BOREL-DENSE BLACKWELL SPACES
ARE STRONGLY BLACKWELL

BY

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0. **Introduction.** In their monograph on Borel structures, Bhaskara Rao and Rao (1981) posed the problem (P4) of whether every Blackwell space is strongly Blackwell. We answer this question in the affirmative for a particular class of Blackwell spaces, namely those Borel-dense (i.e. with totally imperfect complement) in some standard space; in particular, we prove that the constructions given in Orkin (1972) and Section 9 of Bhaskara Rao and Rao (1981) both produce exactly these spaces. Additionally, they are characterised as “Borel-dense of order 2”, as defined below.

For other results on Blackwell spaces, the reader is referred to the works of Maitra (1970), Sarbadhikari (1973) and Ramachandran (1975); the latter gives certain relations with foundational probability.

1. **Preliminaries.** We work exclusively with separable spaces, i.e. measurable spaces $(X, \mathcal{B})$ whose $\sigma$-algebra $\mathcal{B}$ is countably generated (c.g.) and separates points of $X$. Often, the notation of a $\sigma$-algebra is suppressed: the space is called $X$ only, and when needed, its measurable structure is indicated by $\mathcal{B} = \mathcal{B}(X)$. If $\mathcal{C}$ is a sub-$\sigma$-algebra of $\mathcal{B}(X)$, and $A \subset X$, then we use the notations:

$$\mathcal{A}(A) = \{B \cap A: B \in \mathcal{B}(X)\} \quad \text{and} \quad \mathcal{C}(A) = \{C \cap A: C \in \mathcal{C}\}.$$  

A separable space $(S, \mathcal{D})$ is standard if there is a complete separable metric topology on $S$ for which $\mathcal{D}$ is the corresponding Borel structure. If $\mathcal{C}$ and $\mathcal{D}$ are c.g. sub-$\sigma$-algebras of $\mathcal{B}(S)$, then say that $\mathcal{C}$ is proper in $\mathcal{D}$ when:

1) $\mathcal{C} \subset \mathcal{D}$, and

2) there are uncountably many atoms of $\mathcal{C}$ that are not atoms of (i.e. are “split” by) $\mathcal{D}$.

A separable space $X$ is a Blackwell space if whenever $\mathcal{C}$ is a c.g. sub-$\sigma$-algebra of $\mathcal{B}(X)$ that separates points, then $\mathcal{C} = \mathcal{B}(X)$. A separable $X$ is strongly Blackwell if whenever $\mathcal{C} \subset \mathcal{D}$ are c.g. sub-$\sigma$-algebras of $\mathcal{B}(X)$ with
the same atoms, then \( \mathcal{C} = \mathcal{D} \). If \( X \) is a subset of a standard space \( S \), then \( X \) is \((*)\)-Blackwell in \( S \) if whenever \( \mathcal{C} \) is a c.g. sub-\( \sigma \)-algebra of \( \mathcal{B}(S) \) that is proper in \( \mathcal{B}(S) \), then there is some atom \( C \) of \( \mathcal{C} \) such that \( C \cap X \) contains at least two distinct points (i.e. \( \mathcal{C} \) does not separate points of \( X \)). A subset \( X \) is \textit{strongly \((*)\)-Blackwell in} \( S \) if whenever \( \mathcal{C} \) and \( \mathcal{D} \) are c.g. sub-\( \sigma \)-algebras of \( \mathcal{B}(S) \) with \( \mathcal{C} \) proper in \( \mathcal{D} \), then there is some atom \( C \) of \( \mathcal{C} \) and two distinct points in \( C \cap X \) that are separated by \( \mathcal{D} \).

It is not hard to see that the following lattice of implications obtains:

\[
\begin{align*}
\{ X \text{ strongly \((*)\)-Blackwell in } S \} & \supseteq \{ X \text{ strongly Blackwell} \} \\
\{ X \text{ \((*)\)-Blackwell in } S \} & \iff \{ X \text{ Blackwell} \}
\end{align*}
\]


If \( S \) is any set and \( s \in S \), then by a \textit{1-slice of } \( S \times S \text{ over the point } s \) we mean a set of the form \( \{s\} \times S \) or \( S \times \{s\} \); if \( s \) is not specified, then we refer simply to a \textit{1-slice of } \( S \times S \). If \( B \subseteq S \times S \), then by a \textit{1-section of } \( B \) we mean the intersection of \( B \) with a 1-slice of \( S \times S \); if \( C \) is a 1-slice of \( S \times S \) over the point \( s \), then \( B \cap C \) is a \textit{1-section of } \( B \text{ over the point } s \). A 1-section is naturally identified with its one-one projection on one of the \( S \) factors. A subset \( B \) of \( S \times S \) is \textit{symmetric} if \( (s, t) \in B \) implies \( (t, s) \in B \).

Let \( S \) be a standard space; a subset \( X \) of \( S \) is \textit{Borel-dense of order 1 in } \( S \) (or simply \textit{Borel-dense in } \( S \)) if \( S \setminus X \) contains no uncountable members of \( \mathcal{B}(S) \), or what is equivalent, no uncountable analytic sets. Also, \( X \) is \textit{Borel-dense of order 2 in } \( S \) if whenever \( B \subseteq \mathcal{B}(S \times S) \) is a subset of \( (S \times S) \setminus (X \times X) \), then \( B \) is contained in a countable union of 1-slices of \( S \times S \) over points in \( S \setminus X \). It is not hard to see that Borel-density of order 2 implies that of order 1. A more complete study of Borel densities of order \( n \) is (Shortt (1984)).

\textbf{Example 1.} A conventional argument using transfinite induction establishes the existence of a Borel-dense subset \( X \) of the real numbers \( R \) such that \( X \times X \) does not meet the line \( y = -x \) in the plane \( R^2 \). Thus \( X \) is Borel-dense of order 1, but not of order 2.

\textbf{Lemma 1.} Let \( S \) be a standard space; if \( X \subseteq S \) is \((*)\)-Blackwell in \( S \), then \( X \) is Borel-dense (of order 1) in \( S \).

\textbf{Proof.} If \( B \subseteq S \setminus X \) is an uncountable member of \( \mathcal{B}(S) \), then there is an isomorphism \( j \) of \( B \) onto \( B \times B \); let \( f_0 : B \to B \) be the map \( j \) followed by projection onto the first factor of \( B \times B \). Define \( f : S \to S \) by

\[
f(s) = \begin{cases} 
  f_0(s) & \text{for } s \in B, \\
  s & \text{for } s \in S \setminus B
\end{cases}
\]
and put \( \mathcal{G} = \mathcal{B}_f = \{ f^{-1}(A) : A \in \mathcal{B}(S) \} \). Then \( \mathcal{G} \) is a c.g. \( \sigma \)-algebra proper in \( \mathcal{B}(S) \) and separating points of \( X \). The lemma follows by contraposition. Q.E.D.

It is the purpose of our main theorem below to bring attention to the fact that the notions of strongly \((*)\)-Blackwell, \((*)\)-Blackwell, and second-order Borel-density coincide for subsets \( X \) of a standard space \( S \); moreover, such subsets \( X \) are precisely those Blackwell spaces Borel-dense in \( S \). Thus the notions of Blackwell space and strongly Blackwell space coincide for Borel-dense sets. Before proceeding, we require the use of four more lemmas.

**Lemma 2.** Let \( E \) and \( F \) be analytic spaces and let \( A \) be an analytic subset of \( E \times F \). If \( A(y) = \{ x \in E : (x, y) \in A \} \) denotes the 1-section of \( A \) over the point \( y \), then \( \{ y \in F : A(y) \text{ is uncountable} \} \) is an analytic subset of \( F \).

**Proof.** This theorem is originally due to Mazurkiewicz and Sierpiński (1924) and has been generalised by Hoffmann-Jørgensen (1970), III.6.1.

**Lemma 3.** Let \( A \) be a standard subset of the product \( E \times F \) of analytic spaces \( E \) and \( F \). If the 1-sections \( A(x) = \{ y \in F : (x, y) \in A \} \) are countable for all \( x \) in \( E \), then there exist standard sets \( B_n \subset E \) (\( n = 1, 2, \ldots \)), and measurable mappings \( f_n : B_n \to F \) such that:

1) \( f_n(x) \neq f_m(x) \) for all \( x \) in \( B_n \cap B_m \) and \( n \neq m \),

and

2) \( A = \bigcup_{n=1}^{\infty} G(f_n) \), where \( G(f_n) \) is the graph of \( f_n \).

**Proof.** This theorem is essentially due to Lusin (1930) p. 243; a proof is to be found in Hoffmann-Jørgensen (1970), III.6.7.

Let \( E \) and \( F \) be separable spaces and let \( S \) be an uncountable standard subset of \( E \times F \). Given \( x_0 \) in \( E \) and \( y_0 \) in \( F \), define the 1-sections

\[
S_1(x_0) = \{ y \in F : (x_0, y) \in S \},
\]

\[
S_2(y_0) = \{ x \in E : (x, y_0) \in S \}.
\]

**Lemma 4.** Suppose that for each \( x \in E \) and \( y \in F \), one has \( S_1(x) \) and \( S_2(y) \) countable; then there is an uncountable standard subset \( S_0 \) of \( E \) and a one-one measurable function \( f : S_0 \to F \) whose graph \( G(f) \) is contained in \( S \).

**Proof.** Using Lemma 3, we find (for \( n = 1, 2, \ldots \)) standard subsets \( B_n \subset E \) and measurable mappings \( f_n : B_n \to F \) so that \( S = \bigcup_{n=1}^{\infty} G(f_n) \); select \( n \) so that \( G(f_n) \) is uncountable. Notice that since \( g_n : B_n \to S \) defined by \( g_n(x) = (x, f_n(x)) \) is one-one and measurable, its range \( G(f_n) \) belongs to \( \mathcal{M}(S) \) and so is standard.

Apply Lemma 3 once more, this time to the set \( G(f_n) \), using the fact that its "horizontal" sections are countable. There are, for \( m = 1, 2, \ldots \), standard
subsets $C_m \subseteq F$ and measurable mappings $g_m: C_m \to E$ so that $G(f_n) = \bigcup_{m=1}^{\infty} G(g_m)$; select $m$ so that $G(g_m)$ is uncountable.

Since each "vertical" section of $G(f_n)$, hence of $G(g_m)$ is a singleton, $g_m: C_m \to B_n$ is one-one and so bimeasurable. We may put $S_0 = g_m(C_m)$ and $f = g_m^{-1}$ on $S_0$: Q.E.D.

The same argument shows that $S$ is the countable union of such graphs.

**Lemma 5.** Let $X$ be a subset of a standard space $S$ such that $X$ is Borel-dense of order 1 but not of order 2 in $S$; then there is a measurable automorphism $g$ of $S$ onto itself such that

a) $g \circ g$ is the identity map on $S$, and

b) the set $T = \{ (s, g(s)): g(s) \neq s \}$ is uncountable and does not meet $X \times X$.

**Proof.** If $X$ is not Borel-dense of order 2 in $S$, then there is some $B \in \mathcal{B}(S \times S)$ with $B \subset (S \times S) \setminus (X \times X)$ but such that $B$ is not contained in a countable union of 1-slices of $S \times S$. $B$ may be chosen symmetric and, assuming $X$ is order-one dense in $S$, such that $B$ does not meet the diagonal $\Delta$ of $S \times S$: in any case via projection onto one co-ordinate, $B \cap \Delta$ would be isomorphic with a standard subset of $S \setminus X$ and so would be at most countably infinite.

Consider now the 1-sections of $B$: over points of $X$, these are standard subsets of $S \setminus X$ and so, by the Borel-density of $X$, are countable. The set of all points in $S$ for which these 1-sections are uncountable is, by Lemma 2, an analytic subset of $S \setminus X$ and so is countable. Subtract from $B$ the 1-slices of $S \times S$, in each co-ordinate, over the points in this countable set. What remains of $B$ is a subset $B_0$ of $(S \times S) \setminus (X \times X)$ such that:

(i) $B_0$ is symmetric;

(ii) $B_0 \in \mathcal{B}(S \times S)$;

(iii) $B_0 \cap \Delta = \emptyset$;

(iv) each 1-section of $B_0$ is countable;

(v) $B_0$ is uncountable.

Using the isomorphism theorem for standard (or "absolute") Borel spaces, we consider $S$ as a Borel subset of the real line with its usual order and metric structure. Define

$$B_- = \{ (s, t) \in B_0: s > t \}, \quad B_+ = \{ (s, t) \in B_0: s < t \},$$

disjoint, uncountable standard sets with $B_0 = B_- \cup B_+$.

By Lemma 4, there are uncountable standard subsets $D$ and $R$ of $S$ and an isomorphism $h$ of $D$ onto $R$ whose graph $H$ is a subset of $B_-$. then $h(s) < s$ for all $s$ in $D$, and there is some $\varepsilon > 0$ such that

$$D(\varepsilon) = \{ s \in D: h(s) < s - \varepsilon \}$$
is uncountable. Then there is some open interval $N$ of length $\varepsilon$ such that $D_0 = N \cap D(e)$ is uncountable. Whenever $s$ and $t$ are elements of $D_0$, then $h(s) < t$: so $D_0 \cap h(D_0) = \emptyset$.

Define $g: S \rightarrow S$ by the rule

$$
g(s) = \begin{cases} 
   h(s) & \text{if } s \in D_0, \\
   h^{-1}(s) & \text{if } s \in h(D_0), \\
   s & \text{otherwise}.
\end{cases}
$$

Then $g$ is an automorphism of $S$ such that $g \circ g$ is the identity map. Also, $T = \{(s, g(s)): g(s) \neq s\}$ is an uncountable subset of $B_0$ and so does not meet $X \times X$. Q.E.D.

The construction bears comparison with Corollary 2 of Shortt (1984).

2. The Principal Result.

**Theorem.** Let $X$ be a subset of a standard space $S$; then the following statements are equivalent:

1) $X$ is Borel-dense of order 2 in $S$,
2) $X$ is strongly ($\ast$)-Blackwell in $S$,
3) $X$ is ($\ast$)-Blackwell in $S$,
4) $X$ is a Blackwell space and is Borel-dense in $S$.

**Proof.** 1) implies 2). Assume that $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-$\sigma$-algebras of $\mathcal{B}(S)$ with $\mathscr{C}$ proper in $\mathscr{D}$. Let $f$ and $g$ be Marczewski functions for $\mathscr{C}$ and $\mathscr{D}$, respectively, and consider the set

$$
T = \{(s, t) \in S \times S: g(s) \neq g(t) \text{ and } f(s) = f(t)\}.
$$

$T$ is a member of $\mathcal{B}(S \times S)$ which, since $\mathscr{C}$ is proper in $\mathscr{D}$, is not contained in a countable union of 1-slices of $S \times S$. If $X$ is second-order Borel-dense in $S$, then $X \times X$ must intersect $T$; thus $X$ is strongly ($\ast$)-Blackwell in $S$.

2) implies 3). Trivial.

3) implies 1). Assume $X$ is ($\ast$)-Blackwell in $S$; then by Lemma 1, $X$ is Borel-dense (of order 1) in $S$. If, however, $X$ is not second-order Borel-dense in $S$, then by Lemma 5, there is a measurable automorphism $g: S \rightarrow S$ such that:

a) $g \circ g$ is the identity map on $S$;

b) the set $T = \{(s, g(s)): g(s) \neq s\}$ is uncountable and does not meet $X \times X$.

Since $S$ is isomorphic with some Borel subset of the real line, it makes sense to speak of a linear ordering $\leq$ on $S$ that respects (and generates) the Borel structure $\mathcal{B}(S)$. Having fixed such an ordering, we now define $f: S \rightarrow S$
by \( f(s) = s \land g(s) \), the minimum of \( s \) and \( g(s) \). So
\[
T = \{ (s, t) : s \neq t, g(s) = t \}
\]
\[
= \{ (s, t) : s \neq t, s \land g(s) = t \land g(t) \}
\]
\[
= \{ (s, t) : s \neq t, f(s) = f(t) \}
\]
is an uncountable member of \( \mathcal{B}(S \times S) \) not meeting \( X \times X \).

Consider \( \mathcal{A}_f = \{ f^{-1}(B) : B \in \mathcal{B}(S) \} \); the atoms of \( \mathcal{A}_f \) are given by
\[
f^{-1}(t) = \begin{cases} 
\{ t \} & \text{if } t = g(t), \\
\emptyset & \text{if } t > g(t), \\
\{ t, g(t) \} & \text{if } t < g(t), 
\end{cases}
\]
so that \( \mathcal{A}_f \) is c.g. and proper in \( \mathcal{B}(S) \), and, since \( X \times X \) does not meet \( T \), \( \mathcal{A}_f(X) \) is separable. Therefore \( X \) is not \((*)\)-Blackwell in \( S \).

3) implies 4). Immediate from Lemma 1.

4) implies 3). Suppose that \( \mathcal{C} \) is a c.g. sub-\( \sigma \)-algebra of \( \mathcal{B}(S) \) such that \( \mathcal{C}(X) \) is separable. Let \( f \) be a Marczewski function for \( \mathcal{C} \); if \( X \) is a Blackwell space, then \( \mathcal{C}(X) = \mathcal{B}(X) \), and \( f \) is an isomorphism when restricted to \( X \). Thus \( f \) is an isomorphism on some member \( S_0 \) of \( \mathcal{B}(S) \) containing \( X \). If \( X \) is Borel-dense in \( S \), then \( S \setminus S_0 \) is countable. This means that no c.g. sub\( \sigma \)-algebra \( \mathcal{C} \) of \( \mathcal{B}(S) \) can be proper in \( \mathcal{B}(S) \) and still separate points of \( X \), i.e. \( X \) is \((*)\)-Blackwell in \( S \). Q.E.D.

**Corollary.** A Borel-dense subset of a standard space is Blackwell if and only if it is strongly Blackwell.

**Remark.** The following two statements are implied by Corollary 5 and Proposition 13, respectively, in Shortt (1984):

1) If \( X \) is Borel-dense and universally measurable in \( S \), then \( X \) is a strongly Blackwell space.

2) If \( X \) is a non-Borel \((*)\)-Blackwell subset of \( S \), then \( X \) is not isomorphic with the product of any two uncountable spaces.

**Remark.** Returning to example 1, we see that the function \( g : R \to R \) guaranteed to exist in Lemma 5 may be taken to be \( g(s) = -s \). Referring to the proof that \( 3) \Rightarrow 1) \) in the theorem, we find \( f(s) = s \land g(s) = s \land (\neg s) = -|s| \); then \( \mathcal{A}_f \) consists of those Borel sets of \( R \) symmetric about zero, a c.g. \( \sigma \)-algebra proper in \( \mathcal{B}(R) \). Here \( \mathcal{A}_f(X) \) is separable, but cannot coincide with \( \mathcal{B}(X) \): were it so, \( f \) would be an isomorphism on \( X \) and therefore on all but countably many points of \( R \) (by Borel-density), clearly a contradiction.

**Remark.** The \((*)\)-Blackwell sets discussed above are easily seen to be precisely those satisfying condition 2* in the paper of Orkin (1972). Our theorem above shows that this actually coincides with the seemingly stronger construction due to Ryll-Nardzewski and presented in Sarbadhikari (1973).
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