ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 4 (1985), p.549-557

Z. POROSIŃSKI (Wrocław)

OPTIMAL SELECTION OF THE MAXIMUM OF A DISCOUNTABLE SEQUENCE OF INDEPENDENT RANDOM VARIABLES

0. Introduction. Suppose that we observe the realization of the sequence of random variables $\xi_1, \xi_2, \xi_3, \ldots$ and we want to stop the observation at the moment in which the realization achieves the maximal value in the set of all realizations with maximal probability. This problem for a finite sequence of independent identically distributed random variables with a continuous distribution function was posed by Gilbert and Mosteller [3], and they solved it by a heuristic argument. Bojdecki [1] has confirmed this result and he solved also an optimal stopping problem for an infinite sequence with "costs of experiments". He has considered $\xi_n = X_n - cn$ with $X_1, X_2, X_3, \ldots$ being a sequence of independent identically distributed random variables with a continuous distribution function and c (the cost of one experiment) as a fixed positive number.

This paper contains the solutions of the problems of seeking with maximal probability the maximal value of a finite or infinite sequence of independent identically distributed random variables with a continuous distribution function which is discounted by a non-increasing sequence of positive numbers (for a precise formulation see Section 1).

The following situation is considered:

$$\xi_n = c_n X_n \quad \text{or} \quad \xi_n = c_n \max(X_1, \ldots, X_n)$$

and $X_1, X_2, X_3, \ldots$ are interpreted as consecutive results of some experiment. We want to obtain the possibly largest result of the sequence $(X_n)_{n\in N}$ but we take into account also various restrictions (for example: limit of time, costs of experiments) which discourage from a continuation of the observation. The sequence of discounts $(c_n)_{n\in N}$ takes into consideration all those restrictions.

The above problems are reduced to the classical optimal stopping problems for some Markov chains with some reward functions ([4]). It is proved that optimal stopping rules in each case exist and their forms are found.

1. Formulation of problems. Assume that

(1) $X_1, X_2, X_3, \ldots$ is a sequence of independent identically distributed random variables with a continuous distribution function F, defined on the probability space $(\Omega, \mathcal{F}, P)$,

and

(2) $$P(X_1 > 0) > 0.$$

Let be given a numerical sequence $(c_n)_{n=1}^{\infty}$ such that

(3) $$0 < c_{n+1} \leqslant c_n \leqslant 1 \quad \text{for} \quad n \in N.$$

Define $\xi_n = c_n X_n$ for $n \in N$.

Let $\mathcal{F}_n$ be the σ-field of events generated by $\xi_1, \xi_2, \ldots, \xi_n$ (naturally $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$) and let $\mathfrak{M}$ be the set of all stopping times with respect to the family $(\mathcal{F}_n)_{n=1}^{\infty}$.

Consider two problems: the first one:

(S_N) Find a stopping time $\tau^* \in \mathfrak{M}$ such that

$$P(\tau^* \leqslant N; \xi_{\tau^*} = \max_{k \leqslant N} \xi_k) = \sup_{\tau \in \mathfrak{M}} P(\tau \leqslant N; \xi_\tau = \max_{k \leqslant N} \xi_k)$$

where N is a fixed positive integer number,

and the second problem:

(S_∞) Find a stopping time $\tau^* \in \mathfrak{M}$ such that

$$P(\tau^* < \infty; \xi_{\tau^*} = \max_{k} \xi_k) = \sup_{\tau \in \mathfrak{M}} P(\tau < \infty; \xi_\tau = \max_{k} \xi_k)$$

under additional assumptions

(4) $$\mathrm{E}|X_1|^p < \infty,$$

(5) $$\sum_{k=1}^{\infty} c_k^p < \infty,$$

for a certain $p \in N$.

The problem (S_∞) has sense because of the following lemma:

LEMMA 1. *If conditions* (1)–(5) *hold then the relations*

(a) $\lim_{n \to \infty} \xi_n = 0$,

(b) *there exists* $k \in N$ *such that* $\xi_k > 0$,

are fulfilled almost surely (*a.s.*).

Proof. For each $\varepsilon > 0$ we have

$$P\Big(\bigcup_{k=n}^{\infty}\{|\xi_k| > \varepsilon\}\Big) \leqslant \sum_{k=n}^{\infty} P(|\xi_k| > \varepsilon) = \sum_{k=n}^{\infty} P(|X_1| > \varepsilon/c_k) = \frac{\mathrm{E}|X_1|^p}{\varepsilon^p}\sum_{k=n}^{\infty} c_k^p \to 0$$

if $n \to \infty$. Thus, (a) is true. Moreover

$$P(\exists\, k\in N;\ \xi_k > 0) = P\Big(\bigcup_{k=1}^{\infty}\{X_k > 0\}\Big) = \lim_{n\to\infty} P\Big(\bigcup_{k=1}^{n}\{X_k > 0\}\Big)$$

$$= 1 - \lim_{n\to\infty} P\Big(\bigcap_{k=1}^{n}\{X_k \leqslant 0\}\Big) = 1 - \lim_{n\to\infty}[F(0)]^n = 1.$$

The lemma is proved.

In order to apply the method used by Bojdecki ([1]) we define $\xi_n = \xi_N$ for $n > N$. We consider the problem:

(S'_N) Find a stopping time $\tau^* \in \mathfrak{M}$ such that

$$P(\tau^* < \infty,\ \xi_{\tau^*} = \max_k \xi_k) = \sup_{\tau\in\mathfrak{M}} P(\tau < \infty,\ \xi_\tau = \max_k \xi_k),$$

taking the infinite sequence $(\xi_n)_{n\in N}$ defined above.

Naturally, the solutions of the problems (S_N) and (S'_N) are identical. Note also that the problem:

(S'_∞) Find a stopping time $\tau^* \in \mathfrak{M}$ which realizes

$$\sup_{\tau\in\mathfrak{M}} P\big(\tau < \infty,\ c_\tau \max_{k\leqslant\tau} X_k = \sup_n (c_n \max_{k\leqslant n} X_k)\big)$$

(therefore the situation when we have all the "past" results of the experiment) is equivalent to problem (S_∞) because $\sup_n(c_n X_n) = \sup_n(c_n \max_{k\leqslant n} X_k)$. Analogously as above we can define a new problem equivalent to (S_N).

2. Reduction of problems. Denote

$$Z_n = P(\xi_n = \max_k \xi_k | \mathscr{F}_n) \quad \text{for} \quad n\in N, \quad Z_\infty = 0.$$

Thus, we have

$$Z_n = I_{\{\xi_n = \max_{k\leqslant n}\xi_k\}}\, P(\xi_n \geqslant \max_{k>n} \xi_k | \mathscr{F}_n) \stackrel{\mathrm{df}}{=} I_{\{\xi_n = \max_{k\leqslant n}\xi_k\}} \cdot W_n, \tag{6}$$

where I_A denotes the indicator function of the event A.

If we denote $N^* = N$ for the problem (S_N) and $N^* = +\infty$ for (S_∞) then

$$W_n = P(\xi_n \geqslant \sup_{n<k\leqslant N^*} \xi_k | \mathscr{F}_n) = \prod_{k=n+1}^{N^*} F(\xi_n/c_k) \quad \text{for} \quad n < N^*,$$

$$W_n = 1 \quad \text{for} \quad n \geqslant N^*,$$

for both problems.

It suffices to consider stopping times belonging to the set

$$\mathfrak{M}_0 = \{\tau \in \mathfrak{M};\ \tau = n \Rightarrow \xi_n = \max_{k \leqslant n} \xi_k,\ n \in N\}.$$

This is a consequence of the following

LEMMA 2 ([2], [1]). *For every* $\tau \in \mathfrak{M}$ *there exists* $\tau' \in \mathfrak{M}_0$ *such that*

$$P(\tau < \infty,\ \xi_\tau = \max_k \xi_k) \leqslant P(\tau' < \infty,\ \xi_{\tau'} = \max_k \xi_k).$$

Let now $\tau_1 = 1$, $\tau_{j+1} = \inf\{n;\ n > \tau_j,\ \xi_n \geqslant \xi_{\tau_j}\}$ for $j \in N$. Naturally, $\tau_j \in \mathfrak{M}_0$ and $\tau_j \leqslant \infty$.

We define the sequence of random variables

$$Y_j = \begin{cases} (\tau_j, \xi_{\tau_j}) & \text{for} \quad \tau_j < \infty, \\ \partial & \text{for} \quad \tau_j = \infty, \end{cases}$$

where ∂ is a label for the final state. $Y = (Y_n)_{n=1}^{\infty}$ is a homogeneous Markov chain with respect to the σ-fields $(\mathscr{F}_{\tau_n})_{n=1}^{\infty}$ with the state space $(N \times R) \cup \{\partial\}$.

For $m \leqslant N$ the assumptions (1) imply that

$$\begin{aligned} & P(Y_{j+1} \in \{m\} \times (-\infty, y] \mid \mathscr{F}_{\tau_j}) \\ & = \sum_{n=1}^{m-1} I_{\{\tau_j = n\}} P(\tau_{j+1} = m,\ \xi_m \leqslant y \mid \mathscr{F}_n) \\ & = \sum_{n=1}^{m-1} P(\tau_j = n,\ \xi_{n+1} < \xi_n, \ldots, \xi_{m-1} < \xi_n,\ \xi_n \leqslant \xi_m \leqslant y \mid \mathscr{F}_n) \\ & = \begin{cases} \prod_{k=n+1}^{m-1} F\left(\frac{\xi_{\tau_j}}{c_k}\right)\left[F\left(\frac{y}{c_m}\right) - F\left(\frac{\xi_{\tau_j}}{c_m}\right)\right], & \text{for} \quad \tau_j < m \text{ and } y > \xi_{\tau_j}, \\ 0, & \text{for} \quad \tau_j \geqslant m \text{ and/or } y \leqslant \xi_{\tau_j}. \end{cases} \end{aligned}$$

Therefore the transition function of the Markov chain Y is given by

$$(7) \quad P(n, x; m, (-\infty, y]) = P(Y_{j+1} = m,\ \xi_m \leqslant y \mid Y_j = n,\ \xi_n = x)$$

$$= \begin{cases} \prod_{k=n+1}^{m-1} F\left(\frac{x}{c_k}\right)\left[F\left(\frac{y}{c_m}\right) - F\left(\frac{x}{c_m}\right)\right], & \text{for} \quad n < m \leqslant N^* \text{ and } y > x, \\ 0, & \text{for} \quad n \geqslant m \text{ and/or } y \leqslant x, \end{cases}$$

where we adopt the convention that $\prod_{k=n+1}^{m-1} F(x/c_k) = 1$ if $m = n+1$. The transition function for other states can be obtained in a similar way (∂ is an absorbing state).

Next, for any $\tau \in \mathfrak{M}_0$ we define

$$\sigma(\omega) = \begin{cases} j, & \text{for} \quad \omega \in \{\tau = \tau_j < \infty\}, \\ \infty, & \text{for} \quad \omega \in \{\tau = \infty\}. \end{cases}$$

Here σ is a stopping time with respect to $(\mathscr{F}_{\tau_j})_{j=1}^{\infty}$ and (6) implies that

$$Z_\tau = \begin{cases} W_{\tau_\sigma}, & \text{for } \tau < \infty \\ 0, & \text{for } \tau = \infty \end{cases} \stackrel{\text{df}}{=} f(Y_\sigma),$$

where

$$f(n, x) = \begin{cases} \prod_{k=n+1}^{N^*} F(x/c_k) & \text{for } n < N^*, \\ 1 & \text{for } n \geqslant N^*, \end{cases} \tag{8}$$

and $f(\partial) = 0$.

Thus we reduce the initial problems (S_N) and (S_∞) to the problem of optimal stopping of the Markov chain Y with the reward function f. To solve these problems we use the lemma proved by Cowan and Zabczyk [2].

3. Solutions of problems. Let $Y = (Y_n)_{n=1}^{\infty}$ be a homogeneous Markov chain on $(\Omega, \mathscr{F}, P)$ with state space $(\boldsymbol{E}, \mathscr{B})$ and let $p(\cdot;\cdot)$ denote the transition function, i.e. $p(y; B) = P(Y_{n+1} \in B \mid Y_n = y)$ for $B \in \mathscr{B}$. Let $h\colon \boldsymbol{E} \to \boldsymbol{R}$ be a bounded function. Define

$$\boldsymbol{P}h(y) = \int_{\boldsymbol{E}} h(x)\, p(y; dx), \tag{9}$$

$$\boldsymbol{\Gamma} = \{y \in \boldsymbol{E};\ \boldsymbol{P}f(y) \leqslant f(y)\}. \tag{10}$$

LEMMA 3 ([2], [1]). *If*

$$p(y; \boldsymbol{\Gamma}) = 1 \quad \textit{for} \quad y \in \boldsymbol{\Gamma}, \tag{11}$$

$$\sigma_\Gamma \stackrel{\text{df}}{=} \inf\{n;\ Y_n \in \boldsymbol{\Gamma}\} < +\infty \ \textit{a.s.}, \tag{12}$$

then σ_Γ *is the optimal stopping time for stopping the Markov chain* Y *with the reward function* f, *i.e. the expected value* $\mathrm{E}(f(Y_\sigma))$ *is maximal for the stopping time* $\sigma = \sigma_\Gamma$.

Now we prove the following theorem giving the solutions of our problems.

THEOREM. *Under the assumptions* (1)–(3) *there exists a solution of the problem* (S_N) *which has the form*

$$\tau^* = \inf\{n \leqslant N;\ \xi_n = \max_{k \leqslant n} \xi_k,\ \xi_n \geqslant x_n\}, \tag{13}$$

where $x_N = 0$ *and* x_n, $n < N$, *is the least root of the equation*

$$\sum_{m=n+1}^{N} \left(\prod_{k=m}^{N} F\left(\frac{x}{c_k}\right)\right)^{-1} \int_{[x/c_m, +\infty)} \prod_{k=m+1}^{N} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy) = 1. \tag{14}$$

Under the assumptions (1)–(5) *there exists a solution of the problem* (S_∞) (*therefore* (S'_∞), *too*) *which has the form*

$$\tau^* = \inf\{n \in \mathbf{N};\ \xi_n = \max_{k \leqslant n} \xi_k,\ \xi_n \geqslant x_n\}, \tag{15}$$

where x_n *is the least root of the equation*

$$\sum_{m=n+1}^{\infty} \left(\prod_{k=m}^{\infty} F\left(\frac{x}{c_k}\right)\right)^{-1} \int\limits_{[x/c_m, +\infty)} \prod_{k=m+1}^{\infty} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy) = 1. \tag{16}$$

Proof. In order to apply Lemma 3 we have to describe the set $\boldsymbol{\Gamma}$ defined by (10). Taking advantage of (9) we have $\boldsymbol{P}f(\partial) = 0 = f(\partial)$, therefore $\partial \in \boldsymbol{\Gamma}$. For the problem ($S_N$) the equality (8) implies that $\boldsymbol{P}f(n, x) = 1 = f(n, x)$, for $n \geqslant N$, therefore $\{N, N+1, \ldots\} \times \boldsymbol{R} \subset \boldsymbol{\Gamma}$. For $n < N^*$ the equality (7) implies

$$\begin{aligned} \boldsymbol{P}f(n, x) &= \sum_m \int\limits_{\boldsymbol{R}} f(m, y)\, p(n, x; m, dy) \\ &= \sum_{m=n+1}^{N^*} \prod_{k=n+1}^{m-1} F\left(\frac{x}{c_k}\right) \int\limits_{[x/c_m, +\infty)} f(m, c_m \cdot y) F(dy) \\ &= \sum_{m=n+1}^{N^*} \prod_{k=n+1}^{m-1} F\left(\frac{x}{c_k}\right) \int\limits_{[x/c_m, +\infty)} \prod_{k=m+1}^{N^*} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy). \end{aligned}$$

For $n < N^*$ we consider the inequality (see (10))

$$\sum_{m=n+1}^{N^*} \prod_{k=n+1}^{m-1} F\left(\frac{x}{c_k}\right) \int\limits_{[x/c_m, +\infty)} \prod_{k=m+1}^{N^*} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy) \leqslant \prod_{k=n+1}^{N^*} F\left(\frac{x}{c_k}\right). \tag{17}$$

To transform this inequality we use the following

LEMMA 4. (a) *For each* $n \in \mathbf{N}$ $f(n, x)$ *is positive at least for* x *sufficiently large.*

(b) *If* $f(n, x) = 0$ *then the inequality* (17) *is false.*

Proof. For $N^* = N$ the statement (a) is obvious. For $N^* = +\infty$ it is a consequence of the assumptions (4) and (5). In this case for $x > 0$ we have:

$$\sum_{k=1}^{\infty} P\left(X_1 > \frac{x}{c_k}\right) = \sum_{k=1}^{\infty} \left(1 - F\left(\frac{x}{c_k}\right)\right) < \infty.$$

Thus the product $\prod_{k=n+1}^{\infty} F(x/c_k)$ is convergent (i.e. $f(n, x) > 0$) if only $F(x/c_{n+1}) > 0$. Moreover $f(n, x) = 0$ for $x \leqslant 0$, because the condition (4) implies $F(x/c_k) \leqslant F(0) < 1$ for each $k \in \mathbf{N}$.

To prove statement (b) it suffices to show that if $f(n, x) = 0$ then the first term of the left-hand side of (17) is positive, i.e. $I_{n+1} > 0$, where

$$I_m \stackrel{\text{df}}{=} \int\limits_{[x/c_m, +\infty)} \prod_{k=m+1}^{N^*} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy), \quad \text{for} \quad m < N^*.$$

If $f(n, x) = 0$ then there must be $F(x/c_{N^*}) = 0$ when $x < 0$ and $F(x/c_{n+1}) = 0$ when $x \geqslant 0$ (here and in the sequel we adopt the conventions that $c_\infty = 0$, $-1/0 = -\infty$, $0\cdot(-\infty) = 0$). Hence the distribution of X_1 is concentrated on $[x/c_{N^*}, +\infty)$ when $x < 0$ and on $[x/c_{n+1}, +\infty)$ when $x \geqslant 0$. On the other hand, it follows from (a) that

$$\left\{y > 0;\ \prod_{k=n+2}^{N^*} F\left(\frac{c_{n+1} \cdot y}{c_k}\right) = 0\right\} \subset \{y > 0;\ F(y) = 0\}.$$

This property and the assumption (2) imply that $\prod_{k=n+2}^{N^*} F(c_{n+1} \cdot y/c_k) > 0$ on a certain set of positive measure F. Thus the integral I_{n+1} is positive and the lemma is proved.

From this lemma we have immediately that the inequality (17) may be written as

$$\sum_{m=n+1}^{N^*} \left(\prod_{k=m}^{N^*} F\left(\frac{x}{c_k}\right)\right)^{-1} \int\limits_{[x/c_m, +\infty)} \prod_{k=m+1}^{N^*} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy) \leqslant 1.$$

Denote the left-hand side of this inequality by $h_n(x)$. Let $a = \sup\{x;\ F(x) = 0\}$. Naturally $-\infty \leqslant a < +\infty$. The function $h_n(x)$ is well defined on the interval $(c_{n+1} a, +\infty)$ when $a \geqslant 0$ and on the interval $(c_{N^*} a, +\infty)$ when $a < 0$. It is a continuous (as a sum of a uniformly convergent series of continuous functions) and non-increasing function. Moreover

(a) $\lim\limits_{x \to c_{n+1} a+} h_n(x) = +\infty$ when $a \geqslant 0$,

(b) $\lim\limits_{x \to c_{N^*} a+} h_n(x) = +\infty$ when $a < 0$,

(c) $\lim\limits_{x \to c_{n+1} b-} h_n(x) = 0$, where $b = \inf\{x;\ F(x) = 1\}$. Naturally $0 < b \leqslant +\infty$ by (2).

The statement (a) is valid because the integral I_{n+1} is positive and $\prod_{k=n+1}^{N^*} F(x/c_k) \to 0$ when $x \to c_{n+1} a+$. The statement (b) can be proved analogously as (a); a component for $m = N$ or $m = n+1$ is divergent when $N^* = N$ or $N^* = +\infty$, respectively. To prove the statement (c) it suffices to show that each component is convergent to zero. This is a consequence of the fact that the integral I_m is convergent for each m.

These properties of the function $h_n(x)$ imply that a solution of the equation $h_n(x) = 1$ (therefore (14) when $N^* = N$ and (16) when $N^* = +\infty$) exists. A solution of the inequality $h_n(x) \leqslant 1$ is the half-line $[x_n, +\infty)$ where $x_n = \inf\{x;\ h_n(x) = 1\}$.

In effect we obtain for the problem (S_N)

$$\Gamma = \{\partial\} \cup \bigcup_{n=1}^{N-1} (\{n\} \times [x_n, +\infty)) \cup \bigcup_{n=N}^{\infty} (\{n\} \times \boldsymbol{R}),$$

where x_n is the least root of the equation (14), and for the problem (S_∞)

$$\Gamma = \{\partial\} \cup \bigcup_{n=1}^{\infty} (\{n\} \times [x_n, +\infty)),$$

where x_n is the least root of the equation (16).

These sets satisfy the assumptions of Lemma 3. Indeed, the condition (12) holds because for (S_N) we have

$$\sigma_\Gamma \leqslant \inf\{n;\ Y_n \in \{\partial\} \cup \bigcup_{n=N}^{\infty} (\{n\} \times \boldsymbol{R})\} < +\infty \text{ a.s.}$$

For (S_∞) also

$$\sigma_\Gamma \leqslant \inf\{n;\ Y_n = \partial\} < +\infty \text{ a.s.}$$

To prove (11) it suffices to verify that the sequence $(x_n)_{n=1}^{N^*-1}$ is non-increasing. To this effect note that

$$h_n(x_{n+1}) - h_{n+1}(x_{n+1}) = \left(\prod_{k=n+1}^{N^*} F\left(\frac{x_{n+1}}{c_k}\right)\right)^{-1} \int\limits_{[x_{n+1}/c_{n+1}, +\infty)} \prod_{k=n+2}^{N^*} F\left(\frac{c_m \cdot y}{c_k}\right) F(dy) \geqslant 0$$

for $n < N^* - 1$. Hence $h_n(x_{n+1}) \geqslant h_{n+1}(x_{n+1}) = 1 = h_n(x_n)$ and this implies that $x_{n+1} \leqslant x_n$. Finally, by Lemma 2 and condition (6) we obtain that the solutions of the problems (S_N) and (S_∞) are given by (13) and (15), respectively. The theorem is proved.

4. Example. If the sequence $(c_n)_{n=1}^{N}$ is constant then the condition (2) for the problem (S_N) is superfluous and the solution is identical with the results of Gilbert and Mosteller [3] and Bojdecki [1].

References

[1] T. Bojdecki, *On optimal stopping of a sequence of independent random variables – probability maximizing approach*, Stochastic Process. Appl. 6 (1978), p. 153-163.

[2] R. Cowan and J. Zabczyk, *An optimal selection problem associated with the Poisson process*, Теория вер. и её прим. 23 (1978), p. 606-614.

[3] J. Gilbert and F. Mosteller, *Recognizing the maximum of a sequence*, J. Amer. Statist. Assoc. 61 (1966), p. 35-73.

[4] А. Н. Ширяев, *Статистический последовательный анализ*, Москва 1976.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY WROCŁAW
50-370 WROCŁAW

Received on 20. 7. 1983;

revised version on 16. 3. 1984