is obtained from \(G\) by coalescence of the last two rows \(g_{k-1}\) and \(g_k\).

Repeating this procedure with respect to \(A_{k-1}\) we get \(A_{k-2}\), and so on. Finally, we come to the relation
\[
\text{sign } A_k(x) = \text{sign } A_1,
\]
where \(A_1\) is a Taylor matrix
\[
\begin{bmatrix}
\{1, x - a, \ldots, (x - a)^{r-1}\}_{|x=a}^{(j)} \\
\quad j = j_0, \ldots, j_{r-1}
\end{bmatrix}
\]
with \((j_0, \ldots, j_{r-1})\) a certain permutation of \((0, \ldots, r - 1)\). Thus
\[
\text{sign} \ det \ A = (-1)^\sigma,
\]
where \(\sigma\) is the number of transpositions needed to rearrange the numbers \((j_0, \ldots, j_{r-1})\) in the natural order.

It is easily seen that \(\sigma = 0\), i.e., \((j_0, \ldots, j_{r-1}) = (0, \ldots, r - 1)\), if the assumption of the lemma holds for \(i = 1, \ldots, k - 1\). For example, this clearly holds if \(g_i\) contains only one block \(\beta_i := [g_{i,l}, \ldots, g_{i,l+q}]\) of 1-entries \((l\) is the level of \(\beta_i)\) for \(i = 1, \ldots, k\) and the level increases or remains the same when \(i\) increases. This condition holds for Hermitian matrices \(G\).

Another particular case: if the previous assumption holds for \(i = 2, \ldots, k\) and \(i_1, \ldots, i_p\) are the positions of the 1-entries in \(g_1\), then
\[
(j_0, \ldots, j_{r-1}) \equiv (i_1, \ldots, i_p, k_1, \ldots, k_{r-p}),
\]
where \(k_1 < \ldots < k_{r-p}\) and thus
\[
\sigma = (i_1 - 1) + (i_2 - 2) + \ldots + (i_p - p) = p(p + 1)/2 + i_1 + \ldots + i_p.
\]
The lemma is proved.

**Theorem 1.** Let \(x = (x_0, x_1, \ldots, x_{m+1}), a = x_0 < x_1 < \ldots < x_{m+1} = b, E = (e_{ij})_{i=0}^{m+1} E_{j=0}^{r-1}\) and \(|E| = r + N\). Suppose that \(\{(x_k, E_k)\}_{k=1}^{N} \) is an s-regular \((r + 1)\)-partition of \((x, E)\). Then there is a \(\sigma\), depending only on \(E\), such that
\[
(-1)^\sigma \ det \ V[(x, E), (\xi, \nu)] \geq 0
\]
for each choice of the set
\[
\xi = (\tau_1, \ldots, \tau_N) \equiv ((\xi_1, \nu_1), \ldots, (\xi_n, \nu_n))
\]
of points \(\xi_1 < \ldots < \xi_n\) with respective multiplicities \(\nu_1, \ldots, \nu_n\) such that \(1 \leq \nu_i \leq r, i = 1, \ldots, n; \nu_1 + \ldots + \nu_n = N\). Moreover,
\[
(-1)^\sigma \ det \ V[(x, E), (\xi, \nu)] > 0
\]
if and only if \(\tau_i \in \text{supp} \ B[(x_i, E_i); t], i = 1, \ldots, N\).

**Proof.** Clearly the matrix \(V[(x, E), (\xi, \nu)]\) consists of the rows
\[
w_{ij} := \{1, (x - a), \ldots, (x - a)^{r-1}, K(x, \xi_1), \ldots, K(\nu_1 - 1)(x, \xi_n)\}_{x=x_i}^{(j)}
\]
where \((i, j)\) runs over the indices of all 1-entries \(e_{ij}\) in the sequence
\[
e_{00}, \ldots, e_{0,r-1}, \ldots, e_{10}, \ldots, e_{1,r-1}, \ldots, e_{m+1,0}, \ldots, e_{m+1,r-1}
\]
and \(K(x, t) := (x - t)^{r-1}, K(j)(x, t) := (\partial^j / \partial t^j)K(x, t)\). In order to find \(\det V\) we shall perform some elementary transformations in \(V\), writing in row \(r + k\) \((k = 1, \ldots, N)\) a linear combination of rows
\[
v_{r+k} := \sum_{e_{ij} = 1} c_{ij}w_{ij}
\]
where the sum is over the 1-entries of $E_k$ and $\{c_{ij}\}$ are the coefficients in the divided difference

$$D[(x_k, E_k); f] = \sum_{e_{ij} = 1} c_{ij} f^{(j)}(x_i).$$

Denote by $\alpha_k$ the coefficient of the highest derivative at the last point of $x_k$ appearing in $D[(x_k, E_k); f]$. According to Lemma 1,

$$\alpha_k > 0.$$  \hfill (10)

Denote by $V_0$ the matrix obtained from $V$ by the described transformation of rows $r + 1, \ldots, r + N$. Clearly $\det V = \alpha \det V_0$ with $\alpha := 1 / (\alpha_1 \ldots \alpha_N)$, and the $(r + k)$th row of $V$ is of the form

$$v_{r+k}^0 := \{D_k[1], \ldots, D_k[(x-a)^{r-1}], D_k[K(x, \xi_1)], \ldots, D_k[K^{(\nu-1)}(x, \xi_n)]\}$$

where $D_k := D[(x_k, E_k); \cdot]$. Using the property $D_k[f] = 0$ for all $f \in \pi_{r-1}$ and the definition of $B$-splines we see that

$$v_{r+k}^0 = \{0, \ldots, 0, B_k(\xi_1), \ldots, B_k^{(\nu-1)}(\xi_n)\}.$$  

Let $i_1, \ldots, i_p$ be the positions of the 1-entries in $(e_{00}, \ldots, e_{0,r-1})$. Then, by the Laplace formula,

$$\det V = \alpha \det A \cdot \det\{B_k(\tau_j)\}_{k=1}^N_{j=1}$$

where

$$A = \begin{bmatrix}
\{1, (x-a)_{r-1}, (x-a)^{(j)}|_{x=x_i}\} & e_{ij} = 1, & e_{ij} \in E_0
\end{bmatrix}$$

and $E_0$ is obtained from $E_1$ by replacing the last 1-entry (i.e., the last 1 in the last row of $E_1$) by 0. Since $E_1$ was assumed to be $s$-regular, $E_0$ is regular. Then, by Lemma 2, $\det A \neq 0$. Further, by Proposition 2,

$$\Delta := \det\{B_k(\tau_j)\}_{k=1}^N_{j=1} \geq 0$$

and strict inequality holds if and only if $\tau_k \in \supp B_k$, $k = 1, \ldots, N$. Therefore, in view of (10) and (11), $\det V \neq 0$ if and only if $\Delta \neq 0$, and

$$\text{sign } \det V = \text{sign } \det A.$$  \hfill (12)

The theorem is proved.

Next we derive Karlin's total positivity theorem as a particular case of Theorem 1.

**Corollary 1.** Let $(x, E)$ be any pair with $x = (x_0, x_1, \ldots, x_{m+1})$, $a = x_0 < x_1 < \ldots < x_{m+1} = b$, and with a quasi-Hermitian incidence matrix $E = (e_{ij})_{i=0}^{m+1}_{j=0}$ such that $|E| = r + N$. Suppose that $\{(x_k, E_k)\}_{k=1}^N$ is an
s-regular \((r+1)\)-partition of \((x,E)\). Then
\[
\det \left[ \{1,(x-a),\ldots,(x-a)^{r-1},K(x,\xi_1),\ldots,K^{(\nu_n-1)}(x,\xi_n)\}_{j}\big|_{x=x_i} \right]_{e_{ij}} = 1, \quad e_{ij} \in \hat{E}
\]
for each choice of points \(\xi_1 < \ldots < \xi_n\) with respective multiplicities \(\nu_1,\ldots,\nu_n\) such that \(1 \leq \nu_i \leq r, i=1,\ldots,n, \nu_1 + \ldots + \nu_n = N\). Here \(\hat{E}\) is the matrix obtained from \(E\) by replacing the first \(r\) 1-entries by 0 (i.e., annihilating the matrix \(E_0\)). Strict inequality holds if and only if
\[
\tau_i \in \text{supp } B[(x_i,E_i);t], \quad i=1,\ldots,N.
\]

Proof. Denote the determinant considered by \(W\). Clearly, up to a positive constant,
\[
W = (-1)^\sigma \det V = (-1)^\sigma \det A \cdot \det \{B_k(\tau_j)\},
\]
where \(\sigma\) and \(A\) are as in the theorem. Since \(E_1\) is quasi-Hermitian, sign \(\det A = (-1)^\sigma\) and the assertion follows from Theorem 1.

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