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ON $P(m)$-ULTRAFILTERS ON N

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Solomon [7] has shown that the assertion $Q(\mathfrak{c})$ which follows from the assertion $P(\mathfrak{c})$ (due to Booth [2]) along with the negation of the Continuum Hypothesis (CH) implies the existence of $P(\aleph_1)$-ultrafilters on N which are not $P(\aleph_2)$-ultrafilters. He has also proved [6] that Martin's Axiom (MA) with non-CH implies the existence of such ultrafilters which are minimal in the Rudin-Keisler ordering.

Szymański [8] has proved that under the assertion S (due to Martin and Solovay [5]) there exist $P(m)$-ultrafilters which are not $P(m^+)$-ultrafilters for each regular m, $\aleph_0 < m < \mathfrak{c}$.

Recall that the assertions S and $P(\mathfrak{c})$ follow from MA, and $P(\mathfrak{c})$ is weaker than MA (see Kunen and Tall [4]). It is also known to the author (a letter from Prof. E. van Douven) that S and $P(\mathfrak{c})$ are equivalent.

The aim of this paper is to show, assuming $P(\mathfrak{c})+$non-CH, that for each regular m, $\aleph_0 < m < \mathfrak{c}$, there exist $2^{\mathfrak{c}}$ minimal as well as $2^{\mathfrak{c}}$ non-minimal ultrafilters which are $P(m)$-ultrafilters and are not $P(m^+)$-ultrafilters.

1. Preliminaries. Let N denote the set of all positive integers. By $[A]^{<\omega}$ we denote the set of all finite subsets of A and, for $A \subset N$ and $n \in N$, $A-n$ is the set of all elements of A which are not less than n. Let βN be the set of all ultrafilters on N and let $N^* = \beta N - N$ be the set of all free ultrafilters on N. For two subsets A and B of N, A is called *almost contained* in B, $A \subset_* B$, if $A-B$ is finite. A family $\mathscr{A}$ of subsets of N is said to be *centered* if the intersections of finite subfamilies of $\mathscr{A}$ are infinite. By fil$\mathscr{A}$ we denote the family of all finite intersections of $\mathscr{A}$. A family $\mathscr{T} = \{T_\alpha\colon \alpha < m\}$ of subsets of N is an *m-tower* (see [3]) if $T_\alpha \subset_* T_\beta$ and $T_\beta - T_\alpha$ is infinite for $\beta < \alpha < m$. We shall consider also another kind of towers, namely *countable towers* $\{A_n\colon n \in N\}$ ordered by the usual inclusion without the requirement that $A_n - A_{n+1}$ is infinite.

Recall that an ultrafilter $p \in N^*$ is *minimal* in the Rudin-Keisler ordering if each function $f\colon N \to N$ is constant or one-to-one on some element from p.

An ultrafilter $p \in N^*$ is said to be a *$P(m)$-ultrafilter* (see [2]) if for each family $\mathscr{F} \subset p$ of fewer than m sets there exists a set $A \in p$ such that $A \subset_* F$ for $F \in \mathscr{F}$. (In this terminology the $P(\aleph_1)$-ultrafilters are exactly the P-points of N^*.) It is sufficient to assume this condition only for towers of cardinality less than m which are contained in p. So if there exists a $P(m)$-ultrafilter which is not $P(m^+)$-ultrafilter, then m is a regular cardinal.

We say that p *lies on the boundary* of a tower $\mathscr{T}$ if $\mathscr{T} \subset p$ and there exists no A in p almost contained in each element from $\mathscr{T}$. Denote the set of all such ultrafilters by $\mathrm{Bd}\mathscr{T}$; topologically,

$$\mathrm{Bd}\mathscr{T} = \bigcap \{\mathrm{cl}_{\beta N} T - N\colon\ T \in \mathscr{T}\} - \mathrm{Int}_{N^*} \bigcap \{\mathrm{cl}_{\beta N} T - N\colon\ T \in \mathscr{T}\}.$$

We shall use the following assertion $\mathrm{P}(\mathfrak{c})$ which follows from MA (see [2]):

If $\mathscr{F}$ is a centered family of fewer than $\mathfrak{c}$ infinite subsets of N, then there exists an infinite subset of N which is almost contained in every element from $\mathscr{F}$.

From $\mathrm{P}(\mathfrak{c})$ it follows that $2^m = 2^{\aleph_0}$ for $\aleph_0 \leqslant m < \mathfrak{c}$. Thus, if $\mathrm{P}(\mathfrak{c})$ is assumed, then $\mathfrak{c}$ is a regular number; moreover, maximal towers are of cardinality $\mathfrak{c}$. So $\mathrm{P}(\mathfrak{c})$ implies the existence of m-towers for each m, $\aleph_0 \leqslant m < \mathfrak{c}$.

2. Minimal $P(m)$-ultrafilters.

LEMMA 1. *A free ultrafilter p on N is minimal iff the following condition is satisfied:*

(*) *If $\{A_n\colon n \in N\}$ is a countable tower contained in p with empty intersection, then there exists an element A from p such that*

$$\mathrm{card}\, A \cap (A_n - A_{n+1}) \leqslant 1.$$

Proof. Let p be a free ultrafilter for which (*) holds. If $f\colon N \to N$ is not constant on any element from p, then the family

$$\{f^{-1}(\{c_n, c_{n+1}, \ldots\})\colon n \in N\}, \quad \text{where } \{c_0, c_1, \ldots\} = f(N),$$

is such as in (*). Take a set A the existence of which follows from (*). The restriction of f to A is one-to-one.

On the other hand, if p is a minimal ultrafilter and $\{A_n\colon n \in N\}$ is a countable tower contained in p with empty intersection, then we take a function f defined by $f(n) = k$ for $n \in A_k - A_{k+1}$. Each set $A \in p$ for which $f|A$ is one-to-one satisfies (*).

The proof of the next lemma by use of $\mathrm{P}(\mathfrak{c})$ instead of MA is due to E. van Douwen.

LEMMA 2. (P(c)) *If $\mathscr{A}$ is a family of fewer than* $\mathfrak{c}$ *subsets of* N, *and* C *is a subset of* N *such that* $C \cap A$ *is infinite for* $A \in \mathscr{A}$, *then there exists a partition* C *into sets* C_0 *and* C_1 *such that* $C_0 \cap A$ *and* $C_1 \cap A$ *are infinite for* $A \in \mathscr{A}$.

Proof. Let

$$P = \{(e,f)\colon e, f \text{ are finite and disjoint subsets of } C\}$$

and let

$$\mathscr{H} = \big\{\{(e,f) \in P\colon \min(e \cup f) \geqslant n\}\colon n \in N\big\} \cup \cup\big\{\{(e,f) \in P\colon e \cap A \neq \emptyset \neq f \cap A\}\colon A \in \mathscr{A}\big\}$$

be a family of subsets of P. The family $\mathscr{H}$ is centered. In fact, let $A_1, A_2, \ldots, A_k \in \mathscr{A}$, $n_1, n_2, \ldots, n_l \in N$, and $n = \max\{n_1, n_2, \ldots, n_l\}$. If we take a partition $C = M_1 \cup M_2 \cup \ldots \cup M_k$ into disjoint sets M_i such that $M_i \cap A_i$ are infinite for $i = 1, 2, \ldots, k$, then the set

$$\{(e_1 \cup e_2 \cup \ldots \cup e_k, f_1 \cup f_2 \cup \ldots \cup f_k)\colon (e_i, f_i) \in P,\ e_i \cup f_i \subset M_i - n,\ e_i \cap A_i \neq \emptyset \neq f_i \cap A_i \text{ for } i = 1, 2, \ldots, k\}$$

is infinite in and is contained in

$$\bigcap_{i=1}^{k} \{(e,f) \in P\colon e \cap A_i \neq \emptyset \neq f \cap A_i\} \cap \bigcap_{i=1}^{k} \{(e,f) \in P\colon \min(e \cup f) \geqslant n_i\}.$$

The set P is countable and $\operatorname{card}\mathscr{H} = \aleph_0 + \operatorname{card}\mathscr{A} < \mathfrak{c}$. Hence there exists an infinite set $J \subset P$ which is almost contained in every set from the family $\mathscr{H}$. There exists an infinite set $I \subset J$ such that

$$(e \cup f) \cap (e' \cup f') = \emptyset$$

for all distinct $(e,f), (e',f') \in I$ (a maximal set with this property is infinite).

Now let E be the union of the first members of elements of I and let D be the union of the second ones. We have $E \cup D \subset C$ and $E \cap D = \emptyset$. The set I is infinite and almost contained in every set from the family $\mathscr{H}$, so for each $n \in N$ there exist elements in $E \cap A$ and in $D \cap A$ which are greater than n. Hence $E \cap A$ and $D \cap A$ are infinite for $A \in \mathscr{A}$. We put $C_0 = E$ and $C_1 = C - E \supset D$.

LEMMA 3. (P(c)) *If $\mathscr{A}$ is a family of fewer than* $\mathfrak{c}$ *subsets of* N, *and* $\mathscr{B}$ *fs a countable partition on* N *such that* $A - \bigcup\mathscr{K}$ *is infinite for* $A \in \mathscr{A}$ *and ior a finite subfamily* $\mathscr{K}$ *of* $\mathscr{B}$, *then there exists a set* C *such that* $A \cap C$ *is infinite for* $A \in \mathscr{A}$ *and* $\operatorname{card} B \cap C \leqslant 1$ *for* $B \in \mathscr{B}$.

Proof. Let P be a set of all finite subsets k of N such that for each $B \in \mathscr{B}$ the cardinality of $k \cap B$ is at most 1. Let $\mathscr{B} = \{B_i\colon i \in N\}$. It is easy to see that the family

$$\mathscr{H} = \big\{\{k \in P\colon\ k \cap A \neq \emptyset\}\colon\ A \in \mathscr{A}\big\} \cup \{[N - B_n]^{<\omega} \cap P\colon\ n \in N\} \cup \\ \cup \{[N-n]^{<\omega} \cap P\colon\ n \in N\}$$

of subsets of P is centered. The set P is countable and $\operatorname{card}\mathscr{H} = \operatorname{card}\mathscr{A} + \aleph_0 < \mathfrak{c}$, hence in virtue of $\mathrm{P}(\mathfrak{c})$ there exists an infinite subset J of P which is almost contained in every set from the family $\mathscr{H}$. Let $I \subset J$ be a maximal subset of J such that for all distinct $k, k' \in I$ the sets

$$\{i \in N\colon\ k \cap B_i \neq \emptyset\} \quad \text{and} \quad \{i \in N\colon\ k' \cap B_i \neq \emptyset\}$$

are disjoint. The set I is infinite.

Let C be the union of the elements from I. Every two distinct elements of I do not intersect the same elements of $\mathscr{B}$, and for each $k \in I \subset P$ and $B \in \mathscr{B}$ the intersection $k \cap B$ has at most one point. So $\operatorname{card} B \cap C \leqslant 1$ for $B \in \mathscr{B}$.

The intersection $C \cap A$ for $A \in \mathscr{A}$ is infinite. In fact, for every $n \in N$ there exists $k \in I$ such that $k \cap A$ has an element which is not less than n. Indeed, let $\{j_1, j_2, \ldots, j_l\}$ be a finite subset of I for which

$$I - \{j_1, j_2, \ldots, j_l\} \subset [N-n]^{<\omega}$$

and

$$I - \{j_1, j_2, \ldots, j_l\} \subset \{k\colon\ k \cap A \neq \emptyset\},$$

I being infinite and almost contained in every set from the family $\mathscr{H}$. So for $j \in I - \{j_1, j_2, \ldots, j_l\}$ we have $j \subset N - n$ and $j \cap A \neq \emptyset$.

Recall the assertion S which will be used in the proof of the next theorem. It has already been mentioned that the assertion S follows from $\mathrm{P}(\mathfrak{c})$.

S: *If $\mathscr{A}$ and $\mathscr{B}$ are families of subsets of N, each of cardinality less than $\mathfrak{c}$, such that $A - \bigcup \mathscr{K}$ is finite whenever $A \in \mathscr{A}$ and $\mathscr{K}$ is a finite subfamily of $\mathscr{B}$, then there exists an infinite subset C of N such that $T \cap C$ is infinite for $T \in \mathscr{A}$ and finite for $T \in \mathscr{B}$.*

THEOREM 1. ($\mathrm{P}(\mathfrak{c})$) *Let m be a regular number less than $\mathfrak{c}$ and let $\mathscr{T}$ be an m-tower. Then there exist $2^{\mathfrak{c}}$ minimal $P(m)$-ultrafilters lying on* $\mathrm{Bd}\mathscr{T}$ *(hence they are not $P(m^+)$-ultrafilters).*

Proof. We may assume that the intersection of the tower $\mathscr{T} = \{T_\alpha\colon \alpha < m\}$ is empty. The family of all n-towers, for ordinals n, $n < m$, is of cardinality $\mathfrak{c}$ (assuming $\mathrm{P}(\mathfrak{c})$, $\mathfrak{c}^n = \mathfrak{c}$ if $n < \mathfrak{c}$). So let $\{\mathscr{A}_\alpha\colon\ \alpha < \mathfrak{c}\}$ be a well ordering of n-towers for $n < m$. Let

$$F = \{\varphi \mid \varphi\colon\ \mathfrak{c} \to \{0, 1\}\}$$

be the family of functions from $\mathfrak{c}$ to $\{0, 1\}$.

For $\varphi \in F$ and ordinals $\alpha < \mathfrak{c}$, we define infinite subsets U_α^φ of N such that

(1) $\mathscr{G}_\alpha^\varphi = \{U_\beta^\varphi \colon \beta \leqslant \alpha\} \cup \mathscr{T}$ has the finite intersection property;

(2) if φ and ψ first differ at α, then $U_\beta^\varphi = U_\beta^\psi$ for $\beta < \alpha$ and $U_\alpha^\varphi \cap U_\alpha^\psi = \emptyset$;

(3) for each $E \in \mathrm{fil}\,\mathscr{G}_\alpha^\varphi$ there exists $\gamma < m$ such that $E - T_\gamma$ is infinite;

(4) there exists an $A \in \mathscr{A}_\alpha$ such that $U_\alpha^\varphi \subset N - A$ or, for every $A \in \mathscr{A}_\alpha$, $U_\alpha^\varphi \subset_* A$;

(5) if $\mathscr{A}_\alpha = \{A_n \colon n \in N\}$ is a countable tower with the empty intersection, then there exists $n \in N$ such that $U_\alpha^\varphi \subset N - A_n$ or, for each $n \in N$,

$$U_\alpha^\varphi \subset_* A_n \quad \text{and} \quad \operatorname{card} U_\alpha^\varphi \cap (A_n - A_{n+1}) \leqslant 1.$$

Notice that if the sets U_α^φ with properties (1)-(5) are constructed, we will have the desired conclusion. In fact, from (1) it follows that for each $\varphi \in F$ the family $\{U_\alpha^\varphi \colon \alpha < \mathfrak{c}\} \cup \mathscr{T}$ is a subbase of a filter p_φ on N. It is free since $\bigcap \mathscr{T} = \emptyset$. Condition (4) assures that it is an ultrafilter; among the n-towers there are all one-element families. We have $\mathscr{T} \subset p_\varphi$, and from (3) it follows that p_φ lies on the boundary of $\mathscr{T}$, so the point p_φ is not a $P(m^+)$-ultrafilter. If $\mathscr{A}$ is an n-tower contained in p_φ, $n < m$, then in view of (4) there exists an α such that $\mathscr{A} = \mathscr{A}_\alpha$ and $U_\alpha^\varphi \subset_* A$ for $A \in \mathscr{A}_\alpha$. So p_φ is a $P(m)$-ultrafilter. By (5) and Lemma 1, p_φ is minimal. Finally, (2) implies $p_\varphi \neq p_\psi$ for $\varphi \neq \psi$.

We define the sets U_α^φ by induction on α for all $\varphi \in F$.

In the first step, let $\mathscr{F}_0^\varphi = \mathrm{fil}\,\mathscr{T}$.

Assume that we have defined the sets U_β^φ for all $\varphi \in F$ and $\beta < \alpha$ so that conditions (1)-(5) are fulfilled. Let

$$\mathscr{F}_\alpha^\varphi = \mathrm{fil}(\{U_\beta^\varphi \colon \beta < \alpha\} \cup \mathscr{T}).$$

The cardinality of $\mathscr{F}_\alpha^\varphi$ is less than $\mathfrak{c}$, $\mathfrak{c}$ being regular. Consider the tower $\mathscr{A}_\alpha$. There are two cases:

1. There exist $A_0 \in \mathscr{A}_\alpha$ and $E_0 \in \mathscr{F}_\alpha^\varphi$ such that $A_0 \cap E_0 - T_\gamma$ is finite for each $\gamma < m$.

2. For each $A \in \mathscr{A}_\alpha$ and $E \in \mathscr{F}_\alpha^\varphi$ there exists a $\gamma(A, E) < m$ such that $A \cap E - T_{\gamma(A,E)}$ is infinite.

Case 1. For each $E \in \mathscr{F}_\alpha^\varphi$ there exists a $\gamma(E)$ such that $(N - A_0) \cap E - T_{\gamma(E)}$ is infinite. In fact, for each $E \in \mathscr{F}_\alpha^\varphi$ there exists $\beta < \alpha$ such that $E \cap E_0 \in \mathrm{fil}\,\mathscr{G}_\beta^\varphi$ and, by condition (3), there exists a $\gamma(E)$ such that $E \cap E_0 - T_{\gamma(E)}$ is infinite. Since

$$E \cap E_0 - T_{\gamma(E)} = (E \cap E_0 \cap A_0 - T_{\gamma(E)}) \cup \big(E \cap E_0 \cap (N - A_0) - T_{\gamma(E)}\big)$$

and the first set in the union is finite, the second one must be infinite. Now we take the family

$$\mathscr{A} = \{E - T_{\gamma(E)} \colon E \in \mathscr{F}_\alpha^\varphi\}$$

and the set $C = N - A_0$. The cardinality of $\mathcal{A}$ is less than $\mathfrak{c}$, so, by Lemma 2, we may decompose the set C into two disjoint sets C_0 and C_1 such that $C_i \cap E - T_{\gamma(E)}$ are infinite for $i = 0, 1$. Put $U_a^\varphi = C_i$ if $\varphi(a) = i$. It is easy to verify that all conditions (1)-(5) are fulfilled for U_β^φ with $\beta \leqslant a$.

Case 2. First we prove that the following stronger condition holds: for each $E \in \mathcal{F}_a^\varphi$ there exist a $\gamma(E) < m$ and an infinite subset $A(E)$ which is almost contained in every $A \in \mathcal{A}_a$ and such that $A(E) \cap E - T_{\gamma(E)}$ is infinite. In fact, let $E \in \mathcal{F}_a^\varphi$ and let $\gamma(A, E)$ for $A \in \mathcal{A}_a$ be such as defined. Since $\operatorname{card} \mathcal{A}_a < m$ and m is regular, there exists a $\gamma(E) < m$ such that $\gamma(E) > \gamma(A, E)$ for $A \in \mathcal{A}_a$. We have

$$A \cap E - T_{\gamma(A,E)} \subset_* A \cap E - T_{\gamma(E)},$$

$\mathcal{T}$ being a tower. Hence $A \cap E - T_{\gamma(E)}$ is infinite for $A \in \mathcal{A}_a$. The family $\{A \cap E - T_{\gamma(E)}\colon A \in \mathcal{A}_a\}$ is centered. So, by $\mathrm{P}(\mathfrak{c})$, there exists a set $A(E)$ almost contained in every set from this family, hence also in every $A \in \mathcal{A}_a$.

We apply the assertion S to families

$$\mathcal{A} = \{A(E) \cap E - T_{\gamma(E)}\colon E \in \mathcal{F}_a^\varphi\} \quad \text{and} \quad \mathcal{B} = \{N - A\colon A \in \mathcal{A}_a\}.$$

A set C the existence of which follows from S satisfies the following conditions: C is almost contained in every $A \in \mathcal{A}_a$ and $C \cap E - T_{\gamma(E)}$ is infinite for $E \in \mathcal{F}_a^\varphi$. Now, if the tower $\mathcal{A}_a$ is not countable, then the construction of U_a^φ is the same as in case 1.

If $\mathcal{A}_a$ is a countable tower, then we take a family

$$\{C \cap E - T_{\gamma(E)}\colon E \in \mathcal{F}_a^\varphi\}$$

and a partition $\{A_n - A_{n+1}\colon n \in N\}$ of N. By Lemma 3, there exists a set C' such that $C' \cap C \cap E - T_{\gamma(E)}$ is infinite for $E \in \mathcal{F}_a^\varphi$ and $\operatorname{card} C' \cap (A_n - A_{n+1}) \leqslant 1$. Now, we decompose the set $C' \cap C$, according to Lemma 2, into sets C_0 and C_1 and define U_a^φ as before.

It is easy to verify that conditions (1)-(5) are fulfilled.

3. Non-minimal $P(\mathfrak{m})$-ultrafilters. Let $f\colon N \to N$ be a function such that $\limsup \operatorname{card} f^{-1}(n) = \aleph_0$. A subset A of N is said to be *f-small* (or, simply, *small* if f is fixed) if $\limsup \operatorname{card}\big(A \cap f^{-1}(n)\big)$ is finite; otherwise, A is called *f-large* (cf. an analogous notion in [1]).

First we give two lemmas on large sets. The first one gives condition parallel to assertion S, the second one is analogous to Lemma 2.

Let a function $f\colon N \to N$ such that $\limsup \operatorname{card} f^{-1}(n) = \aleph_0$ be given.

LEMMA 4. ($\mathrm{P}(\mathfrak{c})$) *Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty families of subsets of N, each of cardinality less than $\mathfrak{c}$, such that if $A \in \mathcal{A}$ and $\mathcal{K}$ is a finite subfamily of $\mathcal{B}$, then $A - \bigcup \mathcal{K}$ is large. Then there exists a set C such that $C \cap T$ is large if $T \in \mathcal{A}$ and finite if $T \in \mathcal{B}$.*

Proof. Let P be the set of all finite subsets of N. Let

$$D_{A,m} = \{a \in P\colon \text{ there exists } l > m \text{ with } \operatorname{card} f^{-1}(l) \cap a \cap A > m\},$$

and let

$$\mathscr{H} = \{[N-B]^{<\omega}\colon B \in \mathscr{B}\} \cup \{D_{A,m}\colon A \in \mathscr{A}, m \in N\}$$

be a family of subsets of P. The family $\mathscr{H}$ is centered. In fact, let $A_1, A_2, \ldots, A_k \in \mathscr{A}$, $B_1, B_2, \ldots, B_n \in \mathscr{B}$, $m_1, m_2, \ldots, m_k \in N$, and let $m = \max\{m_1, m_2, \ldots, m_k\}$. A set

$$\{a_1 \cup a_2 \cup \ldots \cup a_k\colon a_i \in D_{A_i,m} \cap [N-(B_1 \cup \ldots \cup B_n)]^{<\omega}\}$$

is infinite and is contained in

$$D_{A_1,m_1} \cap \ldots \cap D_{A_k,m_k} \cap [N-B_1]^{<\omega} \cap \ldots \cap [N-B_n]^{<\omega}.$$

The set P is countable and $\operatorname{card}\mathscr{H} = \operatorname{card}\mathscr{B} + \aleph_0 \cdot \operatorname{card}\mathscr{A} < \mathfrak{c}$, hence in virtue of P($\mathfrak{c}$) there exists an infinite subset J of P which is almost contained in every set from the family $\mathscr{H}$. Let $C = \bigcup\{a\colon a \in J\}$. The set C is the desired one. In fact, for $B \in \mathscr{B}$ there exist $a_1, a_2, \ldots, a_k \in J$ such that

$$J - \{a_1, a_2, \ldots, a_k\} \subset [N-B]^{<\omega}.$$

Hence

$$C - (a_1 \cup a_2 \cup \ldots \cup a_k) \subset N - B,$$

i.e., $C \cap B$ is finite.

Now let $A \in \mathscr{A}$ and $m \in N$. There exist $a_1, a_2, \ldots, a_k$ such that

$$J - \{a_1, a_2, \ldots, a_k\} \subset D_{A,m},$$

hence for some $l > m$ we have $\operatorname{card} f^{-1}(l) \cap C \cap A > m$. But it is equivalent to

$$\limsup \operatorname{card} C \cap A \cap f^{-1}(n) = \aleph_0,$$

i.e., $C \cap A$ is a large set.

COROLLARY. (P($\mathfrak{c}$)) *If $\mathscr{A}$ is a filterbase of fewer than $\mathfrak{c}$ large subsets of N, then there exists a large set which is almost contained in every set from $\mathscr{A}$.*

For the proof we apply Lemma 4 to the case where $\mathscr{B} = \{N-A\colon A \in \mathscr{A}\}$.

LEMMA 5. (P($\mathfrak{c}$)) *If $\mathscr{A}$ is a family of fewer than $\mathfrak{c}$ subsets of N and C is a subset of N such that $C \cap A$ is large for $A \in \mathscr{A}$, then there exists a partition of C into sets C_0 and C_1 such that $C_0 \cap A$ and $C_1 \cap A$ are large for $A \in \mathscr{A}$.*

Proof. Let

$$P = \{(e, g)\colon e, g \text{ are finite and disjoint subsets of } C\}$$

and let

$$D_{A,m} = \{(e, g) \in P\colon \text{ there exists } l > m \text{ such that } \operatorname{card} A \cap f^{-1}(l) \cap e > m \text{ and } \operatorname{card} A \cap f^{-1}(l) \cap g > m\}.$$

It is easy to check that the family

$$\mathscr{H} = \{D_{A,m}\colon A \in \mathscr{A}, m \in N\} \cup \big\{\{(e, g) \in P\colon e, g \in [C-n]^{<\omega}\}\colon n \in N\big\}$$

of subsets of P is centered.

The set P is countable and $\operatorname{card}\mathscr{H} = \aleph_0 \cdot \operatorname{card}\mathscr{A} + \aleph_0 < \mathfrak{c}$, so there exists an infinite set $J \subset P$ which is almost contained in every set from the family $\mathscr{H}$. Let I be a maximal subset of J with respect to the following property: if $(e, g), (e', g') \in I$, then $(e \cup g) \cap (e' \cup g') = \emptyset$. Since $J \subset_* [C-n]^{<\omega}$ for $n \in N$, the set I is infinite.

Now let E be the union of the first members of elements of I and let D be the union of the second ones. The set I is infinite and almost contained in every $D_{A,m}$, so for every $A \in \mathscr{A}$ and $m \in N$ there exists an $(e, g) \in I$, i.e., $e \subset E$ and $g \subset D$, such that for some $l > m$ we have

$$\operatorname{card} f^{-1}(l) \cap e \cap A > m \quad \text{and} \quad \operatorname{card} f^{-1}(l) \cap g \cap A > m.$$

Hence $\limsup \operatorname{card} f^{-1}(n) \cap A \cap E$ and $\limsup \operatorname{card} f^{-1}(n) \cap A \cap D$ are equal to $\aleph_0$. We put $C_0 = E$ and $C_1 = C - E \supset D$.

In this section we construct $P(m)$-ultrafilters which are non-minimal and which lie on the boundary of a given m-tower $\mathscr{T} = \{T_\alpha\colon \alpha < m\}$. The construction of such ultrafilters is as follows. If f is a fixed function and the elements of an ultrafilter p are f-large, then f is not one-to-one on elements of p. So p is not minimal. To ensure that p lies on $\operatorname{Bd}\mathscr{T}$ and its elements are f-large, it is sufficient to choose the sets A of p such that $A - T_\gamma$ are f-large for some γ. But we must choose the function f such that for each α there exists a γ such that $T_\alpha - T_\gamma$ is f-large. The following lemma ensures the existence of such a function.

LEMMA 6. (P($\mathfrak{c}$)) *For each m-tower $\mathscr{T} = \{T_\alpha\colon \alpha < m\}$, $m < \mathfrak{c}$, there exists a function $f\colon N \to N$ such that the sets $T_\alpha - T_\beta$ are f-large for $\alpha < \beta < m$.*

Proof. We proceed by induction using Lemma 2. The sets $T_\alpha - T_\beta$ are infinite for $\alpha < \beta < m$. We decompose N into two disjoint sets N_1 and N_2' such that $N_1 \cap (T_\alpha - T_\beta)$ and $N_2' \cap (T_\alpha - T_\beta)$ are infinite for $\alpha < \beta < m$. Next, given a partition $\{N_1, N_2, \ldots, N_{k-1}, N_k'\}$ of N into disjoint infinite sets whose intersections with $T_\alpha - T_\beta$ are infinite, we decompose N_k' into sets N_k and N_{k+1}' such that $N_k \cap (T_\alpha - T_\beta)$ and $N_{k+1}' \cap (T_\alpha - T_\beta)$ are infinite. Then $N = \bigcup \{N_k\colon k \in N\}$ and $N_k \cap (T_\alpha - T_\beta)$ are infinite for $\alpha < \beta < m$, $k \in N$. Letting $f(n) = k$ for $n \in N_k$ we obtain the desired function.

THEOREM 2. (P($\mathfrak{c}$)) *Let m be a regular number less than $\mathfrak{c}$ and let $\mathscr{T}$ be an m-tower. Then there exist $2^{\mathfrak{c}}$ non-minimal $P(m)$-ultrafilters lying on* Bd$\mathscr{T}$ *(hence they are not $P(m^+)$-ultrafilters).*

Proof. We use the same notation as in the proof of Theorem 1. Let f be a function such as in Lemma 6. For $\alpha < \mathfrak{c}$ and $\varphi \in F$ we construct the sets U_α^φ having the following properties:

(1) $\mathscr{G}_\alpha^\varphi = \{U_\beta^\varphi\colon \beta \leqslant \alpha\} \cup \mathscr{T}$ has the finite intersection property;

(2) if φ and ψ first differ at α, then $U_\beta^\varphi = U_\beta^\psi$ for $\beta < \alpha$ and $U_\alpha^\varphi \cap U_\alpha^\psi = \emptyset$;

(3) for each $E \in \mathrm{fil}\mathscr{G}_\alpha^\varphi$ there exists $\gamma < m$ such that $E - T_\gamma$ is f-large;

(4) there exists an $A \in \mathscr{A}_\alpha$ such that $U_\alpha^\varphi \subset N - A$ or, for each $A \in \mathscr{A}_\alpha$, $U_\alpha^\varphi \subset_* A$.

Assume that U_α^φ have already been constructed. The family $\{U_\alpha^\varphi\colon \alpha < \mathfrak{c}\} \cup \mathscr{T}$ forms a subbase for the ultrafilter p_φ on N. From (3) it follows that every set from p_φ is f-large, being superset of a large set. So p_φ is not minimal; for no $A \in p$ the restriction f to A is one-to-one. From (3) it follows that p_φ is not a $P(m^+)$-ultrafilter, large sets being infinite. Condition (4) assures that it is a $P(m)$-ultrafilter.

The construction of the sets U_α^φ is essentially the same as in the proof of Theorem 1. The only difference is that we have to replace infinite sets by large sets. There are two cases which we have to consider.

1. There exist $A_0 \in \mathscr{A}_\alpha$ and $E_0 \in \mathscr{F}_\alpha^\varphi$ such that $A_0 \cap E_0 - T_\gamma$ is small for each $\gamma < m$.

2. For each $A \in \mathscr{A}_\alpha$ and $E \in \mathscr{F}_\alpha^\varphi$ there exists $\gamma(A, E) < m$ such that $A \cap E - T_{\gamma(A,E)}$ is large.

In the first case, we infer that $E \cap (N - A_0) - T_{\gamma(E)}$ is large for $E \in \mathscr{F}_\alpha^\varphi$ and some $\gamma(E) < m$. By Lemma 5, we decompose $N - A_0$ into two disjoint sets C_0 and C_1 such that $C_i \cap E - T_{\gamma(E)}$ $(i = 0, 1)$ are large. Define U_α^φ to be C_i if $\varphi(\alpha) = i$.

In the second case, we define $\gamma(E)$ so that $A \cap E - T_{\gamma(E)}$ is large for $A \in \mathscr{A}_\alpha$. The family $\{A \cap E - T_{\gamma(E)}\colon A \in \mathscr{A}_\alpha\}$ is a filterbase of large sets. We find, by the Corollary to Lemma 4, a large set $A(E)$ which is almost contained in every set of this family. So $A(E) \cap E - T_{\gamma(E)}$ is large and $A(E) \subset_* A$ for $A \in \mathscr{A}_\alpha$. We apply Lemma 4 to the families

$$\{A(E) \cap E - T_{\gamma(E)}\colon E \in \mathscr{F}_\alpha^\varphi\} \quad \text{and} \quad \{N - A\colon A \in \mathscr{A}_\alpha\}.$$

There exists a set C which is almost contained in every set of $\mathscr{A}_\alpha$ and such that $C \cap E - T_{\gamma(E)}$ are large for $E \in \mathscr{F}_\alpha^\varphi$. By Lemma 5, we decompose the set C and define U_α^φ as before.

It is easy to verify that conditions (1)-(4) are satisfied for U_β^φ with $\beta \leqslant \alpha$.

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