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Symmetric derivative of nowhere monotone functions, I

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Let f be a real function defined on the real line R. For $\xi \in R$, let

$$\Phi(\xi, h) = \frac{f(\xi+h)-f(\xi-h)}{2h}, \quad h \in R, h \neq 0.$$

Then $\limsup_{h\to 0} \Phi(\xi, h)$ and $\liminf_{h\to 0} \Phi(\xi, h)$ are called the *upper* and the *lower symmetric derivative* of f at ξ and are denoted by $\overline{f^{(\prime)}}(\xi)$ and $\underline{f^{(\prime)}}(\xi)$, respectively [3]. If $\overline{f^{(\prime)}}(\xi)$ and $\underline{f^{(\prime)}}(\xi)$ are equal and finite, then the function f is said to be *symmetrically differentiable* at ξ and the common value is called the *symmetric derivative* [2] or the *Schwarz derivative* [4] (p. 36) of f at ξ and is denoted by $f^{(\prime)}(\xi)$. It is clear that if the ordinary derivative $f'(\xi)$ at ξ exists, then $f^{(\prime)}(\xi)$ also exists and they are equal; but the converse is not true.

We introduce the following definitions:

A function f is *symmetrically increasing* (*decreasing*) at a point ξ iff there exists a real number $h_\xi > 0$ such that

$$f(\xi+t) > f(\xi-t) \quad [f(\xi+t) < f(\xi-t)] \quad \text{for all } t,\ 0 < t < h_\xi.$$

The function f is *symmetrically non-decreasing* (*non-increasing*) at ξ iff there exists $h_\xi > 0$ such that

$$f(\xi+t) \geqslant f(\xi-t) \quad [f(\xi+t) \leqslant f(\xi-t)] \quad \text{for all } t,\ 0 < t < h_\xi.$$

A function f is said to be *symmetrically increasing* (resp. *non-decreasing, decreasing, non-increasing*) on an interval I iff f is symmetrically increasing (resp. non-decreasing, decreasing, non-increasing) at each point of I.

A function f is said to be *nowhere symmetrically monotone* in an interval I iff there is no subinterval of I in which f is symmetrically monotone.

A function f is *symmetrically oscillating* at a point ξ iff f is neither symmetrically non-increasing nor symmetrically non-decreasing at ξ. That is, a point ξ is a point of symmetric oscillation for the function f iff given any $h > 0$, there are t_1, t_2 satisfying $0 < t_1 < h$, $0 < t_2 < h$, such that

$$f(\xi+t_1) > f(\xi-t_1) \quad \text{and} \quad f(\xi+t_2) < f(\xi-t_2).$$

Remarks. 1. The following definitions are known [1].

A function f is said to be *increasing* (*decreasing*) at a point ξ on the right iff there exists a real number $h_\xi > 0$ such that

$$f(x) > f(\xi) \quad [f(x) < f(\xi)] \quad \text{for } \xi < x < \xi + h_\xi .$$

The function f is said to be *non-decreasing* (*non-increasing*) at ξ on the right iff there exists $h_\xi > 0$ such that

$$f(x) \geqslant f(\xi) \quad [f(x) \leqslant f(\xi)] \quad \text{for } \xi < x < \xi + h_\xi .$$

If the point ξ is such that in every right neighbourhood of ξ there are points x and y, where $f(x) < f(\xi)$ and $f(y) > f(\xi)$, then f is said to be *oscillating* on the right of ξ.

The above concepts on the left of ξ are similarly defined. Clearly, if a function f is symmetrically increasing at a point ξ, then it need not be increasing on the right at ξ or on the left at ξ. The function

$$f(x) = x, \quad x \neq 0; \quad f(0) = 1$$

is such that f is symmetrically increasing at the point $x = 0$ but f is not increasing on the right at 0. But if a function f is increasing at a point both on the right and on the left, then f is symmetrically increasing at ξ. Thus if f is an increasing function on an interval I, then f is also symmetrically increasing on I; but the converse is not true.

2. The two ideas viz. "a function f is symmetrically oscillating at a point ξ" and "a function f is oscillating symmetrically on both sides of a point ξ" should be carefully distinguished. The function

$$f(x) = x \sin\frac{1}{x}, \quad x \neq 0; \quad f(0) = 0$$

is such that it is oscillating on both sides of the point $x = 0$ satisfying $f(x) = f(-x)$ for all x. So, although f is oscillating symmetrically on both sides of 0, yet f is not symmetrically oscillating at 0 according to our definition.

THEOREM 1. *If f is continuous and nowhere monotone* (1) *in an intervall I, then the set of points in I, where f is symmetrically non-decreasing, is of the first category.*

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(1) A function f is called *nowhere monotone* iff there is no interval in which f is monotone. Since a non-decreasing function is also symmetrically non-decreasing, every function which is nowhere symmetrically monotone is a nowhere monotone function. So we shall prove our results for the class of nowhere monotone functions which, however, will include the class of all nowhere symmetrically monotone functions.

Proof. Let f be a continuous nowhere monotone function in I and let E be the set of all points of I, where f is symmetrically non-decreasing. Then for each $x \epsilon E$, there is $h_x > 0$ such that

$$f(x+t) \geqslant f(x-t) \quad \text{whenever } 0 < t < h_x.$$

For each positive integer n, let E_n denote the set of all points x of I such that

$$f(x+t) \geqslant f(x-t) \quad \text{whenever } 0 < t < 1/n.$$

Then

$$E = \bigcup_{n=1}^{\infty} E_n.$$

We shall show that E_n is nowhere dense in I for each n. Let n be fixed and let $I' = [a, b]$ be any subinterval of I. We may suppose that $b-a < 1/n$. Since f is nowhere monotone, there are two points $c, d \epsilon I'$, $c < d$, such that $f(c) > f(d)$. Let $\inf\limits_{x \epsilon [c,d]} f(x) = m$. Since f is continuous, the set

$$S = \{x\colon\ f(x) = m;\, c \leqslant x \leqslant d\}$$

is a bounded non-void closed set. Let $k = \inf S$. Then $k \epsilon S$. Also $f(c) > m$ and hence $c \notin S$. Therefore $c < k \leqslant d$. Clearly $\{x\colon c \leqslant x < k\} \cap S = 0$. Since f is continuous,

$$f(x) > m \quad \text{for all } x,\, c \leqslant x < k. \tag{1}$$

Let $c' = \dfrac{c+k}{2}$. Then $c' \notin S$. Choose any real number c'' such that $c' < c'' < k$. Then we shall show that $[c', c''] \cap E_n = 0$. Let $x \epsilon [c', c'']$. Then

$$a \leqslant c < c' \leqslant x \leqslant c'' < k \leqslant d \leqslant b$$

and hence

$$0 < k - x < b - a < 1/n. \tag{2}$$

Also

$$x-(k-x) = 2x-k \geqslant 2c'-k = c$$

and

$$x-(k-x) = 2x-k < 2k-k = k.$$

Thus

$$c \leqslant x-(k-x) < k.$$

Hence from (1)

$$f\big(x-(k-x)\big) > m = f(k) = f\big(x+(k-x)\big).$$

From (2) and from the construction of the set E_n, we conclude that $x \notin E_n$. Since x is any arbitrary point of $[c', c'']$, it follows that $[c', c''] \cap E_n = 0$.

Thus E_n is nowhere dense in I and consequently the set E is of the first category.

The following theorem can be similarly proved:

THEOREM 2. *If f is continuous and nowhere monotone in an interval I, then the set of points in I, where f is symmetrically non-increasing, is of the fist category.*

Combining Theorems 1 and 2 we get

THEOREM 3. *If f is continuous and nowhere nonotone in I, then the set of points, where f is symmetrically oscillating in I, is a residual set.*

Let f be a continuous nowhere monotone function defined in an interval I and let G be the residual set in I, where f is symmetrically oscillating. Clearly,

$$\underline{f^{(')}}(x) \leqslant 0 \leqslant \overline{f^{(')}}(x) \quad \text{for each } x \in G.$$

Thus we get

THEOREM 4. *If f is continuous and nowhere monotone, then except a set of the first category, the following relation is true:*

$$\underline{f^{(')}}(x) \leqslant 0 \leqslant \overline{f^{(')}}(x).$$

COROLLARY 1. *If f is continuous, nowhere monotone and everywhere symmetrically differentiable, then the symmetric derivative $f^{(')}(x)$ vanishes at a residual set of points.*

Note 1. If f is continuous and nowhere monotone, then the sets

$$\{x\colon \underline{f^{(')}}(x) < 0\} \quad \text{and} \quad \{x\colon \overline{f^{(')}}(x) > 0\}$$

are everywhere dense. For if there is an interval I such that $I \cap \{x\colon \underline{f^{(')}}(x) < 0\} = 0$, then $\underline{f^{(')}}(x) \geqslant 0$ for all $x \in I$ and hence from Theorem 3 of [3] f would be non-decreasing in I. If f is continuous, nowhere monotone and everywhere symmetrically differentiable, then in view of the above corollary the sets

$$\{x\colon f^{(')}(x) < 0\} \quad \text{and} \quad \{x\colon f^{(')}(x) > 0\}$$

are everywhere dense and of the first category.

Note 2. If in the above corollary we assume the existence of $f^{(')}(x)$ on a residual set only, then also the conclusion remains valid.

COROLLARY 2. *Let f be continuous and symmetrically differentiable. If the sets*

$$\{x\colon f^{(')}(x) > \lambda\} \quad \text{and} \quad \{x\colon f^{(')}(x) < \lambda\}$$

are everywhere dense, then the set

$$\{x\colon f^{(')}(x) = \lambda\}$$

is a resudual set, where λ is any real number.

Proof. Under the hypothesis, the function $f(x)-\lambda x$ is continuous nowhere monotone and everywhere symmetrically differentiable and the result follows from Corollary 1.

COROLLARY 3. *If a continuous symmetrically differentiable function f defined in an interval I is such that the set*

$$E = \{x\colon f^{(')}(x) \neq 0,\ x \in I\}$$

is of the second category in I, then there exists a subinterval of in I in which f is monotone.

Consequently, if the set E is of the second category in every subinterval of I, then there exists an everywhere dense set of intervals in I in each of which f is monotone.

COROLLARY 4. *If f is a continuous function such that there exist two real numbers r_1 and r_2, $r_1 < r_2$, for which $f(x)-r_1x$ and $f(x)-r_2x$ are nowhere monotone, then the set of points x, where $f^{(')}(x)$ exists, is of the first category.*

Proof. From the above theorem the sets

$$G_1 = \{x\colon \underline{f^{(')}}(x) \leqslant r_1 \leqslant \overline{f^{(')}}(x)\} \quad \text{and} \quad G_2 = \{x\colon \underline{f^{(')}}(x) \leqslant r_2 \leqslant \overline{f^{(')}}(x)\}$$

are residual. Hence $G_1 \cap G_2$ is also residual. Now for $x \in G_1 \cap G_2$

$$\underline{f^{(')}}(x) \leqslant r_1 < r_2 \leqslant \overline{f^{(')}}(x).$$

So, for $x \in G_1 \cap G_2$, $f^{(')}(x)$ does not exist and this completes the proof.

From Theorem 4 it is clear that if f is continuous and if $f(x)-rx$ is nowhere monotone for some real number r, then the set

$$\{x\colon \underline{f^{(')}}(x) \leqslant r \leqslant \overline{f^{(')}}(x)\}$$

is a residual set. If, however, the behaviour of the function $f(x)-rx$ regarding its monotonicity is not known but there is a sequence of real numbers $\{r_n\}$ such that $r_n \to r$ as $n \to \infty$ and $f(x)-r_nx$ is nowhere monotone for each n, then also the above result holds. In fact, we prove the following theorem:

THEOREM 5. *Let f be continuous and let $\{r_n\}$ be a sequence of real numbers such that $r_n \to r$ as $n \to \infty$. Let $f(x)-r_nx$ be nowhere monotone for each n. Then the set*

$$\{x\colon \underline{f^{(')}}(x) \leqslant r \leqslant \overline{f^{(')}}(x)\}$$

is a residual set.

Proof. Since $f(x)-r_nx$ is continuous and nowhere monotone for each n, it follows from Theorem 4 that the set

$$G_n = \{x\colon \underline{f^{(')}}(x) \leqslant r_n \leqslant \overline{f^{(')}}(x)\}$$

is a residual set. Hence the set $G = \bigcap_{n=1}^{\infty} G_n$ is also residual. But

$$G \subset \{x:\ \underline{f}^{(')}(x) \leqslant r \leqslant \overline{f}^{(')}(x)\}.$$

Since every subset of a set of the first category is again a set of the first category, the set

$$\{x:\ \underline{f}^{(')}(x) \leqslant r \leqslant \overline{f}^{(')}(x)\}.$$

is a residual set.

COROLLARY 1. *Let f be continuous and let $\{r_n\}$ and $\{s_n\}$ be two sequences of real numbers such that $r_n \to r$, $s_n \to s$ as $n \to \infty$, where $r < s$. Let $f(x) - r_n x$ and $f(x) - s_n x$ be nowhere monotone for each n. Then the set*

$$\{x:\ \underline{f}^{(')}(x) \leqslant r < s \leqslant \overline{f}^{(')}(x)\}$$

is a residual set.

THEOREM 6. *Let f be continuous and let $\{r_n\}$ and $\{s_n\}$ be two sequences of real numbers such that $r_n \to -\infty$ and $s_n \to +\infty$ as $n \to \infty$. If $f(x) - r_n x$ and $f(x) - s_n x$ are nowhere monotone for each n, then the set*

$$\{x:\ \underline{f}^{(')}(x) = -\infty;\ \overline{f}^{(')}(x) = +\infty\}$$

is a residual set.

Proof. From Theorem 4 we get that the sets

$$G_n = \{x:\ \underline{f}^{(')}(x) \leqslant r_n \leqslant \overline{f}^{(')}(x)\} \quad \text{and} \quad H_n = \{x:\ \underline{f}^{(')}(x) \leqslant s_n \leqslant \overline{f}^{(')}(x)\}$$

are both residual. Hence the set $F_n = G_n \cap H_n$ is also residual. Set $G = \bigcap_{n=1}^{\infty} F_n$. Then G is a residual set. Also

$$G \subset \{x:\ \underline{f}^{(')}(x) = -\infty;\ \ \overline{f}^{(')}(x) = +\infty\}.$$

This completes the proof.

COROLLARY 1. *If f is continuous and if each of the sets*

$$\{x:\ \underline{f}^{(')}(x) = -\infty\} \quad \text{and} \quad \{x:\ \overline{f}^{(')}(x) = +\infty\}$$

is everywhere dense in an interval I, then the set

$$\{x:\ \underline{f}^{(')}(x) = -\infty;\ \ \overline{f}^{(')}(x) = +\infty\}$$

is residual in I.

Proof. Since the sets $\{x:\ \underline{f}^{(')}(x) = -\infty\}$ and $\{x:\ \overline{f}^{(')}(x) = +\infty\}$ everywhere dense, the functions $f(x) - rx$ are nowhere monotone for all real number r. Hence by the above theorem the set

$$\{x:\ \underline{f}^{(')}(x) = -\infty;\ \overline{f}^{(')}(x) = +\infty\}$$

is residual in I.

COROLLARY 2. *If f is continuous, then the set*

$$\{x:\ \underline{f^{(')}}(x) = -\infty;\ \overline{f^{(')}}(x) = +\infty\}$$

is either nowhere dense or it is of the second category which is residual in every interval in which it is everywhere dense.

COROLLARY 3. *If f is continuous function such that the set*

$$\{x:\ -\infty < \underline{f^{(')}}(x) \leqslant \overline{f^{(')}}(x) < +\infty\}$$

is of the second category, then there is a positive real number N such that at least one of the following is true:

(i) *for each member $f(x)-rx$ of the family $\{f(x)-rx:\ r \geqslant N\}$ there exists at least one subinterval in which $f(x)-rx$ is monotone;*

(ii) *for each member $f(x)-rx$ of the family $\{f(x)+rx:\ r \geqslant N\}$ there exists at least one subinterval in which $f(x)+rx$ is monotone.*

COROLLARY 4. *Let f be continuous and let $f^{(')}(x)$ exists and be finite on a set which is of the second category. Then there exists a positive number N such that for each member $f(x)-rx$ of the family $\{f(x)-rx:\ |r| \geqslant N\}$ there exists a subinterval in which $f(x)-rx$ is monotone.*

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