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INTERSECTING PENCIL OF HYPERBOLIC CIRCLES

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Intersecting pencil of circles in the Euclidean plane may be defined in the following way. Each member of the pencil is a locus of points X which together with the two given points U and V (the common points of all circles of the pencil) form a triangle UVX with the property

$$\sphericalangle XVU + \sphericalangle XUV = \begin{cases} \delta \text{ on one side of the line } UV, \\ \pi - \delta \text{ on the other side, } 0 < \delta < \pi. \end{cases}$$

In order to obtain the whole pencil we must include points U and V. Applying the definition in the hyperbolic plane, we obtain curves which may be called *hyperbolic circles* (although, of course, other curves pretend to be hyperbolic circles, too). The aim of this note is to give some properties of them. They differ from Euclidean circles

(i) topologically, in general being, almost all of them, unclosed curves;

(ii) in their groups of symmetry being, in general, symmetrical by reflexion in one line only.

1. Let U and V be two fixed points. The line UV dissects the plane into two half-planes M_1 and M_2. Let $\bar{M}_1$ and $\bar{M}_2$ denote $M_1 \cup$ the line UV and $M_2 \cup$ the line UV, respectively. Our considerations in sections 1, 2 and 3 are fully equivalent for both half-planes. Except the definitions and the theorem, we will write therefore M instead of M_1 and M_2, and $\bar{M}$ instead of $\bar{M}_1$ and $\bar{M}_2$.

Let l_0 be the ray in $\bar{M}$ from the middle Q of UV perpendicular to UV and let l_ε, $|\varepsilon| < \pi/2$, be the ray from Q such that the oriented angle (l_0, l_ε) is equal to ε. Points on the ray l_ε or the end of it (i.e., the point in the infinity) will be denoted by X_ε. Let k_a or k'_a $(0 < a < \pi)$ be the rays in $\bar{M}$ from U and from V such that the non-oriented angle $(k_a, \overrightarrow{UV})$ or $(k'_a, \overrightarrow{VU})$ is equal to a. By $\overrightarrow{UV}$ we mean the ray from U through V.

LEMMA 1. *Let δ_1 and δ_2 be the angles in the asymptotic triangle UVX_ε at U and at V, respectively; then*

$$\operatorname{ctg}\frac{\delta_1+\delta_2}{2} = \cos\varepsilon \operatorname{sh}\frac{a}{2}, \tag{1}$$

where $a = |UV|$.

Proof. Consider the case in which U separates Q and the foot S of the perpendicular from X_ε to the line UV (Fig. 1). Then $d = |QS| > a/2$.

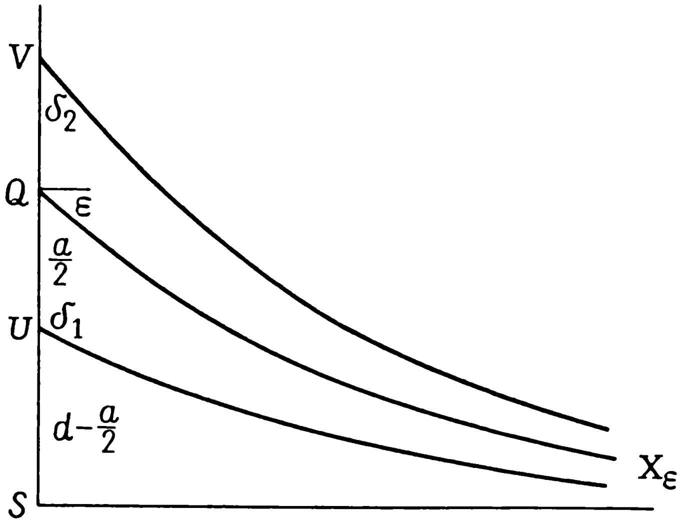

Fig. 1

Applying Lobachevsky's formula to the triangles USX_ε, VSX_ε, and QSX_ε we obtain

$$\operatorname{ctg}\frac{1}{2}(\pi-\delta_1) = e^{d-a/2}, \tag{2}$$

$$\operatorname{ctg}\frac{\delta_2}{2} = e^{d+a/2}, \tag{3}$$

and

$$\operatorname{ctg}\frac{1}{2}\left(\frac{\pi}{2}-|\varepsilon|\right) = e^{d}. \tag{4}$$

From the formula expressing $\operatorname{ctg}(\delta_1+\delta_2)/2$ by $\operatorname{ctg}\delta_1/2$ and $\operatorname{ctg}\delta_2/2$ we have, in virtue of (2) and (3),

$$\operatorname{ctg}\frac{\delta_1+\delta_2}{2} = \frac{e^{a/2}-e^{a/2}}{e^{d}+e^{-d}}. \tag{5}$$

Substituting now in (5) instead of e^d the left-hand side of (4), we obtain (1).

It is equally easy to prove (1) in the case when S lies on the ray $\overrightarrow{UV}$.

Remark. For the sake of simplicity we set the curvature of the plane equal to -1. In general case, when the curvature $\varkappa < 0$ is arbitrary, it is enough to set, in what follows, $a/(-\varkappa)$ instead of a in all symbols in which a is the distance UV.

2. Definition 1. The locus of a point $X \epsilon M_i$ $(i = 1, 2)$ such that in any triangle UVX the sum of the angles at U and at V is equal to δ, $0 < \delta < \pi$, is said to be the *curve* $L_i(a, \delta)$.

In what follows $L(a, \delta)$ will stand for $L_1(a, \delta)$ and $L_2(a, \delta)$.

From Definition 1 it follows that $L(a, \delta)$ is symmetric by reflection in the line containing l_0. Let us denote this property of $L(a, \delta)$ by Ps.

X is a continuous function of δ_1 and δ_2, where δ_1 and δ_2 are the angles at U and at V in the triangle UVX. In fact, let two pairs of rays r_1, r_1' and r_2, r_2' from U and from V, respectively, form with the rays $\overrightarrow{UV}$ and $\overrightarrow{VU}$ angles δ_1, δ_2 and δ_1', δ_2' such that $|\delta_1 - \delta_1'|$ and $|\delta_2 - \delta_2'|$ are small. Then $X = r_1 \cdot r_2$ and $X' = r_1' \cdot r_2'$, if they exist, are close in a natural topology of the plane(1). For $\delta = \text{const}$, X, if it exists, is a continuous function of one variable, say δ_1 (then $\delta_2 = \delta - \delta_1$), and the set $\{X(\delta_1)\}$ is equal to $L(a, \delta)$ for $0 < \delta_1 < \delta$. It is clear that through every point of M passes a curve $L(a, \delta)$. Thus we have

LEMMA 2. *If a finite open arc belongs to* $L(a, \delta)$, *so does its closure in* M.

LEMMA 3. *If* $\delta_1 \to \delta$, *then* $X(\delta_1) \to U$, *and if* $\delta_1 \to 0$, *then* $X(\delta_1) \to V$.

Proof. Consider the first case (the proof of the second is analogous). If $X(\delta_1)$ tended to a point $X_0 \neq U$, then $X_0 \epsilon k_\delta = \overrightarrow{UX_0}$ and, of course, $X_0 \epsilon M$. In virtue of Lemma 2 there is $X_0 \epsilon L(a, \delta)$. But then the sum of angles at U and at V in the triangle UVX would be greater than δ contrary to the definition of $L(a, \delta)$.

Consequently, we adjoint to every $L(a, \delta)$ the points U and V; thus we formulate

Definition 2. The union $L_i(a, \delta) \cup U \cup V$ $(i = 1, 2)$ is said to be the *curve* $\bar{L}_i(a, \delta)$.

From Lemmas 3 and 2 we obtain

LEMMA 2'. *If a finite open arc belongs to* $\bar{L}(a, \delta)$, *so does its closure in* $\bar{M}$.

Take a point X on $\bar{L}(a, \delta)$ close to U. Let X tend to U along $\bar{L}(a, \delta)$. Then the ray $\overrightarrow{UX}$ tends to the limit ray k_δ which is a one-sided tangent. The same is true for V with the limit ray k_δ'. Let us denote this property of $\bar{L}(a, \delta)$ by $P\delta$. To show $P\delta$, it is enough to observe that the limit ray k_α, could not be neither such that $\alpha > \delta$ nor such that $\alpha < \delta$.

(1) K. Borsuk and W. Szmielew, *Foundation of Geometry*, Amsterdam 1960, p. 64.

3. We now define two other properties of $\bar{L}(a, \delta)$.

$\bar{L}(a, \delta)$ is said to have the *property Pc* if every l_ε cuts the curve in a point, and the property $P\varepsilon_0$ if: (i) two lines (or one line in the limit case, $\varepsilon_0 = 0$) containing $l_{\varepsilon_0}, l_{-\varepsilon_0}$, where $0 \leqslant \varepsilon_0 < \pi/2$, are the asymptotes (or is an asymptote) of the curve; (ii) the rays (or a ray) l_ε for $|\varepsilon| < \varepsilon_0$ do not cut the curve; (iii) any l_ε for $\varepsilon_0 < |\varepsilon| < \pi/2$ cuts the curve in a point.

LEMMA 4. *If*

$$\operatorname{ctg}\frac{\delta}{2} > \operatorname{sh}\frac{a}{2}, \tag{6}$$

then $\bar{L}(a, \delta)$ has the property Pc.

Proof. Denote by $\delta(X_\varepsilon^\infty)$ the sum of the angles at U and at V in the asymptotic triangle UVX_ε^∞, $|\varepsilon| < \dfrac{\pi}{2}$. From (1) we have

$$\operatorname{ctg}\frac{\delta(X_\varepsilon^\infty)}{2} = \cos\varepsilon \operatorname{sh}\frac{a}{2}. \tag{7}$$

(6) may be written as follows:

$$\operatorname{ctg}\frac{\delta}{2} = \operatorname{sh}\frac{a}{2} + c, \qquad c > 0. \tag{8}$$

In the degenerate triangle UVQ we have

$$\operatorname{ctg}\frac{\delta(Q)}{2} = \infty, \tag{9}$$

where $\delta(Q) = \sphericalangle QUV + \sphericalangle QVU = 0$. If X_ε runs from Q to the end of l_ε, then the sum $\delta(X_\varepsilon)$ of angles at U and at V in the triangle UVX_ε increases monotonically from $\delta(Q) = 0$ to $\delta(X_\varepsilon^\infty)$. Therefore it follows from (7), (8) and (9) that there exists a finite point X'_ε (different from Q) such that in the triangle UVX'_ε (6) is satisfied. Hence $\bar{L}(a, \delta)$ has the property Pc provided a and δ satisfy (6).

If the quotient $\operatorname{sh}\frac{1}{2}a/\operatorname{ctg}\frac{1}{2}\delta$ (< 1) is close to 1, then the point X'_0 of the intersection of $\bar{L}(a, \delta)$ and l_0 is far; if $\operatorname{sh}\frac{1}{2}a/\operatorname{ctg}\frac{1}{2}\delta$ diminishes to zero, X_0 tends to Q.

LEMMA 5. *If*

$$\operatorname{ctg}\frac{\delta}{2} = \operatorname{sh}\frac{a}{2}, \tag{10}$$

then $\bar{L}(a, \delta)$ has the property $P0$ (i.e. $P\varepsilon_0$ for $\varepsilon_0 = 0$).

Proof. The same reasoning as in the proof of Lemma 4 shows that if a and δ satisfy (10), then on every l_ε, there is a point $X'_\varepsilon(\neq 0) \in \bar{L}(a, \delta)$ for $|\varepsilon| > 0$; in (8) we set in this case $c = 0$.

$X = X(\delta_1)$ (section 2) is now a continuous function in two open intervals $I_1 = (0, \frac{1}{2}\delta)$ and $I_2 = (\frac{1}{2}\delta, \delta)$. Suppose that if δ_1 tends to $\frac{1}{2}\delta$ in I_1 or I_2, then X tends to a finite point X_0 on l_0 (symmetrically from both sides of l_0). Then, in virtue of Lemma 2′, $\measuredangle\, UVX_0 + \measuredangle\, VUX_0 = \delta$. But it follows from (10) and Lemma 1 that for $\varepsilon = 0$ the sum of angles at U and at V in the asymptotic triangle, with l_0 as its asymptote, is equal to δ. The contradiction shows that l_0 must be the asymptote of $\bar{L}(a, \delta)$. In other words, $\bar{L}(a, \delta)$ has the property $P0$, when (10) is satisfied.

LEMMA 6. *If*

$$0 < \operatorname{ctg}\frac{\delta}{2} < \operatorname{sh}\frac{a}{2}, \tag{11}$$

then $\bar{L}(a, \delta)$ *has the property* $P\varepsilon_0$, *where* ε_0 *is the positive solution of* (1) *for* $\delta_1 + \delta_2 = \delta$.

Proof. It is sufficient to modify slightly the proof of Lemma 5 to show that (i) on any l_ε for $\varepsilon_0 < |\varepsilon| < \pi/2$ there exists one and only one point X such that δ and a in UVX satisfy (11); (ii) X describes a curve with asymptotes $l_{-\varepsilon_0}$ and l_{ε_0}.

(11) may be written as follows:

$$\operatorname{ctg}\frac{\delta}{2} = \cos\varepsilon_0 \operatorname{sh}\frac{a}{2} - c, \qquad 0 < c < \cos\varepsilon_0 \operatorname{sh}\frac{a}{2}.$$

In the asymptotic triangle UVX_ε^∞ for $|\varepsilon| < \varepsilon_0$ we have (7) and in the degenerate triangle UVQ (9) holds. Thus, for every X_ε with $|\varepsilon| < \varepsilon_0$,

$$\cos\varepsilon \operatorname{sh}\frac{a}{2} < \operatorname{ctg}\frac{\delta(X_\varepsilon)}{2} < \infty, \tag{12}$$

where $\delta(X_\varepsilon)$ is the sum of angles at U and at V in UVX_ε. Since

$$\cos\varepsilon_0 \operatorname{sh}\frac{a}{2} - c < \cos\varepsilon \operatorname{sh}\frac{a}{2}, \qquad |\varepsilon| < \varepsilon_0,$$

we infer from (12) that there is no point X on l_ε, $|\varepsilon| < \varepsilon_0$, such that δ and a in UVX would satisfy (11). And this is the end of the proof.

If the quotient $\operatorname{sh}\frac{1}{2}a/\operatorname{ctg}\frac{1}{2}\delta$ (> 1) is close to 1, the angle $2\varepsilon_0$ between the asymptotes $l_{-\varepsilon_0}$ and l_{ε_0} of $\bar{L}(a, \delta)$ is small; when $\operatorname{sh}\frac{1}{2}a/\operatorname{ctg}\frac{1}{2}\delta$ increases to the infinity, $2\varepsilon_0$ tends to π.

Our previous considerations for one half-plane $\bar{M}_1$ or $\bar{M}_2$ can be summarized in the following

THEOREM. *For every* $\delta \in (0, \pi)$ *there exists a continuous locus* $\bar{L}_i(a, \delta)$ *in* $\bar{M}_i$ $(i = 1, 2)$ *and if* (i) $\operatorname{ctg}\frac{1}{2}\delta > \operatorname{sh}\frac{1}{2}a$ *or* (ii) $\operatorname{ctg}\frac{1}{2}\delta = \operatorname{sh}\frac{1}{2}a$ *or* (iii) $\operatorname{ctg}\frac{1}{2}\delta < \operatorname{sh}\frac{1}{2}a$, *then* $\bar{L}_i(a, \delta)$ *has the properties* (i) $Ps, P\delta, Pc$ *or* (ii) $Ps, P\delta, P0$ *or* (iii) $Ps, P\delta, P\varepsilon_0$, *where* $\cos\varepsilon_0 = \operatorname{sh}\frac{1}{2}a/\operatorname{ctg}\frac{1}{2}\delta$, *respectively.*

Case	If	then $L_1(a, \delta)$ has the property	Consequently,	If, moreover,	and $L_2(a, \pi-\delta)$ has the property	then $L_2(a, \pi-\delta)$ has the property	Number of asymptotes of $L^*(a, \delta)$	Number of asymptotic rays of $L^*(a, \delta)$
$\operatorname{sh}\frac{a}{2} < 1$	$\operatorname{ctg}\frac{\delta}{2} > \operatorname{sh}\frac{a}{2}$	Pc		$\operatorname{ctg}\frac{\pi-\delta}{2} > \operatorname{sh}\frac{a}{2}$		Pc	0	0
				$\operatorname{ctg}\frac{\pi-\delta}{2} = \operatorname{sh}\frac{a}{2}$		$P0$	1	1
				$\operatorname{ctg}\frac{\pi-\delta}{2} < \operatorname{sh}\frac{a}{2}$		$P\varepsilon_0'$	2	2
	$\operatorname{ctg}\frac{\delta}{2} = \operatorname{sh}\frac{a}{2}$	$P0$	$\operatorname{ctg}\frac{\pi-\delta}{2} > \operatorname{sh}\frac{a}{2}$		Pc		1	1
	$\operatorname{ctg}\frac{\delta}{2} < \operatorname{sh}\frac{a}{2}$	$P\varepsilon_0$	$\operatorname{ctg}\frac{\pi-\delta}{2} > \operatorname{sh}\frac{a}{2}$		Pc		2	2
$\operatorname{sh}\frac{a}{2} = 1$	$\operatorname{ctg}\frac{\delta}{2} > \operatorname{sh}\frac{a}{2}$	Pc	$\operatorname{ctg}\frac{\pi-\delta}{2} < \operatorname{sh}\frac{a}{2}$		$P\varepsilon_0'$		2	2
	$\operatorname{ctg}\frac{\delta}{2} = \operatorname{sh}\frac{a}{2}$	$P0$	$\operatorname{ctg}\frac{\pi-\delta}{2} = \operatorname{sh}\frac{a}{2}$		$P0$		1	2
	$\operatorname{ctg}\frac{\delta}{2} < \operatorname{sh}\frac{a}{2}$	$P\varepsilon_0$	$\operatorname{ctg}\frac{\pi-\delta}{2} > \operatorname{sh}\frac{a}{2}$		Pc		2	2

$\operatorname{sh}\frac{a}{2} > 1$	$\operatorname{ctg}\frac{\delta}{2} > \operatorname{sh}\frac{a}{2}$	Pc	$\operatorname{ctg}\frac{\pi-\delta}{2} < \operatorname{sh}\frac{a}{2}$		$P\varepsilon_0'$		2	2
	$\operatorname{ctg}\frac{\delta}{2} = \operatorname{sh}\frac{a}{2}$	$P0$	$\operatorname{ctg}\frac{\pi-\delta}{2} < \operatorname{sh}\frac{a}{2}$		$P\varepsilon_0'$		3	3
	$\operatorname{ctg}\frac{\delta}{2} < \operatorname{sh}\frac{a}{2}$	$P\varepsilon_0$		$\operatorname{ctg}\frac{\pi-\delta}{2} > \operatorname{sh}\frac{a}{2}$		Pc	2	2
				$\operatorname{ctg}\frac{\pi-\delta}{2} = \operatorname{sh}\frac{a}{2}$		$P0$	3	3
				$\operatorname{ctg}\frac{\pi-\delta}{2} < \operatorname{sh}\frac{a}{2}$		$P\varepsilon_0'$	2 or 4(*)	4
	$\cos\varepsilon_0 = \operatorname{ctg}\frac{\delta}{2}/\operatorname{sh}\frac{a}{2}, 0 < \varepsilon_0 < \frac{\pi}{2}$		$\cos\varepsilon_0' = \operatorname{ctg}\frac{\pi-\delta}{2}/\operatorname{sh}\frac{a}{2}, \quad 0 < \varepsilon_0' < \frac{\pi}{2}$					

(*) 2, when $\delta = \frac{\pi}{2}$; then $\varepsilon_0 = \varepsilon_0'$. 4 in other cases.

4. Definition 2, when applied to both half-planes of the Euclidean plane, provides a member $C^*(a, \delta)$ of the intersecting pencil of circles. The hyperbolic circle $L^*(a, \delta)$, which is the counterpart of $C^*(a, \delta)$, is then, in the hyperbolic plane, equal to the union of the two curves $\bar{L}_1(a, \delta)$ and $\bar{L}_2(a, \pi - \delta)$:

$$L^*(a, \delta) = \bar{L}_1(a, \delta) \cup \bar{L}_2(a, \pi - \delta).$$

The lines tangent to $L^*(a, \delta)$ at U and at V are $k_{1\delta} \cup k_{2,\pi-\delta}$ and $k'_{1\delta} \cup k'_{2,\pi-\delta}$ respectively, where k_{ia}, $k'_{ia} \in \bar{M}_i$.

$L^*(a, \delta)$ may have none, one, two, three or four asymptotes. All possibilities are tabled on p. 270 and 271. The table is a collection of theorems; we consider in it three cases: $\operatorname{sh} a/2 < 1, \operatorname{sh} a/2 = 1$ and $\operatorname{sh} a/2 > 1$; in each of them some subcases are to be distinguished.

In two cases only such as $\operatorname{ctg} \delta/2 = 1, \operatorname{sh} a/2 = 1$ and $\operatorname{sh} a/2 > 1$, $\delta = \pi/2$, the hyperbolic circles have two reflexion lines.

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