SOME REMARKS

ON INFINITESIMAL PROJECTIVE TRANSFORMATIONS
IN RECURRENT AND RICCI-RECURRENT SPACES

BY

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1. An $n$-dimensional ($n > 2$) Riemannian space is called recurrent [3], if the curvature tensor satisfies the condition

$$R_{hkij,l} = c_l R_{hkij} \neq 0$$

for some vector $c_l$, where the comma indicates the covariant derivative.

Contracting (1) with $g^{hk}$ we see that for a recurrent space the relation

$$R_{ij,l} = c_l R_{ij}$$

holds.

Spaces whose Ricci-tensor $R_{ij}$ satisfies (2) for some vector $c_l$, where $n > 2$, are called Ricci-recurrent. We assume, moreover, that $R_{ij} \neq 0 \neq c_l$.

Thus every recurrent space with $R_{ij} \neq 0$ is Ricci-recurrent.

Let a Riemannian space admit an infinitesimal projective transformation with respect to the vector field $v^i$. Denote by $\mathcal{L}$ the Lie derivative with respect to this field. Then we have [4]:

$$\mathcal{L}T^k_{jk} = \delta^k_j A_k + \delta^k_A j,$$

$$\mathcal{L}R^{hij}_{ijk} = \delta^h_j A_{i,k} - \delta^h k A_{i,j},$$

$$\mathcal{L}R_{ij} = (1-n)A_{i,j},$$

$$\mathcal{L}P^h_{ijk} = 0,$$

where $T^k_{jk}$ are the Christoffel symbols, $A_j$ is a gradient vector field, and $P^h_{ijk}$ denotes the projective curvature tensor, i.e.

$$P^h_{ijk} = R^h_{ijk} - \frac{1}{n-1} \left( \delta^h_k R_{ij} - \delta^h_j R_{ik} \right).$$
If \( A_j = 0 \), the infinitesimal projective transformation is an affine one.

Infinitesimal projective transformations in recurrent and Ricci-recurrent spaces, not necessarily of \( \neq 0 \) recurrent vector but with definite metric, have been studied by Prvanovitch [1].

For such recurrent and Ricci-recurrent spaces Prvanovitch proved:

(a) If a Ricci-recurrent (recurrent) space admits an infinitesimal projection transformation, then it is an Einstein space or the vector field \( A_i \) satisfies the condition

\[
A_i R_{ij} R^{ij} + A_i R_{ij} R^{ij} + \frac{1}{2} R_{ij} R^{ij} \mathcal{L} c_i = 0.
\]

(b) If a compact Ricci-recurrent space, which is not an Einstein space, admits an infinitesimal projective transformation such that \( \mathcal{L} c_j = 0 \), then this transformation is a motion.

In the present paper we shall investigate infinitesimal projective transformations in recurrent and Ricci-recurrent spaces, whose definitions have been given above.

Formulas

(8) \[ \mathcal{L} T_{jk}^i = \delta_j^i A_k + \delta_k^i A_j, \quad A_j = -\frac{1}{2} \mathcal{L} c_j \]

(Theorem 1) as well as Theorems 2-4 are proved without assuming the definiteness of the metric. Further theorems are valid under the hypothesis that the metric is positive definite.

2. Lemma 1. If \( e_i \) and \( T_{ij} \) are numbers satisfying

(9) \[ e_i T_{mj} + e_j T_{mi} = 0, \]

then either all the \( e_i \) are zero, or all the \( T_{ij} \) are zero.

Proof. Suppose that one of the \( e_i \)'s, say \( e_q \), is non-zero. Then (9) with \( i = j = q \) gives \( 2 e_q T_{mq} = 0 \), and therefore \( T_{mq} = 0 \) for all \( m \).

Putting \( i = q \) in (9) we now have \( e_q T_{mq} = 0 \), and therefore \( T_{mi} = 0 \) for all \( m \) and \( j \).

Lemma 2. If the recurrent vector of a Ricci-recurrent space is a gradient, then the equation

(10) \[ R_{ri} R^{rij} = \frac{1}{2} R R_{rij} \]

holds, where \( R = g^{ij} R_{ij} \).

Proof. It can be easily verified that, since \( c_i \) is a gradient, (2) gives

\[ R_{ij,lm} - R_{ij,ml} = (c_{l,m} - c_{m,l}) R_{ij} = 0, \]

whence using Ricci's identity we obtain

(11) \[ R_{rf} R^{rf}_{ilm} + R_{tr} R^{tr}_{ilm} = 0. \]
By differentiating (11) covariantly and taking into consideration (2), we find
\[ c_k (R_{ij} R_{ilm} + R_{ir} R_{jlm}) + R_{ij} R_{ilm,k} + R_{ir} R_{jlm,k} = 0. \]

This, because of (11), yields
\[ (12) \quad R_{ij} R_{ilm,k} + R_{ir} R_{jlm,k} = 0. \]

Contracting (12) with \( g^{jk} \) and making use of the relation
\[ R_{ijkl,r} = R_{ijkl} - R_{il,j}, \]
which follows easily from Bianchi’s identity, we get
\[ R_{ij}(R_{mr,i} - R_{mi,r}) + R_{ir}(R_{mr,j} - R_{mj,r}) = 0. \]

But the last equation, in virtue of (2) and
\[ (13) \quad c_r R_{rj} = \frac{1}{2} R c_j, \]
which is an immediate consequence of (2), and the formula \( R_{ij,r} = \frac{1}{2} R_{ij} \) (see [5], p. 19, equation (1.61)) is equivalent to
\[ (14) \quad c_i (R_{rm} R_{rj} - \frac{1}{2} R R_{mj}) + c_j (R_{rm} R_{ri} - \frac{1}{2} R R_{mi}) = 0. \]

Putting \( T_{ij} = R_{ri} R_{rj} - \frac{1}{2} R R_{ij} \) we see that (14) can be written as
\[ c_i T_{mj} + c_j T_{mj} = 0, \]
and is therefore of the form (9).

Hence, since \( c_i \neq 0 \), Lemma 1 gives \( T_{ij} = 0 \), which proves our lemma.

**Lemma 3.** If the recurrent vector of a Ricci-recurrent space admitting an infinitesimal projective transformation is a gradient, then the Ricci-tensor satisfies the equation
\[ (15) \quad A_{rk} R_{rj} = A_{rj} R_{rk}. \]

**Proof.** Applying to (11) the Lie derivative and making use of (4) and (5), we find
\[
(1 - n) A_{ir} R_{jim} + A_{im} R_{ir} - A_{ij} R_{im} + A_{im} R_{ij} - A_{ij} R_{im} +
(1 - n) A_{rj} R_{rj,lm} = 0.
\]

The contraction of the last equation with \( g^{ij} \) gives
\[
(1 - n) A_{rj} R_{rj,lm} + A_{rm} R_{rj} - A_{rj} R_{rm} + A_{rm} R_{rj} - A_{rj} R_{rm} +
(1 - n) A_{rj} R_{rj,lm} = 0.
\]

Whence, since \( A_{ij} \) is symmetric and, consequently, \( A_{rj} R_{rj,lm} = 0 \), we obtain (15).
Lemma 4. If the scalar curvature of a Ricci-recurrent space admitting infinitesimal projective transformation is \( \neq 0 \) then the vector field \( A_i \) satisfies the condition

\[
A_{i,r}^r = 0.
\]

Proof. M. Prvanovitch ([1], equations (2.11) and (2.12)) proved that if the Ricci-recurrent space, whose recurrent vector is a gradient, admits an infinitesimal projective transformation, then the following equations hold:

\[
A_{r}^r R_{st}^s R_{st}^t - R A_{r,s} R_{rs}^s = 0,
\]

\[
R A_{r,s}^r - n A_{r,s} R_{rs}^s = 0.
\]

Contracting (2) with \( g^{ij} \), we obtain \( R_{ij} = c_i R_i \), whence it follows that \( c_i \) is a gradient. Hence, the equations (10) and (15) are satisfied.

It is easy to see that (10) gives

\[
R^s R_{rs} = \tfrac{1}{2} R^2.
\]

Substituting (19) into (17), we get \( \tfrac{1}{2} R^2 A_{r}^r R_{st}^s R_{st}^t - R A_{r,s} R_{rs}^s = 0 \). This, together with (18), yields \( (n-2) R A_{r,s} R_{rs}^s = 0 \). But the last equation, in virtue of (18), implies \( R A_{r,s}^r = 0 \), which proves our lemma.

Lemma 5. If the Ricci-recurrent space with \( R \neq 0 \) admits an infinitesimal projective transformation, then its Ricci-tensor satisfies the equation

\[
(\mathcal{L}_c)_i R_{ij} = -3 A_r R_{ij}^r.
\]

Proof. Applying to (2) the Lie derivative and using (5), (3), and the well-known formula

\[
\mathcal{L}_T T_{ij,l} = (\mathcal{L}_T T_{ij})_l - T_{ij} \mathcal{L}_T T_{ij} - T_{ij} \mathcal{L}_T T_{ij},
\]

we get

\[
(1 - n) A_{i,j} - A_i R_{ij} - 2 A_i R_{ij} - A_j R_{ij} = (\mathcal{L}_c)_i R_{ij} + (1 - n) c_i A_i.
\]

The contraction of (21) with \( g^{ij} \), in virtue of (16), yields

\[
R \mathcal{L}_c = -2 R A_i - 2 A_r R_{ij}^r.
\]

Contracting further this with \( R_{ij}^l \) and taking into consideration (10), we obtain easily (20).

Lemma 6. If the Ricci-recurrent space with \( R \neq 0 \) admits an infinitesimal projective transformation, then the vector field \( A_i \) satisfies the following equation:

\[
A_r R_{ij}^r = \tfrac{1}{2} R A_i.
\]
Proof. Contracting (21) with $g^{ij}$ and using (20), we obtain
\[(1 - n)(A_{i,j}^r - c_r A_{r,j}^i) = RA_i.\]

Since $A_{i,j}^r = A_{r,j}^i + A_r R_{i,j}^r$, which easily follows from Ricci's identity, the last but one equation, in view of (16), can be written as
\[(24) \quad (1 - n)(A_r R_{i,j}^r - c_r A_{r,j}^i) = RA_i.\]

This, by contraction with $R_{k}^j$, yields
\[(1 - n)(A_r R_{i,j}^r R_{k}^j - c_r A_{r,j} R_{k}^j) = RA_i R_{k}^j,\]

or, because of (15),
\[(1 - n)(A_r R_{i,j}^r R_{k}^j - c_r R_{i,k} A_{r,j}) = RA_i R_{k}^j.\]

Making now use of (10) and (13), we find
\[\frac{1}{4}(1 - n) R (A_r R_{i}^r - c_r A_{r,k}) = RA_r R_{k}^r.\]

But this together with (24) implies $\frac{1}{4} R^2 A_r = RA_r R_{k}^r$, which gives (23). Our lemma is thus proved.

3. Substituting now (23) into (22), we have $R \mathcal{L} c_i = -3 A_i$. But this, when $R \neq 0$, gives
\[(25) \quad \mathcal{L} c_i = -3 A_i.\]

Hence, we have the following

**Theorem 1.** If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation satisfies (5).

**Theorem 2.** If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation is an affine one if and only if $\mathcal{L} R_{ij} = 0$.

Proof. If $A_i = 0$, then the equation $\mathcal{L} R_{ij} = 0$ follows immediately from (5).

Suppose now $\mathcal{L} R_{ij} = 0$. Then, in virtue of (5), we have $A_{i,jk} = 0$, which, using the Ricci identity, gives
\[(26) \quad A_{i,jk} - A_{i,kj} = -A_r R_{i,jk}^r = 0.\]

Contracting (26) with $g^{ij}$ and taking into consideration (23), we obtain $\frac{1}{4} R A_r = 0$. This, since $R \neq 0$, completes the proof.

**Theorem 3.** If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation such that $(\mathcal{L} c_i)_j = 0$, then this transformation is an affine one.

This result follows easily from (25) and Theorem 2.
Theorem 4. If a recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation is always an affine one.

Proof. M. Prvanovitch ([1], equation (2.7), p. 220) proved that the projective curvature tensor of a recurrent space, admitting an infinitesimal projective transformation, satisfies the relation

$$(\mathcal{L}c_i)P_{ijlk} = g_{il}A_rP_{r,ijlk} - 2A_lP_{r,ijlk} - A_iP_{ijlk} - A_jP_{lik} - A_kP_{lijl}.$$ 

Because of $P_{r,rij} = P_{r,irj} = P_{r,ijr} = 0$, the contraction of the last equation with $g^{il}$ gives

$$(\mathcal{L}c_r)P_{r,ijk} = (n-2)A_rP_{r,ijk}.$$ (27)

Substituting (7) into (27) and contracting with $g^{ij}$, we obtain

$$n(\mathcal{L}c_r)R_{r,k} - R\mathcal{L}c_k = (n-2)(nA_rR_{r,k} - RA_k).$$

Making now use of (20), (23) and (25), we obtain easily $(n+1)RA_k = 0$, which proves our theorem.

4. Suppose now that the metric of the investigated space is positive definite. Then, contracting (2) with $R^{ij}$, we get

$$(R^{ij}R_{ij})_{,l} = 2c_iR^{ij}R_{ij},$$

whence, since $R^{ij}R_{ij} \neq 0$, it follows that $c_i$ is a gradient.

Taking into consideration (19), which now holds, it can be easily verified that the scalar curvature is $\neq 0$. Therefore the equations (16), (23) and (25) are satisfied.

Hence, we have

Theorem 5. If the Ricci-recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation satisfies (8).

Since the scalar curvature of a recurrent space with positive definite metric cannot be zero, which is an immediate consequence of the relation $R^{ijkl}R_{klij} = R^2$ (see [2], equation (10)), Theorem 4 yields

Theorem 6. If a recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is always an affine one.

Now, the following theorem will be proved

Theorem 7. If a compact orientable Ricci-recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is a motion.

Proof. Applying to the inner product $v^r c_r$, the Laplace operator $\Delta$, we obtain

$$\Delta(v^r c_r) = g^{ij}(v^{r,i}c_r + v^{r,j}c_{r,j}),$$
which, since \( c_t \) is a gradient, can be written in the form

\[
\Delta (v^t c_r) = g^{ij} (v^t, c_r + v^i c_{r,j}).
\]

Taking into account the well-known formula \( \mathcal{L} a_t = v^t a_{i,r} + v^r a_{i,j} \), we obtain in our case \( \Delta (v^t c_r) = g^{ij} (\mathcal{L} c_i),_j \), or, in virtue of (25), \( \Delta (v^t c_r) = -3 A^r_{,r} \).

But, since the metric is positive definite, equation (16) holds, and therefore \( \Delta (v^t c_r) = 0 \) everywhere in the space.

Making now use of Bochner's theorem ([5], p. 30), we obtain \( v^t c_r = \text{const} \) and \( \mathcal{L} c_t = (v^t c_r),_j = 0 \).

Hence, because of (25), this transformation is an affine one. But it is known ([5], p. 58) that an infinitesimal affine transformation in a compact orientable Riemannian space is always a motion. This remark completes the proof of our theorem.

The following theorem is a consequence of Theorem 6:

**Theorem 8.** If a compact orientable recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is a motion.

**REFERENCES**


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