A CHARACTERIZATION OF SPHERES AMONG COMPACT 3-BODIES SATISFYING CROFTON’S THEOREM

BY

HANS HERDA (BOSTON, MASSACHUSETTS)

The aim of this note is to show the following result.

THEOREM. Let \( K \) be a compact subset of \( \mathbb{E}^3 \) having finite surface area \( s \). Let \( q \) be the infimal surface-area-bisecting cross-section of \( K \). If \( K \) satisfies Crofton’s theorem, then \( s \geq 4q \), and \( s = 4q \) implies that \( K \) is a sphere.

In particular, this result holds if \( K \) is assumed to be the closure of a bounded open set \( B \) such that \( \partial B \) is rectifiable.

In the proof we essentially follow Steinhaus [4]. Steinhaus interpreted Crofton’s [2] probabilistic methods for measuring arc length in a modern setting, but instead we use a 3-dimensional approach.

Proof. Consider the set of all lines \( \mathcal{L} \) in a Euclidean 3-space \( \Omega \). Let \( OQ \subset \Omega \) be a fixed segment in a fixed plane \( \pi \), \( P \) the foot of the perpendicular from \( O \) to a given line \( L \in \mathcal{L} \), \( P' \) the perpendicular projection of \( P \) onto \( \pi \). Characterize each line \( L \in \mathcal{L} \) by the parameter triple \((\varphi, \theta, p)\), where \( 0 \leq \varphi < \pi \) measures the angle \( P'OP \) in a plane perpendicular to \( \pi \), \( 0 \leq \theta < \pi \) measures the angle \( QOP' \) in the plane \( \pi \), and \( -\infty < p < +\infty \) measures the distance \( OP \). The one-to-one correspondence

\[
L \in \mathcal{L} \leftrightarrow (\varphi, \theta, p) \in \mathcal{S}
\]

is evident, where \( \mathcal{S} = [0, \pi) \times [0, \pi) \times (-\infty, +\infty) \), an infinite rod with square cross-section.

Define the measure \( \mu(Z) \) of any set of lines \( Z \subset \mathcal{L} \) by setting \( \mu(Z) \) equal to the 3-dimensional Lebesgue measure of \( Z^* \), the image of \( Z \) in \( \mathcal{S} \). This is a modern version of Crofton’s basic result. Consider a compact body \( K \subset \Omega \), and let \( \mathcal{A}_k (k = 1, 2, \ldots) \) be the sets of lines \( L \in \mathcal{L} \) which cut the surface \( \partial K \) in exactly \( k \) points. Crofton’s theorem in this case takes the form (\( s \) being the surface area of \( K \))

\[
s = \frac{1}{\pi} \sum_{k=1}^{\infty} k\mu(A_k),
\]
if this sum is finite (for example, if $K = \bar{B}$, where $B$ is bounded and open, and $\partial B$ is rectifiable).

Restrict $\Omega$ to the unit sphere $U$ and assume that $K \subset U$. Call $L(\varphi, \theta, p)$ the image $L \in \mathcal{L}$ of the point $(\varphi, \theta, p) \in S$. The number of intersections of $L(\varphi, \theta, p)$ with $\partial K$ in $\Omega$ is a function $f(\varphi, \theta, p)$ of three real variables.

We can now write (1) in the form

\begin{equation}
    s = \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{0}^{\pi} \int_{0}^{\pi} f(\varphi, \theta, p) \, d\varphi \, d\theta \, dp,
\end{equation}

where the left-hand side is understood as Jordan content and the right-hand side as a Lebesgue integral. An analogous formula is developed, from the integral geometry viewpoint, in [1], p. 65.

Let $p(\vec{v})$ denote the measure of the 2-dimensional projection of $K$ onto a plane normal to the direction $\vec{v}$, and write

\[ p = \inf_{\vec{v}} p(\vec{v}). \]

Similarly, let $q(\vec{v})$ denote the measure of that planar cross-section of $K$, normal to $\vec{v}$, which bisects the surface area $s$ of $K$, and write

\[ q = \inf_{\vec{v}} q(\vec{v}). \]

Since the number of intersections $f(\varphi, \theta, p)$ with $\partial K$ is not less than 2 for all lines intersecting $K$ (except a measure zero set), we infer, by use of (2), that $s \geq 4\pi \geq 4p \geq 4q$.

If $s = 4q$, then all lines (except a measure zero set) intersecting $K$ must intersect $\partial K$ in exactly 2 points, that is, $f(\varphi, \theta, p) = 2$ there. Hence $K$ is convex, and in the convex case $K$ is already known to be a sphere [3].

REFERENCES


DEPARTMENT OF MATHEMATICS
BOSTON STATE COLLEGE

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