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A CLASSIFICATION OF H-CLOSED EXTENSIONS

BY

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Porter and Votaw [6] and Fedorčuk [5] have recently presented some results concerning a classification of H-closed extensions of Hausdorff spaces. In particular, Fedorčuk has described the set of all semiregular H-closed extensions in terms of some special families of θ-proximities. The aim of this paper is to give a simple description of the set of all semiregular H-closed extensions in terms of the pseudo-quotient operation of Alexandroff [1], considered recently by Rudolf [7]. Namely, we show that each semiregular H-closed extension is a pseudo-quotient image of the Banaschewski H-closed extension induced by a decomposition into closed disjoint subsets. We give also another description in terms of families of filters.

Semiregular H-closed extensions are said to be also *H-minimal*, for Hausdorff minimal spaces coincide with H-closed semiregular ones. Banaschewski's H-minimal extension bX of a semiregular space X is the semiregularization of the corresponding Katětov extension τX (see [2]). A decomposition of a space means here a family of non-empty mutually disjoint subsets covering it. A decomposition Q of a space X is said to be *irreducible* if for each open non-empty set U, $U \subset X$, there exists a set A from Q such that $A \subset U$. If Q is a decomposition of a space X, then the pseudo-quotient topology in the quotient set X/Q is generated by the sets

$$q[U] = \{x \in X/Q:\ q^{-1}(x) \subset U\},$$

where U runs over the family of all regularly open subsets of X. The map q: $X \to X/Q$ is θ-continuous and the basic sets in X/Q, i.e., the sets $q[U]$, are regularly open. A map f: $X \to Y$ is called *θ-continuous* if for each x, $x \in X$, and each open neighbourhood V of $f(x)$ there exists an open neighbourhood U of x such that $f(\mathrm{cl}\, U) \subset \mathrm{cl}\, V$.

We shall use the following result due to Rudolf [7]: if Q is an irreducible decomposition of a semiregular H-closed space, then the pseudo-quotient space X/Q is semiregular H-closed, and the pseudo-quotient map $q\colon X \to X/Q$ is θ-continuous.

LEMMA 1. *If Q is an irreducible decomposition of a space X consisting of compact sets and $f\colon X \to Y$ is a θ-continuous map such that $f(x) = f(y)$ whenever x and y belong to the same member of Q, then there exists a θ-continuous map $\varphi\colon X/Q \to Y$ such that $\varphi \circ g = f$, where $q\colon X \to X/Q$ is the pseudo-quotient map induced by Q.*

Proof. The map $\varphi\colon X/Q \to Y$ given by $\varphi(q(x)) = f(x)$ for $x \in X$ is the desired one. To prove that φ is θ-continuous let U be an open neighbourhood of $\varphi(q(x))$. Since f is θ-continuous, for each z, $z \in q^{-1}(q(x))$, there exists an open set V such that $z \in V$ and $f(\mathrm{cl}\,V) \subset \mathrm{cl}\,U$. Since $q^{-1}(q(x))$ is compact, there exist open sets $V_1, \ldots, V_n$ such that

$$f(\mathrm{cl}\,V_i) \subset \mathrm{cl}\,U \quad \text{and} \quad q^{-1}(q(x)) \subset V_1 \cup \ldots \cup V_n \quad \text{for } i = 1, \ldots, n.$$

Then

$$f(\mathrm{cl} W) \subset \mathrm{cl}\,U \text{ for } W = \mathrm{Int}\,\mathrm{cl}(V_1 \cup \ldots \cup V_n) \quad \text{and} \quad q^{-1}(q(x)) \subset W.$$

Clearly, $q(x) \in q[W]$. In [7] it was proved that $\mathrm{cl}\,q[W] \subset q(\mathrm{cl} W)$ for each W being regularly open in X. Thus we get

$$\varphi(\mathrm{cl}\,q[W]) \subset \varphi(q(\mathrm{cl}\,W)) = f(\mathrm{cl}\,W) \subset \mathrm{cl}\,U,$$

which completes the proof.

LEMMA 2. *For each H-closed extension rX of a semiregular space X there exists a θ-continuous map*

$$f\colon bX \xrightarrow{\text{onto}} rX$$

such that $f|X = \mathrm{id}_X$, and $Q = \{f^{-1}(x)\colon x \in rX \setminus X\}$ is a decomposition of $bX \setminus X$ consisting of closed subsets of bX.

Proof. The desired map, $f\colon bX \to rX$, is a superposition of the canonical map from the Katětov extension τX onto rX and of a one-to-one θ-continuous map from bX onto τX. Since f is θ-continuous, the preimages of points are closed.

LEMMA 3. *If A is a closed subset of bX and $A \subset bX \setminus X$, then A is compact.*

Proof. It is known [3] that each closed subset of the Fomin H-closed extension σX which is contained in the remainder is compact. Since there is a continuous contraction of σX onto bX, bX has the same property.

THEOREM 1. *An extension rX of a semiregular space X is H-minimal iff there exists a decomposition Q of bX into closed subsets such that each A, $A \in Q$, is a one-point set whenever $A \cap X \neq \emptyset$, and rX is homeomorphic to bX/Q.*

Proof. 1. Since, by Lemma 3, Q consists of compact sets, bX/Q is Hausdorff. Clearly, Q is an irreducible decomposition of bX. Hence, by a theorem of Rudolf [7], the pseudo-quotient map $q\colon bX \to bX/Q$ is θ-continuous and bX/Q is semiregular. Therefore, bX/Q is an H-minimal extension of X.

2. Let rX be an H-minimal extension of X. By Lemma 2, there exists a θ-continuous map: $f\colon bX \to rX$ such that $\{f^{-1}(x)\colon x \in rX \setminus X\}$ is a decomposition of $bX \setminus X$ consisting of closed subsets of bX. Hence, by Lemma 3, $Q = \{f^{-1}(x)\colon x \in rX\}$ is a decomposition of bX consisting of compact sets. To show that rX is homeomorphic to bX/Q let us note that, by Lemma 1, there exists a θ-continuous map $\varphi\colon bX/Q \to rX$ such that the diagram

$$\begin{tikzcd} bX \arrow[r, "q"] \arrow[d, "f"'] & bX/Q \arrow[dl, "\varphi"] \\ rX & \end{tikzcd}$$

commutes. Clearly, φ is one-to-one. Let us note that bX/Q is H-closed and semiregular. This shows that, for each regularly closed subset F of bX/Q, $\varphi(F)$ is closed as a θ-continuous image of an H-closed subspace. Since bX/Q has a base consisting of regularly open sets, $\varphi^{-1}\colon rX \to bX/Q$ is continuous. But rX is a minimal Hausdorff space, and thus φ is a homeomorphism.

Our description of H-minimal extensions can be translated into the language of filters of regularly open sets, and then into the language of covers.

THEOREM 2. *Let X be a semiregular space. There exists a one-to-one correspondence between the family of all H-minimal extensions of X and the family of all collections $\mathcal{R}$ of filters of regularly open sets for which the following conditions hold:*

(1) *if $\mathcal{F} \in \mathcal{R}$, then*

$$\bigcap \{\mathrm{cl}_X U\colon\ U \in \mathcal{F}\} = \emptyset;$$

(2) *if $\mathcal{G}$ is a filter of regularly open sets of X and*

$$\bigcap \{\mathrm{cl}_X U\colon\ U \in \mathcal{G}\} = \emptyset,$$

then there exists a filter $\mathcal{F}$, $\mathcal{F} \in \mathcal{R}$, such that $U \cap V \neq \emptyset$ for each U from $\mathcal{F}$ and V from $\mathcal{G}$;

(3) *if $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{R}$ and $\mathcal{F}_1 \neq \mathcal{F}_2$, then there exist $U_1 \in \mathcal{F}_1$ and $U_2 \in \mathcal{F}_2$ such that $U_1 \cap U_2 = \emptyset$.*

Proof. It is known that the Banaschewski extension is expressed by

$bX = X \cup \{\xi\colon\ \xi$ is an open ultrafilter without adherence points in $X\}$

with the topology generated by the sets of the form

$$b(U) = U \cup \{\xi \in bX \setminus X:\ U \in \xi\},$$

where U runs over the family of all regularly open subsets of bX. It is easy to see that

$$\mathrm{cl}_{bX} b(U) = \mathrm{cl}_X U \cup b(U)$$

(for a similar calculation see [4]).

In virtue of Theorem 1 it suffices to show that there exists a one-to-one correspondence between the set of all decompositions of $bX \setminus X$ consisting of closed subsets of bX and the family of all collections of filters satisfying (1), (2), and (3). Let $\mathscr{R}$ be a given collection for which conditions (1), (2), and (3) hold. For each filter $\mathscr{F}$, $\mathscr{F} \in \mathscr{R}$, write

$$(*) \qquad A(\mathscr{F}) = \bigcap \{\mathrm{cl}_{bX} b(U):\ U \in \mathscr{F}\}.$$

Since $\mathrm{cl}_{bX} b(U) = \mathrm{cl}_X U \cup b(U)$, by (1) we get

$$(**) \qquad A(\mathscr{F}) = \bigcap \{b(U):\ U \in \mathscr{F}\} \subset bX \setminus X.$$

It is easy to see that, by (2) and (3), $\{A(\mathscr{F}):\ \mathscr{F} \in \mathscr{R}\}$ is a decomposition of $bX \setminus X$ consisting of sets which are closed in bX.

Now let Q be a decomposition of $bX \setminus X$ consisting of closed subsets of bX. We shall show that Q appoints a family $\mathscr{R}$ of filters satisfying conditions (1), (2), and (3) and such that the decomposition of $bX \setminus X$ appointed by $\mathscr{R}$ and defined above equals Q. For a given A, $A \in Q$, let

$$\mathscr{F}(A) = \{U:\ U \text{ is regularly open in } X \text{ and } A \subset b(U)\}.$$

We shall show that

$$(***) \qquad \bigcap \{b(U):\ U \in \mathscr{F}(A)\} = A.$$

To do this let us suppose that $y \notin A$. Since, by Lemma 3, A is compact, there exist regularly open sets $U_1, \ldots, U_n$ and V such that $y \in b(V)$ and

$$A \subset b(U_1) \cup \ldots \cup b(U_n) \quad \text{and} \quad b(U_i) \cap b(V) = \emptyset \quad \text{for } i = 1, \ldots, n.$$

It is easy to check that

$$b(U_1) \cup \ldots \cup b(U_n) \subset b\big(\mathrm{Int}\,\mathrm{cl}(U_1 \cup \ldots \cup U_n)\big) \subset bX \setminus b(V).$$

Hence

$$\mathrm{Int}\,\mathrm{cl}(U_1 \cup \ldots \cup U_n) \in \mathscr{F}(A) \quad \text{and} \quad y \notin b\big(\mathrm{Int}\,\mathrm{cl}(U_1 \cup \ldots \cup U_n)\big).$$

Thus

$$\bigcap \{b(U):\ U \in \mathscr{F}(A)\} \subset A.$$

The converse is obvious.

The assertion above shows that the family $\mathscr{R} = \{\mathscr{F}(A): A \in Q\}$ satisfies conditions (1) and (3). Since $\bigcup Q = bX\setminus X$, condition (2) also holds. Clearly, by (**) and (***), $A(\mathscr{F}(A)) = A$ for each $A, A \in Q$. It remains to show that if $\mathscr{F} \in \mathscr{R}$, then $\mathscr{F}(A(\mathscr{F})) = \mathscr{F}$. The inclusion $\mathscr{F} \subset \mathscr{F}(A(\mathscr{F}))$, by (**), is obvious. To show the converse, let $W \in \mathscr{F}(A(\mathscr{F}))$. Since $A(\mathscr{F}) \subset b(W)$, there exists $U, U \in \mathscr{F}$, such that $b(U) \subset b(W)$. Indeed, in the opposite case,

$$b(U)\cap(bX\setminus b(W)) \neq \emptyset \quad \text{for each } U, U \in \mathscr{F}.$$

But $bX\setminus b(W)$, being regularly closed, is H-closed. Hence

$$\bigcap \{\mathrm{cl}_{bX} b(U): U \in \mathscr{F}\}\cap (bX\setminus b(W)) \neq \emptyset$$

which, by (*), contradicts the inclusion $A(\mathscr{F}) \subset b(W)$. Hence there exists $U, U \in \mathscr{F}$, such that $U \subset W$. Thus $W \in \mathscr{F}$ which completes the proof.

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