DEGREES OF NON-DEFINABILITY OF THE SACKS MODEL

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In this note* we show that the Sacks notion of forcing is homogeneous and, therefore, there are two degrees of non-definability in the Sacks model [2].

We start with some topological facts.

Let \( P \) be the set of all non-empty perfect subsets of the interval \( I = [0, 1] \) ordered by inclusion.

**Theorem 1.** If \( F_1, F_2 \in P \), then there is a homeomorphism

\[
    h: I \longrightarrow I
\]

such that \( h(F_1) \cap F_2 \in P \).

Observe that if \( F_1 \) or \( F_2 \) contains an interval, then we can clearly find a required homeomorphism. Thus it remains to consider the case where \( F_1 \) and \( F_2 \) are nowhere dense sets. We divide the proof into some steps. Let us state the following easy proposition:

**Proposition.** Let \( F \in P \) be a nowhere dense set such that \( 0, 1 \in F \) and let \( S(F) \) be the set of centres of all components of \( I - F \). Then

(i) \( S(F) \cap F = 0 \),

(ii) \( S(F) \supseteq F \),

(iii) \( S(F) \) is ordered by \( < \) in type \( \eta \).

**Lemma 1.** If \( F_1, F_2 \in P \) are nowhere dense and \( 0, 1 \notin (F_1 - F_2) \cup (F_2 - F_1) \), then there is an order isomorphism

\[
    T: S(F_1), < \rightarrow S(F_2), <
\]

which can be extended to an order isomorphism

\[
    \hat{T}: I - F_1 \rightarrow I - F_2.
\]

**Proof.** The existence of \( T \) follows easily from the Proposition. Let \( \langle I^j_n : n \in \omega \rangle \) be the enumeration of components of \( I - F_j \) \((j = 1, 2)\) and

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let $S_n^i$ be the centre of $I_n^i$. Now we define

$$\bar{T} \upharpoonright I_n^i : I_n^i \onto \mathcal{I}(S_n^i)$$

as a linear order-preserving mapping. Then $\bar{T}$ has the required properties.

**Lemma 2.** If $F_1, F_2 \in P$ are nowhere dense sets and $0, 1 \notin (F_1 - F_2) \cup (F_2 - F_1)$, then there exists a homeomorphism

$$h : I \onto I$$

such that $h(F_1) = F_2$.

Proof. Take the isomorphism $\bar{T} : I - F_1 \to I - F_2$ from Lemma 1 and put

$$h(x) = \sup \{T(s) : s \in I - F_1 \& s < x\}.$$

**Lemma 3.** If $F_1, F_2 \in P$ are nowhere dense sets, then there is a homeomorphism

$$h : I \onto I$$

such that $h(F_1) \cap F_2 \in P$.

Proof. There is $\varepsilon > 0$ such that $F_1 \cap [\varepsilon, 1 - \varepsilon]$ and $F_2 \cap [\varepsilon, 1 - \varepsilon]$ are elements of $P$.

Thus Theorem 1 follows easily from Lemma 3.

We assume that the reader is familiar with all necessary notation concerning forcing technique as well as with the notion of degrees of non-definability.

In the sequel let $\mathcal{M}$ denote a countable standard model of ZFC and let $P$ be a set of all non-empty perfect subsets of the interval $I$ constructed in the model $\mathcal{M}$ and ordered by inclusion. $P$ is called the Sacks notion of forcing.

**Theorem 2.** Let $G$ be a $P$-generic over $\mathcal{M}$. Then in $\mathcal{M}[G]$ there are only two degrees of non-definability.

Proof. We recall the following classical theorems:

1. (Sacks [2]). If $x \in \mathcal{M}[G] \setminus \mathcal{M}$, then $\mathcal{M}[G] \vdash V = L[x]$.

2. (Lévy [1], p. 127-151). If $C$ is a homogeneous notion of forcing in $\mathcal{M}$, then for every $G$ $C$-generic over $\mathcal{M}$ we have $\mathcal{M}[G] \vdash \text{HOD} = L$.

From the theorem of Lévy we infer that, in the Sacks model, $\mathcal{M}[G] \vdash \text{HOD} = L$. Now let $x \in \mathcal{M}[G] - L$. Then $L[x] = \mathcal{M}[G]$, but $L[x] \subseteq \text{HOD}(x)$. Hence

$$\mathcal{M}[G] \vdash V = \text{HOD}(x).$$
REFERENCES


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