BOHR LOCAL PROPERTIES OF $C_A(T)$

BY

FRANÇOISE LUST-PIQUARD (ORSAY)

Let $Z$ be the group of relative integers, $T$ its dual group, and $A$ a subset of $Z$. We denote by $C_A(T)$ the closed subspace of $C(T)$ (continuous functions on $T$) which is spanned by $e^{i\lambda t}$ ($\lambda \in A$).

We consider the following problem:

(P) is a Bohr local property of $C_A(T)$ if for every $k \in Z$ there exists a neighborhood $\nu(k)$ in the Bohr compactification of $Z$ such that either $A \cap \nu(k)$ is empty or $C_{A \cap \nu(k)}(T)$ has property (P). Does it imply that $C_A(T)$ has property (P)?

We give (Theorem 2) an example of a set $A$ (a subset of prime numbers) such that $C_A(T)$ has a closed subspace isomorphic to $c_0$ and for every $k \in Z$ there exists $\nu(k)$ such that $C_{A \cap \nu(k)}(T)$ is either empty or one-dimensional. This gives a negative answer to the problem for some properties (P) (Theorem 3). We will study more generally sets of first kind (Definition 2 below).

The same kind of problems was studied before by Y. Meyer and G. Godefroy for closed subspaces $L^1_A(T)$ of $L^1(T)$ ($L^1_A(T)$ is spanned by $e^{i\lambda t}$ ($\lambda \in A$)). A set $A \subseteq Z$ is called a Riesz set if $L^1_A(T) = M_A(T)$, where $M_A(T)$ is the space of Radon measures $\mu$ on $T$ such that $\hat{\mu}(k) = 0$ for every $k \in Z \setminus A$. By [8], the property $A$ is a Riesz set is a local property of $L^1_A(T)$. Some other local properties of $L^1_A(T)$ are studied in [3], Theorem 2.3.

A set $A \subseteq Z$ is called a Rosenthal set if $C_A(T) = L^p_A(T)$. One aim of this paper is to answer the following question of G. Godefroy: is the property $A$ is a Rosenthal set a local one for $C_A(T)$?

Notation and definitions are in the first section. Section 2 contains the definition of the announced set $A$ and the properties (P) for which it is a counterexample. In Section 3 we give some sufficient conditions on a set $A \subseteq Z$ for $C_A(T)$ to have a closed subspace isomorphic to $c_0$. They are more general than the ones used in Section 2. We apply them to sets $A \cap \nu(k)$, where $A$ is the set of prime numbers or $A = \{p^s\}_{s \geq 1}$ (s $\geq$ 1 a fixed integer) and $k \in Z$ lies in the closure of $A$ in the Bohr compactification of $Z$.

We thank S. Hartman for having brought [4] to our attention and for very fruitful discussions on this paper.

1. Notation and definitions. We will only consider sets $A \subseteq Z$ such that $A = (\lambda_j)_{j \geq 1}$, where $(\lambda_j)_{j \geq 1}$ is an increasing sequence. It is easy to extend definitions and results to the case where $(\lambda_j)_{j \in Z}$ is increasing.
We denote by $T$ the group $\mathbb{R}/2\pi \mathbb{Z}$ provided with its usual topology, and by $T_d$ the same group provided with the discrete topology. We denote by $Z$ the Bohr compactification of $\mathbb{Z}$, i.e., the dual group of $T_d$.

$Q$ denotes the group of rational numbers in $T$. The spaces $C(T)$ and $M(T)$ were defined in the introduction. $L^2(T)$ is the space of classes of integrable functions on $T$ with respect to Haar measure. $L^p(T)$ denotes its dual space, and $C^\prime(T)$ denotes the bidual of $C(T)$. Let $A$ be a subset of $\mathbb{Z}$. $C_A(T)$, $L^1_A(T)$ and $M_A(T)$ were already defined. $L^\infty_A(T)$ is the dual of $L^1(T)/L^1_{Z\setminus A}(T)$ and $C_A(T)$ is a closed subspace of $L^\infty_A(T)$. $\overline{A}$ is the closure of $A$ in $\mathbb{Z}$.

Let $D$ be a discrete topological space. $c_0(D)$ is the completion of finitely supported functions for the norm

$$||f|| = \sup_{t \in D} |f(t)|.$$ 

The dual of $c_0(D)$ is denoted by $l^1(D)$, the bidual is denoted by $l^\infty(D)$. For $D = \mathbb{N}$ we write only $c_0$, $l^1$, $l^\infty$.

**Definition 1.** A real sequence $(u_j)_{j \geq 1}$ is uniformly distributed modulo 1 if

$$\forall k \in \mathbb{Z}\setminus\{0\} \quad n^{-1} \sum_{j=1}^{n} \exp(2i\pi ku_j) \to 0 \quad (n \to +\infty).$$

**Definition 2 ([4]).** Let $A = (\lambda_j)_{j \geq 1} \subset \mathbb{Z}$ and let $D$ be the set of $t's \in T$ for which $(\lambda_j t)_{j \geq 1}$ is not uniformly distributed modulo 1. We will say that $A$ is a set of first kind if $D$ is countable, and a set of first kind ($Q$) if $D$ is a subset of $Q$ (which will be the case in the examples).

The following sequence of functions is associated to every $A = (\lambda_j)_{j \geq 1}$:

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi \lambda_j t).$$

It is classically used in harmonic analysis (see, e.g., [4] and [7]). It is a uniformly bounded sequence in $C_A(T)$ which converges to 0 at every $t \in T \setminus D$ and to 1 at $t = 0$. If $A$ is a set of first kind, a subsequence $(f_{n_k})_{k \geq 1}$ converges pointwise on $T$ to a function $l$ which is supported by $D$ and such that $l(0) = 1$. In particular, $(f_{n_k})_{k \geq 1}$ is a weak Cauchy sequence which does not weakly converge.

**Definition 3 ([4]).** Let $A = (\lambda_j)_{j \geq 1} \subset \mathbb{Z}$ and $A' \subset A$. For $n \geq 1$ let

$$\delta(n) = \text{Card}\{ j \mid 1 \leq j \leq n, \lambda_j \in A' \}.$$ 

We say that $A'$ has an upper positive density with respect to $A$ if

$$\limsup_{n} n^{-1} \delta(n) > 0.$$ 

If $n^{-1} \delta(n)$ has a positive limit $(n \to +\infty)$, we say that $A'$ has a positive density with respect to $A$. 
2. We first give a sufficient condition for $C_A(T)$ to have a closed subspace isomorphic to $c_0$.

**Theorem 1.** Let $\Lambda = (\lambda_j)_{j \geq 1} \subset Z$ be a set of first kind and let $R$ be an arithmetic progression such that $\Lambda \cap R$ has an upper positive density with respect to $\Lambda$. Let

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi \lambda_j t).$$

Then

(a) $\Lambda \cap R$ is a set of first kind.

(b) Let us assume that a subsequence of $(f_n)_{n \geq 1}$ converges pointwise to $l \in c_0(T)$, then $C_A(T)$ has a closed subspace isomorphic to $c_0$. $C_{A \cap R}(T)$ also has one if either $\Lambda \cap R$ has a positive density with respect to $\Lambda$ or if $(f_n)_{n \geq 1}$ converges pointwise to $l \in c_0(T)$.

**Proof.** (a) Let $R = j_0 + qZ$ ($j_0 \in Z$, $q \in N \setminus 0$). Let $\mu$ be the measure on $T$ defined by

$$\mu = \exp(2i\pi j_0 t)q^{-1} \sum_{p=0}^{q-1} \delta_{pq^{-1}}.$$

Then $\hat{\mu}(j) = 1$ if $j \in R$ and $\hat{\mu}(j) = 0$ if $j \in Z \setminus R$. Let

$$f_n'(t) = \delta^{-1}(n) \sum_{1 \leq j \leq n, \lambda_j \in A \cap R} \exp(2i\pi \lambda_j t).$$

Then

$$f_n' = n\delta^{-1}(n)f_n \mu.$$

By hypothesis, $(f_n)_{n \geq 1}$ converges pointwise to 0 outside a countable set $D \subset T$; hence $(f_n')_{n \geq 1}$ converges pointwise to 0 outside the countable set

$$D' = \bigcup_{0 < p < q^{-1}} (D + pq^{-1}).$$

(b) Let $\mu \in M(T)$ and let $\mu = \mu_1 + \mu_2$, where $\mu_2$ is the atomic part of $\mu$. If $(f_n)_k \geq 1$ converges pointwise to $l \in c_0(T)$, then

$$\langle f_n', \mu \rangle = \langle f_n', \mu_1 + \mu_2 \rangle - \langle l, \mu_2 \rangle = \langle l, \mu \rangle;$$

hence $l$ defines an element of $C_A^\perp(T) \subset C''(T)$.

By Proposition 3.1 of [5] or Lemma 4 of [6], $C_A(T)$ has a closed subspace isomorphic to $c_0$.

If $n^{-1} \delta(n) \rightarrow \delta (\delta > 0)$, the subsequence $(f_n')_k \geq 1$ converges pointwise to $\delta l \mu \delta$ which is again a non-zero function in $c_0(T)$. The same is true if $(f_n)_k \geq 1$ converges pointwise to $l \in c_0(T)$ and $n^{-1} \delta(n) \rightarrow \delta (\delta > 0)$. Thus $\Lambda \cap R$ satisfies the same hypothesis as $\Lambda$ and the same conclusion holds.
EXAMPLES. 1. Let $P = (p_j)_{j \geq 1}$ be the sequence of prime numbers. By Vinogradov's theorem ([2], Theorem 9.4), $P$ is a set of first kind ($\mathcal{Q}$). For every $t \in \mathcal{Q}$ the sequence

$$ f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi p_j t) $$

converges to $l(t)$ and $l \in c_0(\mathcal{Q})$ ([2], p. 349, and the proof of Theorem 9.4).

Let $R = j_0 + q\mathbb{Z}$ ($j_0, q \in \mathbb{N} \setminus \{0\}$). If $j_0$ and $q$ are not relatively coprime, $P \cap R$ is either empty or contains one point. If $j_0$ and $q$ are relatively coprime, $P \cap R$ has a positive density with respect to $P$ ([2], Theorem 2.5 and p. 280).

So $P$ and $P \cap R$ (if $P \cap R$ contains more than one point) satisfy the hypothesis, and hence the conclusion of Theorem 1. The fact that $P \cap R$ is a set of first kind ($\mathcal{Q}$) was already mentioned in the proof of Theorem 4 in [4].

2. Let $s$ be an integer $\geq 1$. Let $A = (f^s_j)_{j \geq 1}$ and $R = j_0 + q\mathbb{Z}$ ($j_0 \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}$). If $A \cap R$ contains $j$, it contains obviously $(j + qN)^s$; hence $A \cap R$ has a positive upper density with respect to $A$.

By Satz 9 of [11], $A = (f^s_j)_{j \geq 1}$ is a set of first kind ($\mathcal{Q}$). Let $t = a/q \in \mathcal{Q}$ ($0 \leq a < q$). Then

$$ f_n \left( \frac{a}{q} \right) = n^{-1} \sum_{j=1}^{n} \exp \left( 2i\pi \frac{a}{q} j \right) = n^{-1} \sum_{l=1}^{q} \exp \left( 2i\pi \frac{a}{q} l \right) A_{n,l,q}, $$

where $A_{n,l,q} = \text{Card } \{ j \mid 1 \leq j \leq n, j \equiv l(q) \}$. As

$$ n/q - 1 \leq A_{n,l,q} \leq n/q, $$

we have

$$ f_n(a/q) \to f_q(a/q) \quad (n \to +\infty). $$

By Lemma 2.4 of [10], for every $\varepsilon > 0$ there exists a constant $C(s, \varepsilon)$ such that

$$ |f_q(a/q)| \leq C(s, \varepsilon)q^{-1/2^{s-1}}, \quad s \geq 2; $$

hence the function $a/q \sim f_q(a/q)$ lies in $c_0(\mathcal{Q})$. If $s = 1$, we have $f_q(0) = 1$ and $f_q(a/q) = 0$ if $a \neq 0$.

So $A = \{ f_j^s \}_{j \geq 1}$ and $A \cap R$ (as soon as it is not empty) satisfy the hypothesis, and hence the conclusion of Theorem 1.

We will need one more property of the set $P$.

LEMMA 1 ([8]). Let $P$ be the set of prime numbers. For every $k \in \mathbb{Z} \setminus \{1, -1\}$ there exists a neighborhood $v(k)$ in $\mathbb{Z}$ such that $v(k) \cap P$ is either empty or reduced to $\{k\}$.

Let us recall the proof: for every $q \in \mathbb{Z} \setminus \{0\}$, $\overline{q\mathbb{Z}}$ is an open set in $\mathbb{Z}$ and $q\mathbb{Z} \cap \mathbb{Z} = q\mathbb{Z}$. Take $v(0) = 4\mathbb{Z}$ and $v(k) = k + 3k\mathbb{Z}$ if $k \notin \{0, 1, -1\}$.

Here is the announced example:
THEOREM 2. Let $P$ be the set of prime numbers, $R = 2 + 5\mathbb{Z}$ and $\Lambda = P \cap R$. Then:

(a) For every $k \in \mathbb{Z}$ there exists a neighborhood $v(k)$ in $\mathbb{Z}$ such that $\Lambda \cap v(k)$ is either empty or reduced to $\{k\}$.

(b) $C_A(T)$ has a closed subspace isomorphic to $c_0$.

Proof. (a) For $k \neq \pm 1$, $v(k)$ is chosen as in Lemma 1. $v(1) = 1 + 5\mathbb{Z}$ and $v(-1) = -1 + 5\mathbb{Z}$ are disjoint from $R$.

(b) follows from the recalled properties of $P$ and Theorem 1.

Let us now look for which properties $(P)$ this example shows that $(P)$ is not a Bohr local property of $C_A(T)$.

A Banach space $X$ has the Schur property if every weak Cauchy sequence in $X$ is norm convergent. Every finite dimensional space as well as $l^1$ have the Schur property but $c_0$ does not. A Banach space $X$ has the Radon–Nikodým property if every bounded linear operator: $l^1(T) \to X$ is representable by a strongly measurable function: $T \to X$. By [5], $C_A(T)$ has this property iff $\Lambda$ is a Rosenthal set (and $L^\infty_A(T)$ has this property iff $\Lambda$ is a Riesz set).

THEOREM 3. Let $\Lambda$ be as in Theorem 2. Then $C_A(T)$ is locally a Rosenthal set and has locally the Schur property, but $C_A(T)$ has neither of these properties.

Proof. By Theorem 2 the result is obvious for the Schur property. If $\Lambda \cap v(k) = \{k\}$, clearly

$$C_{A \cap v(k)}(T) = L^\infty_{A \cap v(k)}(T).$$

By Theorem 2, $C_A(T)$ (hence $L^\infty_A(T)$) has a closed subspace isomorphic to $c_0$. By [1], $L^\infty(A)(T)$, which is a dual space, has a closed subspace isomorphic to $l^\infty$; hence $L^\infty_A(T)$ cannot be the same space as the separable space $C_A(T)$ and $\Lambda$ is not a Rosenthal set.

By using the same example we can solve Problem 1 of [6].

DEFINITION 4 ([6], Lemma 1, Definition 4, Eberlein theorem). A function $F \in L^\infty(T)$ is totally ergodic if there exists a net $(v_\alpha)_{\alpha} \in l^1(T)$ such that

(i) $\|v_\alpha\|_{l^1(T)} = \langle v_\alpha, 1 \rangle = 1$;

(ii) $v_\alpha$ is supported by $v_\alpha(0)$ and $(v_\alpha(0))_\alpha$ is a basis of neighborhoods of $\{0\}$ in $\mathbb{Z}$.

(iii) $\forall k \in \mathbb{Z}$, $(e^{-2\pi i k t} F)\ast v_\alpha \to F(k)$ uniformly on $T$.

Every continuous function on $T$ is totally ergodic. Hence, if $\Lambda$ is a Rosenthal set, every function $F \in L^\infty_A(T)$ is totally ergodic. The converse does not hold since we have

THEOREM 4. Let $\Lambda$ be as in Theorem 2. Every function in $L^\infty_A(T)$ is totally ergodic but $\Lambda$ is not a Rosenthal set.

Proof. We have already proved in Theorem 3 that $\Lambda$ is not a Rosenthal
set. For every \( k \in \mathbb{Z} \) let \( v(k) \) be as in the proof of Theorem 2. Let

\[ G_a = (e^{-2i\pi k \cdot F}) \ast v_a. \]

Hence

\[ \hat{G}_a(n) = \hat{F}(k+n) \hat{v}_a(n) \quad \text{for every } n \in \mathbb{Z}. \]

As soon as \( k + v_a(0) \subset v(k) \) we have \( \hat{G}_a(n) = 0 \) if \( n \neq 0 \) and \( \hat{G}_a(0) = \hat{F}(k) \) (two cases must be considered: either \( v(k) \cap A = \emptyset \) or \( v(k) \cap A = \{k\} \)). This means that \( G_a \) is the constant function \( \hat{F}(k) \) and that \( F \) is totally ergodic.

3. We now generalize Theorem 1 by using an idea of [5].

**Theorem 5.** Let \( A = (\lambda_j)_{j \geq 1} \subset \mathbb{Z} \) be a set of first kind. We assume that

\[ f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi \lambda_j t) \]

converges pointwise on \( T \) to \( l \in c_0(T) \). If \( A' \subset A \) has a positive upper density with respect to \( A \), then \( C_{A'}(T) \) has a closed subspace isomorphic to \( c_0 \).

In Theorem 3.1 of [5] or Theorem 3 of [6] we only considered the case

\[ A = \mathbb{Z}, \quad f_n(t) = (2n+1)^{-1} \sum_{j=-n}^{n} \exp(2i\pi jt); \]

hence \( l(t) = 0 \) for \( t \neq 0 \) and \( l(0) = 1 \).

**Proof.** By assumption there exists a subsequence \( (n_k)_{k \geq 1} \) such that

\[ n_k^{-1} \delta(n_k) \to \delta, \]

where \( \delta \) is the positive upper density of \( A' \) with respect to \( A \). We then have to consider the subsequence \( (f_{n_k})_{k \geq 1} \). We will however write \( n \geq 1 \) instead of \( (n_k)_{k \geq 1} \) in order to simplify the notation.

By Proposition 3.1 of [5], Theorem 5 will be proved if we can exhibit a non-zero function \( l' \in c_0(T) \) and a uniformly bounded net \( (f'_n) \) in \( C_{A'}(T) \) such that

\[ \langle f'_n, \mu \rangle \to \langle l', \mu \rangle \quad \text{for every } \mu \in M(T). \]

Let us define:

\[ v_n = n^{-1} \sum_{1 \leq j \leq n, \lambda_j \in A} \delta_{\lambda_j}, \]

whence \( v_n \in l^1(\mathbb{Z}) \) and \( \hat{v}_n = f_n \).

\[ v'_n = 1_{A'} v_n = n^{-1} \sum_{1 \leq j \leq n, \lambda_j \in A'} \delta_{\lambda_j}, \]

whence \( v'_n \in l^1(\mathbb{Z}) \);

\[ f'_n(t) = n^{-1} \sum_{1 \leq j \leq n, \lambda_j \in A'} \exp(2i\pi \lambda_j t), \]

whence \( \hat{v}'_n = f'_n \). Then

(i) \( \|v_n\|_{l^1(\mathbb{Z})} = \langle v_n, 1 \rangle = 1 = f_a(0) \),
(ii) \( \forall t \in T \), \( \dot{v}_n(t) \to l(t) \),

(iii) \( \|v_n\|_{L^2(\mathbb{Z})} = \langle v_n, 1 \rangle = f_n'(0) = n^{-1} \delta(n) \leq 1 \) and \( n^{-1} \delta(n) \to \delta \).

By (i) and (ii) there exists a positive measure \( \nu \in M(\mathbb{Z}) \) such that

\[
\dot{\nu} = l \quad \text{and} \quad \nu_n \to \nu \quad \sigma(M(\mathbb{Z}), C(\mathbb{Z})).
\]

Obviously, \( \nu \) is supported by \( \bar{\mathbb{Z}} \).

Let us show that \( \langle f_n', \mu \rangle \to 0 \) for every diffuse measure \( \mu \in M(T) \):

\[
|\langle f_n', \mu \rangle| = |\langle v_n', \tilde{\mu} \rangle| = n^{-1} \delta(n)|\langle v_n' \| v_n' \| \tilde{\mu} \rangle| \\
\leq (n^{-1} \delta(n))^{1/2} |\langle v_n', |\tilde{\mu}|^2 \rangle|^{1/2} \leq |\langle v_n', |\tilde{\mu}|^2 \rangle|^{1/2} = |\langle f_n', \mu*\tilde{\mu} \rangle|^{1/2}
\]

(where \( \tilde{\mu}(E) = \mu(-E) \) for every Borel set \( E \subset T \)). By assumption,

\[
\langle f_n', \mu*\tilde{\mu} \rangle \to \langle l, \mu*\tilde{\mu} \rangle = 0.
\]

Let now \( v' \in M(\mathbb{Z}) \) be a limit point of \( (v_n)_{n \geq 1} \) for \( \sigma(M(\mathbb{Z}), C(\mathbb{Z})) \), i.e.,

\[
\langle v_n', \varphi \rangle \to \langle v', \varphi \rangle \quad \text{for every } \varphi \in C(\mathbb{Z}).
\]

Obviously, \( v' \) is supported by \( \bar{\mathbb{Z}} \).

Let us define

\[
l'(t) = \dot{v}'(t) = \lim \dot{v}_n(t) = \lim f_n'(t) \quad (t \in T).
\]

We shall prove that \( l' \in c_0(T_\mathbb{R}) \). As a consequence, \( \langle l', \mu \rangle = 0 \) for every diffuse measure \( \mu \) on \( T \); hence

\[
\langle l', \mu \rangle = 0 = \lim \langle f_n', \mu \rangle,
\]

and for every \( \mu \in M(T) \)

\[
\langle l', \mu \rangle = \lim \langle f_n', \mu \rangle.
\]

We first prove that \( v' \) defines a continuous linear form on \( L^1(\nu) \). For every \( \varphi \in C(\mathbb{Z}) \)

\[
|\langle v', \varphi \rangle| = \lim |\langle v_n', \varphi \rangle| \leq \lim |\langle v_n', |\varphi| \rangle| \\
\leq \lim |\langle v_n', |\varphi| \rangle = |\langle v, |\varphi| \rangle| = \|\varphi\|_{L^1(\nu)}.
\]

In particular, \( v' \) is absolutely continuous with respect to \( \nu \), i.e., \( v' \in L^1(\nu) \); hence there exists a sequence \( (g_k)_{k \geq 1} \) in \( C(\mathbb{Z}) \) such that

\[
\|g_kv - v'\|_{M(\mathbb{Z})} \to 0
\]

and we may assume that every \( g_k \) is finitely supported on \( T \). A fortiori,

\[
\|\hat{g_k}v - l'\|_{L^\infty(T_\mathbb{R})} \to 0.
\]

As \( l \in c_0(T_\mathbb{R}) \) and \( \hat{g_k}v = \hat{g_k} \ast l \), we have proved that \( l' \in c_0(T_\mathbb{R}) \).
Remark 1. Let \( \Lambda \subset \mathbb{Z} \); let us recall that \( \Lambda \) is an \( M_0 \)-set in \( \mathbb{Z} \) if there exists a measure \( v \in M(\mathbb{Z}) \), supported by \( \Lambda \), such that \( \tilde{\mathcal{N}} \subset c_0(T) \). Under the assumptions of Theorem 5 we have proved that \( \Lambda \) and \( \Lambda' \) are \( M_0 \)-sets in \( \mathbb{Z} \). In particular, the sets \( \tilde{P} \) and \( \{ \tilde{I}_{j} \}_{j \geq 1} \) (\( s \in \mathbb{N}\setminus\{0\} \)) are \( M_0 \)-sets in \( \tilde{Z} \). By [9], \( \tilde{P} \) is a set of Haar measure 0.

We could not answer the following question:

Let \( \Lambda \subset \mathbb{Z} \) be such that \( \Lambda \) is an \( M_0 \)-set. Does \( C_{\Lambda}(T) \) have a closed subspace isomorphic to \( c_0 \)? (P 1374)

We now give a sufficient condition in order that \( \Lambda' = \Lambda \cap v(k) \) has a positive upper density with respect to \( \Lambda \) when \( \Lambda \) is a set of first kind \( (Q) \), \( k \in \mathbb{Z} \) and \( v(k) \) is any neighborhood of \( k \) in \( \mathbb{Z} \). Obviously, \( k \) must be in \( \Lambda \). We generalize the method used in [4] (Lemma 3 and the proof of Theorem 4) for \( k = 0 \) and \( \Lambda = \{ \tilde{I}_{j} \}_{j \geq 1} \) on one hand, 'for \( k = +1 \) and \( \Lambda = \tilde{P} \) on the other hand.

**Theorem 6.** Let \( \Lambda = \{ \lambda_{j} \}_{j \geq 1} \subset \mathbb{Z} \) be a set of first kind \( (Q) \). Let \( k \in \mathbb{Z} \) be such that, for every \( R = k + q \mathbb{Z} \) \( (q \geq 1) \), \( \Lambda \cap \tilde{R} \) has a positive upper density with respect to \( \Lambda \). Then \( k \in \tilde{\Lambda} \) and, for every neighborhood \( v(k) \) in \( \mathbb{Z} \), \( \Lambda \cap v(k) \) has a positive upper density with respect to \( \Lambda \).

**Proof.** For every \( k \in \mathbb{Z} \) and every \( v(k) \) there exist ([4], Lemma 3) an integer \( q \geq 1 \), irrational numbers \( \beta_1, \ldots, \beta_L \) which are independent over \( Q \) and \( \delta > 0 \) such that \( v(k) \) contains the set

\[
E_k = \{ n \in \mathbb{Z} \mid n \in k + q \mathbb{Z} = R \} \text{ and } |\exp(2\pi i n \beta_l q^{-1}) - \exp(2\pi i k \beta_l q^{-1})| < \delta, 1 \leq l \leq L \}.
\]

By assumption and Theorem 1(a) the sequence \( (\lambda_{j} \beta_l q^{-1}) \}_{j \geq 1} \) is uniformly distributed for every \( t \in Q \). Let us put \( \Lambda \cap R = \{ \lambda_{j} \}_{m \geq 0} \). Thus for every \( h_1, \ldots, h_L \in \mathbb{Z} \) not all zero we have

\[
r^{-1} \sum_{m=1}^{r} \exp(2\pi i \lambda_{j} \beta_l q^{-1}) \rightarrow 0 \quad (r \rightarrow +\infty).
\]

Let

\[
(t_{m}^{(l)})_{l=1}^{L} = (\lambda_{j} \beta_l q^{-1})_{l=1}^{L} \in T^L \quad \text{and} \quad \mu_r = r^{-1} \sum_{m=1}^{r} \delta_{t_{m}^{(1)}, \ldots, t_{m}^{(L)}}.
\]

We have just proved that for every \( f \in C(T^L) \)

\[
\langle \mu_r, f \rangle \rightarrow \int f dt^{(1)} \ldots dt^{(L)}.
\]

By taking a non-zero \( f \) with values in \([0, 1]\), supported by a suitably chosen neighborhood \( (k \beta_l q^{-1} \) \) we see that \( \Lambda \cap E_k \) has a positive density with respect to \( \Lambda \cap R \), and hence a positive upper density with respect to \( \Lambda \).

**Theorem 7.** (a) Let \( \tilde{P} \) be the set of prime numbers. For every neighborhood \( v(+1) \in \tilde{Z} \), \( C_{\tilde{P} \cap \mathbb{N}(1)}(T) \) has a closed subspace isomorphic to \( c_0 \). The same is true for every \( v(-1) \).
(b) Let $\Lambda = \{j^s\}_{j \geq 1}$. If $s$ is an even integer,

$$\overline{\Lambda} \cap \mathbb{Z} = \Lambda \cup \{0\}.$$  

If $s$ is an odd integer,

$$\overline{\Lambda} \cap \mathbb{Z} = \Lambda \cup (-\Lambda) \cup \{0\}.$$  

For every $k \in \overline{\Lambda}$ and every neighborhood $v(k)$ in $\mathbb{Z}$, $\Lambda \cap v(k)$ has a positive upper density with respect to $\Lambda$ and $C_{\Lambda \cap v(k)}(T)$ has a closed subspace isomorphic to $c_0$.

Proof. (a) $P \cap v(1)$ or $P \cap v(-1)$ has a positive upper density with respect to $P$ by the proof of Theorem 4 in [4] or by Theorem 6 and the properties of $P \cap R$ recalled in Example 1. We conclude by Theorem 5.

(b) The fact that $\{j^2\}_{j \geq 0} \cap \mathbb{Z} = \{j^2\}_{j \geq 0}$ is proved in Lemma 3.6.2 of [3] by considering two cases: $k < 0$ and $k \in N \setminus \Lambda$. We follow the same method. Let $s$ be an integer $\geq 2$ and $k \in N \setminus \Lambda$, $k \neq 0$. There exist a prime number $p$, $n \geq 0$, $n'$, $k' \geq 1$ such that $k = p^{n+n'}k'$, $1 \leq n' < s$, and $p$ does not divide $k'$. Let

$$R = k + p^{(n+1)}\mathbb{Z};$$

then $\Lambda \cap R$ is empty. Let $s$ be $\geq 2$ and $k < 0$. Let

$$R = k + 3|k|^s\mathbb{Z}.$$  

If $\Lambda \cap R$ is not empty, there exist $j \in N$ and $q \in \mathbb{Z}$ such that

$$j^s = k(1 - 3|k|^{s-1}q) = |k|^{s-1}q.$$  

These two numbers are coprime, and hence $|k|, |1 + 3|k|^{s-1}q| \in \Lambda$. If $s$ is an even integer, this is impossible because the equation $-1 \equiv j^2(3)$ has no solution. (This is taken from [8].) Hence

$$\overline{\Lambda} \cap \mathbb{Z} = \Lambda \cup \{0\}.$$  

If $s$ is odd, this implies $k \in -\Lambda$. Hence

$$\overline{\Lambda} \cap \mathbb{Z} = \Lambda \cup (-\Lambda) \cup \{0\}.$$  

If $k \in \Lambda \cup \{0\}$ and $s \geq 1$, $\Lambda \cap R$ is not empty for every $R = k + q\mathbb{Z}$ ($q \geq 1$) and has a positive upper density with respect to $\Lambda$ as it was recalled in Example 2. If $k \in -\Lambda$ and $s$ is odd, for every $R = k + q\mathbb{Z} = -j_0 + q\mathbb{Z}$ let $a \in N$ be such that $aq - j_0 > 0$. Then

$$(aq - j_0)^s \in \Lambda \cap R;$$  

hence $\Lambda \cap R$ has a positive upper density with respect to $\Lambda$. In both cases, by Theorem 6, $v(k) \cap \Lambda$ is not empty for every $v(k)$; hence $k \in \overline{\Lambda}$. Theorem 6 again and Theorem 5 now conclude the proof.
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