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THE EXOCENTER OF A DOUBLE HEYTING ALGEBRA

BY

W. BOWEN (OXFORD)

0. The *exocenter* E of a distributive lattice L is defined by Epstein [4] as the sublattice of L generated by all pairs $b, c \in L$ which satisfy

E1. c is the greatest element x in L such that $x \wedge b \leqslant c$,

E2. c is the least element x in L such that $x \vee b \geqslant c$,

E3. b is the greatest element x in L such that $x \wedge c \leqslant b$,

E4. b is the least element x in L such that $x \vee c \geqslant b$.

This definition has an inherent drawback: it is non-constructive in the sense that elements of E are not constructed from elements of L. This is a great hindrance to the discovery of properties of E in relation to L.

In this paper we overcome this drawback by giving a constructive definition of the exocenter E of a double Heyting algebra L (Theorem 3.8). In Sections 4 and 5 we derive properties of the exocenter of a double Heyting algebra L of order n and, in particular, we show that the exocenter is the smallest sublattice with respect to which L is chain based.

1. Definitions and notation. Throughout this paper we shall be working in the category of bounded distributive lattices. By a *distributive lattice* we mean a distributive lattice that has a zero element 0 and a unit element 1. By a *sublattice* we mean a sublattice containing $\{0, 1\}$. By a *homomorphism between distributive lattices* we mean a lattice homomorphism that preserves 0 and 1. For all undefined lattice theoretic terms see Balbes and Dwinger [2].

It is immediate from Epstein's definition that, in the case of a double Heyting algebra L, the exocenter E of L is the sublattice generated by all pairs $b, c \in L$ which satisfy

HE1. $b \rightarrow c = c$,

HE2. $b \leftarrow c = c$,

HE3. $c \rightarrow b = b$,

HE4. $c \leftarrow b = b$.

It is also obvious that the center B of a distributive lattice L is contained within the exocenter [4].

Recall that an element a of a double Heyting algebra L is *dense* in L if $a \to 0 = 0$.

Definition 1.1. Let L be a double Heyting algebra. Suppose that L contains a finite chain

$$0 = e_0 < e_1 < \dots < e_{n-2} < e_{n-1} = 1$$

of n elements such that for each $i = 1, \dots, n-1$, e_i is the least dense element of $[e_{i-1}, 1]$. Then we say that $(L, e_0, e_1, \dots, e_{n-2}, e_{n-1})$ is a *double Heyting algebra of order* n.

Note: a chain

$$0 = e_0 < e_1 < \dots < e_{n-2} < e_{n-1} = 1$$

in which e_i is the least dense element of $[e_{i-1}, 1]$ is called a *Heyting chain* [4]. When we say L is a double Heyting algebra of order n, we mean that L is a double Heyting algebra which contains a Heyting chain. This is unambiguous as a Heyting chain, if it exists, is clearly uniquely determined.

Definition 1.2. Suppose L is a distributive lattice which contains a finite chain of elements

$$0 = f_0 < f_1 < \dots < f_{n-2} < f_{n-1} = 1.$$

We say L is *chain based with respect to a sublattice* Q if L is generated by $Q \cup \{f_0, f_1, \dots, f_{n-2}, f_{n-1}\}$. That is, any element a of L has a representation

$$a = \bigvee \{f_i \wedge r_i \colon r_i \in Q,\ i = 1, \dots, n=1\}.$$

Distributive lattices that are chain based with respect to their center have been studied in [5] under the name P_0-lattices. However consider the following example given in [4].

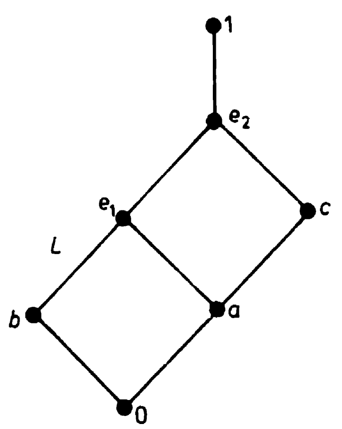

Fig. 1

L is a double Heyting algebra of order 4 with Heyting chain $0 < e_1 < e_2 < 1$ and center $\{0, 1\}$.

Clearly L is not chain based with respect to its center, but L is (trivially) chain based with respect to itself. Epstein noticed that L is also chain based with respect to proper sublattices; for instance L is chain based with respect to $\{0, b, c, e_2, 1\}$. This latter sublattice is in fact the exocenter of L. Epstein wished to extend the class of P_0-lattices by defining an extension of the center of a distributive lattice and considering those sublattices which were chain based with respect to this new sublattice. By considering finite examples, such as the above, Epstein was able to give his definition of the exocenter. In this paper we show that any double Heyting algebra of order n is chain based with respect to its exocenter.

2. Priestley duality theory. We assume that the reader is familiar with Priestley duality for distributive lattices [6], [7], [8], [9]. The notation used here may differ from the above, but it should be self-explanatory.

We shall use this duality theory in Section 3 to give a new characterization of the generators of the exocenter of a double Heyting algebra. In Sections 4 and 5 we apply this characterization to double Heyting algebras of order n, so we need a description of the dual spaces of these lattices. The following lemma is easily deduced from the work of Priestley [9].

LEMMA 2.1. *If L is a double Heyting algebra and L has a least dense element e_1, then e_1 is represented by the set X_1 of minimal points of X_L.*

By trivial extension it follows that, if $(L, e_0, e_1, \ldots, e_{n-1})$ is a double Heyting algebra of order n, then the dual space X_L of L consists of $n-1$ disjoint clopen sets $X_1, X_2, \ldots, X_{n-1}$ where X_i is the set of minimal points of $X_L \setminus \bigcup \{X_j: j \leqslant i-1\}$. The chain element e_i is represented by the set $\bigcup \{X_j: j \leqslant i\}$.

Since the layers $X_1, X_2, \ldots, X_{n-1}$ partition the space X_L, we can define the *order* $o(x)$ of an element x of X_L by

$$o(x) = i \text{ if and only if } x \in X_i.$$

If $o(x) = i$, it is clear that $d(x) \cap X_k \neq \emptyset$ $\forall k \leqslant i$ (X_k is the set of minimal points of $\bigcup \{X_j: j \geqslant k\}$.

Since the set of minimal points of any ordered space is discretely ordered, we see that if $(L, e_0, e_1, \ldots, e_{n-1})$ is a double Heyting algebra of order n, then each of the intervals $[e_{i-1}, e_i]$ is a Boolean algebra.

Definition 2.2. Let L be a distributive lattice. We say that L is a *distributive lattice of order n* if there exists a chain

$$0 = e_0 < e_1 < \ldots < e_{n-2} < e_{n-1} = 1$$

of elements of L such that each interval $[e_{i-1}, e_i]$, $i = 1, \ldots, n-1$, is a Boolean algebra, and n is the smallest number for which such a chain exists.

As we mentioned in Section 1, we shall be concerned with sublattices of a distributive lattice. We describe two ways of visualizing sublattices in dual space terms.

Firstly, and most obviously, there is a bitopological approach. If Q is a subset of a distributive lattice L, then $\{a: a \in Q\}$ is a collection of clopen decreasing subsets of X_L. We may use these sets to generate a new topology $\mathcal{Q}$ on X_L by taking $\{a: a \in Q\} \cup \{X_L \setminus a: a \in Q\}$ as a basis for the open sets in $\mathcal{Q}$. This topology is called the *clopen topology generated by* Q: it is compact, but not in general Hausdorff. If Q is a sublattice of L, then for $a \in L$ we have $a \in Q$ if and only if a is $\mathcal{Q}$-clopen and decreasing in X_L.

Adams ([1], Definition 2.1) describes another technique for dealing with sublattices, by means of separating sets. Given a sublattice Q of a distributive lattice L, we define the set

$$S_Q = \{(x, y) \in X_L \times X_L: x \not\geqslant y \text{ and for any } q \in Q, (x \in q \text{ implies } y \in q)\}.$$

S_Q is called the *separating set* for the sublattice Q. A subset a of X_L is said to be *compatible* with S_Q provided $(x, y) \in S_Q$ and $x \in a$ imply $y \in a$. Clearly, for a clopen decreasing subset a of X_L, $a \in Q$ if and only if a is compatible with S_Q. Also if Q_1, Q_2 are sublattices of L with separating sets S_1, S_2, then $Q_1 \subseteq Q_2$ if and only if $S_1 \supseteq S_2$.

The following lemma is trivial to prove.

LEMMA 2.3. *Let L be a distributive lattice, Q a sublattice and $\varphi: X_L \to X_Q$ the dual of the inclusion. The separating set S_Q for Q is given by*

$$S_Q \in \{(x, y) \in X_L \times X_L: x \not\geqslant y \text{ and } \varphi(x) \geqslant \varphi(y)\}.$$

Armed with this information we can give a theorem that characterizes precisely those subsets of $X_L \times X_L$ that are separating sets for a sublattice of L.

THEOREM 2.4. *Let L be a distributive lattice and S a subset of $X_L \times X_L$. S is a separating set for a sublattice of L if and only if the following conditions hold.*

(1) $S \cap \geqslant = \emptyset$.

(2) $S \cup \geqslant$ *is a transitive set (i.e.* $(x, y), (y, z) \in S \cup \geqslant$ *implies* $(x, z) \in S \cup \geqslant$).

(3) *Given* $(x, y) \notin S \cup \geqslant$, *there is a clopen decreasing subset b of X_L such that* $(x, y) \in b \times (X_L \setminus b) \subseteq (X_L \times X_L) \setminus (S \cup \geqslant)$.

Proof. The necessity of (1) is obvious. Condition (2) is a matter of checking the four possible cases. For condition (3), let $(x, y) \notin S \cup \geqslant$; then there is an element b of Q which contains x but not y; b clearly satisfies (3).

Conversely, if S satisfies the conditions of the theorem, define a relation ϱ on X_L by $x \varrho y$ if and only if $(x, y), (y, x) \in S \cup \geqslant$. We can check that ϱ is an equivalence relation. Let $(X_Q, \mathcal{Q}, \leqslant_1)$ be the quotient space (and φ: X_L

$\to X_Q$ the associated quotient map) ordered by $\varphi(x) \leqslant_1 \varphi(y)$ if and only if $(y, x) \in S \cup \geqslant$. ($\leqslant_1$ is reflexive, antisymmetric after the quotienting, and transitive by condition (2).) If $\varphi(x) \not\geqslant_1 \varphi(y)$, then $(x, y) \in S \cup \geqslant$, so by (3) there is a clopen decreasing subset b of X_L such that

$$(*) \qquad (x, y) \in b \times (X_L \backslash b) \subset (X_L \times X_L) \backslash (S \cup \geqslant).$$

If $z \in \varphi^{-1} \circ \varphi(b)$, then there exists $w \in b$ such that $w \varrho z$. So $(w, z) \in S \cup \geqslant$; thus as $w \in b$, by $(*)$ we have $z \notin (X_L \backslash b)$. That is, $z \in b$ and so $\varphi^{-1} \circ \varphi(b) \subset b$; the reverse inclusion is obvious so $\varphi^{-1} \circ \varphi(b) = b$. Hence since φ is a quotient map, $\varphi(b)$ is clopen in X_Q, $\varphi(x) \in \varphi(b)$ and $\varphi(y) \notin \varphi(b)$. Also if $\varphi(z) \in \varphi(b)$ and $\varphi(w) \leqslant_1 \varphi(z)$, then $(z, w) \in S \cup \geqslant$ and $z \in b$, so by $(*)$ $w \notin (X_L \backslash b)$. That is, $w \in b$ and $\varphi(w) \in \varphi(b)$. Thus $\varphi(b)$ is decreasing. We have shown that $(X_Q, \mathcal{Q}, \leqslant_1)$ is totally order disconnected; it is also compact and since φ is continuous, surjective and (clearly) increasing, X_Q is the dual space of a sublattice Q of L. Since

$$S = \{(x, y) \in X_L \times X_L : x \not\geqslant y \text{ and } \varphi(x) \geqslant_1 \varphi(y)\},$$

we see by Lemma 2.3 that S is the separating set for Q.

The proof of this theorem also reveals the procedure for constructing the dual space for a sublattice from its separating set.

Although not every subset of $X_L \times X_L$ is a separating set, any subset T of $X_L \times X_L$ does generate a sublattice Q_T in a similar fashion:

$$Q_T = \{a \in L : (x, y) \in T \text{ and } x \in a \text{ implies } y \in a\}.$$

It is clear that S_{Q_T} is the smallest separating set containing $T \backslash \geqslant$; we say S_{Q_T} is the separating set *generated* by T.

LEMMA 2.5. *Let L be a distributive lattice. Suppose $x, y \in X_L$; then the separating set S generated by $\{(x, y)\}$ is given by*

$$S = \{(w, z) \in X_L \times X_L : w \not\geqslant z,\ w \in i(x) \text{ and } z \in d(y)\}.$$

Proof. This is simply a matter of checking that S is a separating set (use Theorem 2.4), and that $S \cup \geqslant$ is the transitive closure of $\{(x, y)\} \cup \geqslant$.

3. The exocenter. In this section we use Priestley duality theory to give an explicit construction for the generators of the exocenter of a double Heyting algebra L. In fact, we show (Theorem 3.8) that given $p, q \in L$, the elements $(p \to q) \to (q \leftarrow p)$ and $(q \leftarrow p) \leftarrow (p \to q)$ are a pair of generators for the exocenter and, moreover, that any pair of generators is of this form for some $p, q \in L$ (Corollary 3.9).

Many of the results in this section are of a rather technical nature and their proofs require some complex manipulation of formulae. To help shorten these proofs we now list some standard identities and inequalities that hold in all double Heyting algebras.

Throughout this section let L be a double Heyting algebra.

THEOREM 3.1. *For a, b, $c \in L$, the following hold*:

H1. $b \leqslant a \to b$,

H2. $a \leqslant b \Rightarrow a \to c \geqslant b \to c$,
$c \to a \leqslant c \to b$,

H3. $(a \wedge b) \to c = a \to (b \to c)$,

H4. $a \wedge (a \to b) = a \wedge b$,

H5. $(b \leftarrow a) \to b = a \to b$,

D1. $a \geqslant b \leftarrow a$,

D2. $a \leqslant b \Rightarrow a \leftarrow c \geqslant b \leftarrow c$,
$c \leftarrow a \leqslant c \leftarrow b$,

D3. $(a \vee b) \leftarrow c = a \leftarrow (b \leftarrow c)$,

D4. $a \vee (a \leftarrow b) = a \vee b$,

D5. $(a \to b) \leftarrow a = b \leftarrow a$.

Proof. These are all standard formulae. See, e.g., [2], Theorem XI.2.3.
For p, $q \in L$, define

$$u = p \to q \quad \text{and} \quad v = q \leftarrow p,$$

and then let

$$b = u \to v \quad \text{and} \quad c = v \leftarrow u.$$

LEMMA 3.2. $u \leftarrow v = v$; $v \to u = u$.

Proof.
$$\begin{aligned} u \leftarrow v &= (p \to q) \leftarrow (q \leftarrow p) \\ &= ((p \to q) \vee q) \leftarrow p && \text{by D3} \\ &= (p \to q) \leftarrow p && \text{by H1} \\ &= q \leftarrow p && \text{by D5} \\ &= v && \text{by definition.} \end{aligned}$$

The other equality is proved dually.

COROLLARY 3.3. $(u \to v) \wedge (v \to u) = u \wedge v$; $(u \leftarrow v) \vee (v \leftarrow u) = u \vee v$.

Proof. Lemma 3.2 and H4.
The other equality is proved dually.

COROLLARY 3.4. $b \leftarrow c = c$; $c \to b = b$.

Proof. Replace p by u, etc. in the proof of Lemma 3.2.

LEMMA 3.5. $b \wedge c \leqslant u \wedge v \leqslant p \wedge q$; $b \vee c \geqslant u \vee v \geqslant p \wedge q$.

Proof.
$$\begin{aligned} b \wedge c &= (u \to v) \wedge (v \leftarrow u) \\ &\leqslant (u \to v) \wedge u && \text{by D1} \\ &= u \wedge v && \text{by H4} \\ &\leqslant p \wedge q && \text{by replacing } b \text{ by } u, \\ & && \text{etc. and repeating this method.} \end{aligned}$$

The other inequality is proved dually.

COROLLARY 3.6. $b \wedge q \leqslant p \wedge q$; $b \vee q \geqslant p \vee q$.

Proof.
$$\begin{aligned} b \wedge q &\leqslant b \wedge (p \to q) && \text{by H1} \\ &= (u \to v) \wedge u && \text{by definition} \\ &= u \wedge v && \text{by H4} \\ &\leqslant p \wedge q && \text{by Lemma 3.5.} \end{aligned}$$

The other inequality is proved dually.

There are many other similar inequalities involving various combinations of p, q, u, v, b, c; the above corollary is included for future use.

LEMMA 3.7. *In the dual space X_L of L, $p \backslash q \subseteq b \backslash c$.*

Proof. $b = u \to v \geqslant v = q \leftarrow p = d(p \backslash q) \supseteq p \backslash q$.

$c = v \leftarrow u \leqslant u = p \to q = X_L \backslash i(p \backslash q) \subseteq X_L \backslash (p \backslash q)$.

THEOREM 3.8. *If L is a double Heyting algebra and p, $q \in L$, then the elements $b = (p \to q) \to (q \leftarrow p)$ and $c = (q \leftarrow p) \leftarrow (p \to q)$ are a pair of generators for the exocenter E of L.*

Proof. By definition, we are required to show $c \to b = b = c \leftarrow b$ and $b \to c = c = b \leftarrow c$. By Corollary 3.4, we already know that $b \leftarrow c = c$ and $c \to b = b$.

1. By H1, $b \to c \geqslant c$. Suppose there exists $x \in (b \to c) \backslash c$. If $x \in u$, then $x \in b \vee c$ by Lemma 3.5; but $x \notin c$, so $x \in b \wedge (b \to c) = b \wedge c$. This is a contradiction as $x \notin c$.

If $x \notin u$, then $x \notin X_L \backslash i(p \backslash q)$, so $x \in i(p \backslash q)$. Thus there exists $y \leqslant x$ such that $y \in (p \backslash q)$. By Lemma 3.7, we have $y \in (b \backslash c)$ and $y \in (b \to c)$ as $y \leqslant x$. So $y \in b \wedge (b \to c) = b \wedge c$. This is a contradiction as $y \notin c$. Thus $b \to c = c$.

2. By D1, $c \leftarrow b \leqslant b$. Suppose there exists $x \in b \backslash (c \leftarrow b)$. Then $x \in b \subset b \vee c = (c \leftarrow b) \vee c$, so $x \in b \wedge c$ as $x \notin (c \leftarrow b)$.

If $x \notin v$, then $x \notin u \wedge v$, so by Lemma 3.5, $x \notin b \wedge c$. This is a contradiction to the above.

If $x \in v$, then $x \in d(p \backslash q)$, so there exists $y \geqslant x$ such that $y \in (p \backslash q)$. By Lemma 3.7, $y \in (b \backslash c)$, and $y \notin (c \leftarrow b)$ as $y \geqslant x$. So $y \notin c \vee (c \leftarrow b)$. This is a contradiction as $y \in b \subset b \vee c = c \vee (c \leftarrow b)$. Thus $c \leftarrow b = b$.

COROLLARY 3.9. *Every pair of generators of E is of the form $(p \to q) \to (q \leftarrow p)$ and $(q \leftarrow p) \leftarrow (p \to q)$ for some p, $q \in L$.*

Proof. If b and c are a pair of generators for E, then

$$b = c \to b = (b \to c) \to (c \leftarrow b) \text{ and } c = b \leftarrow c = (c \leftarrow b) \leftarrow (b \to c).$$

COROLLARY 3.10. (a) *Every generator of the exocenter is of the form $(p \to q) \to (q \leftarrow p)$ for some p, $q \in L$*

(b) *Every generator of the exocenter is of the form $(q \leftarrow p) \leftarrow (p \to q)$ for some p, $q \in L$*

Proof. Let b and c be as in Corollary 3.9.

(a) $c = b \to c = (c \to b) \to (b \leftarrow c)$.

(b) $b = c \leftarrow b = (b \leftarrow c) \leftarrow (c \to b)$.

4. Double Heyting algebras of order n. With the help of this new constructive description of the genrators of the exocenter, we may prove the result that motivated Epstein [4] to define this sublattice: if L is a double Heyting algebra of order n, then L is chain based with respect to its exocenter.

Throughout this section let L be a double Heyting algebra of order n with Heyting chain $0 = e_0 < e_1 < \dots < e_{n-2} < e_{n-1} = 1$.

To elucidate our method of proof, consider the following diagram.

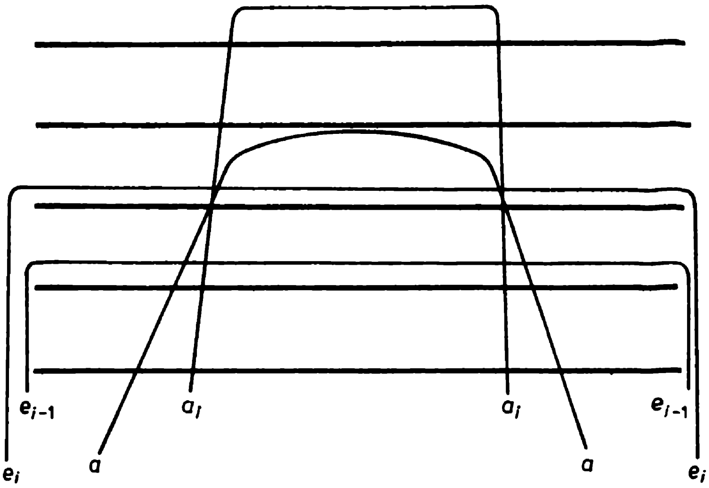

Fig. 2

As noted in Section 2, the dual space X_L of L consists of $n-1$ layers, $X_1, X_2, \dots, X_{n-1}$, of Boolean spaces (heavy black horizontal lines).

Let $a \in L$; then a is a clopen decreasing subset of X_L. For each $i = 1, \dots, n-1$, e_i is the clopen decreasing subset $\bigcup \{X_j : j \leqslant i\}$. For each i, let $a_i = (a \to e_{i-1}) \to (e_{i-1} \leftarrow a)$.

By plotting a_i on the diagram, Fig. 2, it appears that $a_i \wedge e_i \leqslant a \wedge e_i$ and further that $a_i \cap X_i = a \cap X_i$. We prove these two results in Lemma 4.2 and Corollary 4.5; from them we may easily deduce that $a = \bigvee \{a_i \wedge e_i : i = 1, \dots, n-1\}$, and thus, since each a_i is a generator for the exocenter, that L is chain based with respect to its exocenter. To understand the proofs it may be helpful to bear Fig. 2 in mind.

LEMMA 4.1. $e_i \wedge ((a \to e_{i-1}) \to a) \leqslant e_i \wedge a$.

Proof. If $x \in e_i \backslash a$, then $x \notin i(a \backslash e_{i-1})$ since $i(a \backslash e_{i-1}) \wedge e_i = a \cap X_i \sqsubset a$. So $x \in a \to e_{i-1}$, but $x \notin a$, and thus $x \notin (a \to e_{i-1}) \to a$. Hence

$$e_i \wedge ((a \to e_{i-1}) \to a) \leqslant e_i \wedge a.$$

LEMMA 4.2. $e_i \wedge a_i \leqslant e_i \wedge a$.

Proof.
$$\begin{aligned} e_i \wedge a &= e_i \wedge ((a \to e_{i-1}) \to (e_{i-1} \leftarrow a)) \\ &\leqslant e_i \wedge ((a \to e_{i-1}) \to a) && \text{by D1, H2} \\ &\leqslant e_i \wedge a && \text{by Lemma 4.1.} \end{aligned}$$

LEMMA 4.3. $(a \to e_{i-1}) \vee (e_{i-1} \leftarrow a) \geqslant e_i \vee a$.

Proof. Let $x \in e_i \vee a$ and suppose $x \notin (a \to e_{i-1})$, i.e. $x \in i(a \backslash e_{i-1})$. Then there exists $y \in (a \backslash e_{i-1})$ such that $y \leqslant x$. Clearly $x \notin e_{i-1}$ as $y \notin e_{i-1}$ and $y \leqslant x$.

Either $x \in a$, in which case $x \in a \backslash e_{i-1} \sqsubset d(a \backslash e_{i-1}) = e_{i-1} \leftarrow a$, or $x \in e_i$, so $y \in e_i \backslash e_{i-1}$, and hence $y = x$ as $e_i \backslash e_{i-1}$ is discretely ordered. Thus $x \in a \backslash e_{i-1}$

$\sqsubset d(a \backslash e_{i-1}) = e_{i-1} \leftarrow a$, as before. Hence

$$e_i \vee a \leqslant (a \to e_{i-1}) \vee (e_{i-1} \leftarrow a).$$

LEMMA 4.4. $a_i \vee (a \to e_{i-1}) \geqslant a \vee e_i$.

Proof. Let $u = a \to e_{i-1}$ and $v = e_{i-1} \leftarrow a$, so $a_i = u \to v$. We have $u \to v \geqslant v \geqslant u \leftarrow v$ by H1, D1 and $u \geqslant v \leftarrow u$ by D1. So

$$\begin{aligned} a_i \vee (a \to e_{i-1}) &= (u \to v) \vee u \geqslant (u \leftarrow v) \vee (v \leftarrow u) \\ &= u \vee v && \text{by Corollary 3.3} \\ &\geqslant e_i \vee a && \text{by Lemma 4.3.} \end{aligned}$$

COROLLARY 4.5. $a \cap X_i = a_i \cap X_i$.

Proof. Clearly by Lemma 4.2, $a_i \cap X_i \sqsubset a \cap X_i$. If $x \in X_i \backslash a_i$, then by Lemma 4.4, $x \in X_i \backslash (a \to e_{i-1})$. But $a \wedge (a \to e_{i-1}) \leqslant e_{i-1}$, so $x \notin a$. Hence $a_i \cap X_i = a \cap X_i$.

THEOREM 4.6. *If L is a double Heyting algebra of order n, then L is chain based with respect to its exocenter E.*

Proof. Let $a \in L$. For each $i = 1, \ldots, n-1$ define $a_i = (a \to e_{i-1}) \to (e_{i-1} \leftarrow a)$; by Theorem 3.8, a_i is a generator for the exocenter of L. By Lemma 4.2, $e_i \wedge a_i \leqslant e_i \wedge a$ for each i, so

$$\bigvee \{e_i \wedge a_i \colon i = 1, \ldots, n-1\} \leqslant \bigvee \{e_i \wedge a \colon i = 1, \ldots, n-1\} = a.$$

By Corollary 4.5, $a_i \cap X_i = a \cap X_i$, so

$$\begin{aligned} \bigvee \{e_i \wedge a_i \colon i = 1, \ldots, n-1\} &\geqslant \bigcup \{X_i \cap a_i \colon i = 1, \ldots, n-1\} \\ &= \bigcup \{X_i \cap a \colon i = 1, \ldots, n-1\} = a. \end{aligned}$$

Hence $\bigvee \{e_i \wedge a_i \colon i = 1, \ldots, n-1\} = a$ and so L is chain based with respect to its exocenter.

5. Separating sets. Having shown that any double Heyting algebra of order n is chain based with respect to its exocenter, it seems natural to enquire whether the exocenter is the smallest such sublattice (if such exists). By constructing a suitable separating set, we are able to show that such a sublattice does exist and that it is indeed the exocenter. Thus, at one stroke, we find the separating set for the exocenter, and we give it a new characterization as the smallest sublattice with respect to which the lattice itself is chain based.

The following is standard in Epstein and Horn [5].

LEMMA 5.1. *Let $(L, e_0, \ldots, e_{n-1})$ be a double Heyting algebra of order n and let Q be a sublattice. If L is chain based with respect to Q, then every element a of L has a monotonic representation*

$$a = \bigvee \{e_i \wedge q_i \colon i = 1, \ldots, n-1\},$$

where each $q_i \in Q$ and $q_1 \geqslant q_2 \geqslant \ldots \geqslant q_{n-1}$.

Let L be a double Heyting algebra of order n and let Q be a sublattice of L. Let $\mathscr{Q}$ be the clopen topology generated on X_L by Q. L is chain based with respect to Q if and only if L is generated by Q and the chain elements. Translating this definition into a topological criterion on the dual space, we have: L is chain based with respect to Q if and only if the clopen topology $\mathscr{S}$, generated on X_L by Q and the chain elements, is the whole topology $\mathscr{T}$ of X_L. $\mathscr{S}$ will always be coarser than $\mathscr{T}$, so if $\mathscr{S}$ is totally order disconnected, then the two are equal. Given $x, y \in X_L$ with $x \ngeq y$, then either $x < y$, in which case there is a chain element containing x but not y, or $x \nleq y$. Thus a sufficient condition for L to be chain based with respect to Q is: given $x \in X_L$, then for all $y \in X_L$ such that $x \| y$ (i.e. $x \nleq y$ and $y \nleq x$) there is a $\mathscr{Q}$-clopen decreasing subset q such that $x \in q$ and $y \notin q$. In the case of a double Heyting algebra, this condition is also necessary.

THEOREM 5.2. *Let L be a double Heyting algebra of order n with Heyting chain*

$$0 = e_0 < e_1 < \ldots < e_{n-1} = 1.$$

Let Q be a sublattice of L and let $\mathscr{Q}$ be the clopen topology generated on X_L by Q. Then L is chain based with respect to Q if and only if, given $x \in X_L$, for all $y \in X_L$ such that $x \| y$ there is a $\mathscr{Q}$-clopen decreasing subset q such that $x \in q$ and $y \notin q$.

Proof. Suppose that L is chain based with respect to Q. Given $x, y \in X_L$ such that $x \| y$, we have two cases to consider.

1. If $o(x) \geq o(y)$, suppose $o(y) = k$. As $x \ngeq y$, there is a $\mathscr{T}$-clopen decreasing set a with $x \in a$ and $y \notin a$. Now a has a monotonic representation as

$$a = \bigvee \{e_i \wedge q_i \colon i = 1, \ldots, n-1\}.$$

If $y \in q_k$, then as $y \in e_k$, $y \in a$, which is a contradiction. So $y \notin q_k$. If $x \notin q_k$, then $x \notin q_j$ $\forall j \geq k$ as the representation is monotonic. Also $x \notin e_i$ $\forall i < k$ as $o(x) \geq k$. Thus $x \notin a$, which is a contradiction. Thus q_k is a $\mathscr{Q}$-clopen decreasing set containing x but not y.

2. If $o(y) > o(x)$, suppose $o(x) = k$. Let $c = d(y) \cap X_k$. c is non-empty (as we are considering a Heyting chain) and closed. Also $x \notin c$ and $x \| z$ $\forall z \in c$, so there is a $\mathscr{T}$-clopen decreasing set a containing x such that $a \cap c = \emptyset$. Now a has a monotonic representation

$$a = \bigvee \{e_i \wedge q_i \colon i = 1, \ldots, n-1\}$$

and, just as in 1, we can show that $x \in q_k$ but $y \notin q_k$.

Sufficiency is clear by the discussion preceding the theorem.

Now suppose that L is a double Heyting algebra of order n which is chain based with respect to a sublattice Q. Recall from the definition that the separating set for Q is just the set of pairs (x, y) of elements of X_L with $x \ngeq y$

for which there is no $\mathscr{D}$-clopen decreasing set containing x but not y. We wish to find a candidate separating set S_A for the smallest sublattice A with respect to which L is chain based. That is, we wish to characterize the largest separating set that does not contain a pair of incomparable points.

1. Suppose $x \not\leqslant y$. $x \not\geqslant y$ by hypothesis so $x \| y$.

2. Suppose $x < y$ but $[x, y]$ is not a chain (i.e. $\exists w, z \in [x, y]$ with $w \| z$). If $(x, y) \in S$, (w, x), $(y, z) \in \geqslant$, so $(w, z) \in S \cup \geqslant$. But $(w, z) \notin \geqslant$, so $(w, z) \in S$.

3. Suppose $[x, y]$ is a chain but $d(y) \neq d(x) \cup [x, y]$ (i.e. $\exists z \leqslant y$ with $z \| x$). If $(x, y) \in S$, $(y, z) \in \geqslant$, so $(x, z) \in S \cup \geqslant$. But $(x, z) \notin \geqslant$, so $(x, z) \in S$.

4. Suppose $[x, y]$ is a chain but $i(x) \neq i(y) \cup [x, y]$ (i.e. $\exists w \geqslant x$ with $w \| y$). If $(x, y) \in S$, $(w, x) \in \geqslant$, so $(w, y) \in S \cup \geqslant$. But $(w, y) \notin \geqslant$, so $(w, y) \in S$.

We take as our candidate separating set, pairs of the remaining type; namely

$$S_A = \{(x, y) \in X_L \times X_L \colon [x, y] \text{ is a chain},\quad i(x) = i(y) \cup [x, y] \text{ and } d(y) = d(x) \cup [x, y]\}.$$

First we show that we are on the right track.

LEMMA 5.3. *Let L be a double Heyting algebra. If $(x, y) \in S_A$ and if b is a generator for E_L, then $x \in b$ implies $y \in b$.*

Proof. Suppose b is a generator for E_L and $x \in b$. Assume that $y \notin b$. Let c be any generator for E_L such that $b = c \rightarrow b = c \leftarrow b$, that is, $b = X_L \setminus i(c \setminus b)$. Then as $y \notin b$, $y \in i(c \setminus b)$, so there exists $z \in (c \setminus b)$ such that $z \leqslant y$. Now $(x, y) \in S_A$, so $d(y) = d(x) \cup [x, y]$. $z \in d(y)$ but $z \notin d(x)$ as $z \notin b$. Hence $z \in [x, y]$. $x \in b = d(b \setminus c)$, so there exists $w \in (b \setminus c)$ such that $w \geqslant x$. As $(x, y) \in S_A$, $i(x) = i(y) \cup [x, y]$. Now $w \notin i(y)$ as $w \in b$ and $y \notin b$, so $w \in [x, y]$. But $[x, y]$ is a chain, so there are two cases: either $w \geqslant z$, in which case $z \in b$ as $w \in b$. This is a contradiction. Or $z \geqslant w$, in which case $w \in c$ as $z \in c$. This is also a contradiction. Thus our assumption that $y \notin b$ is wrong; that is $y \in b$.

We can now show that S_A is indeed a separating set for a sublattice of L.

THEOREM 5.4. *Let L be a double Heyting algebra. The set S_A is a separating set for a sublattice of L.*

Proof. We use the characterization given in Theorem 2.4. Trivially $S_A \cap \geqslant = \emptyset$, and it is easy to check that $S_A \cup \geqslant$ is a transitive set. It remains to prove condition 3. Given $(x, y) \notin S_A \cup \geqslant$, there are four cases to consider.

1. Suppose $x \not\leqslant y$. Since $x \not\geqslant y$, we may pick clopen decreasing subsets p, q of X_L such that $x \in p$, $y \notin p$ and $x \notin q$, $y \in q$. Let $b = (p \rightarrow q) \rightarrow (q \leftarrow p)$; then, by Corollary 3.6, $b \wedge q \leqslant p \wedge q$, which implies $y \notin b$, and $b \vee q \geqslant p \vee q$, which implies $x \in b$. Thus $(x, y) \in b \times (X_L \setminus b)$. By Lemma 5.3, $b \times (X_L \setminus b) \cap S_A = \emptyset$, and since b is decreasing, $b \times (X_L \setminus b) \cap \geqslant = \emptyset$. Thus the clopen decreasing set b satisfies condition (3) of Theorem 2.4.

2. Suppose $x < y$, but $[x, y]$ is not a chain. That is, $\exists w, z \in [x, y]$ with $w \| z$. Pick clopen decreasing subsets p, q of X_L such that $w \in p$, $z \notin p$ and $w \notin q$, $z \in q$. Let $b = (p \to q) \to (q \leftarrow p)$; then we can check (as in 1) that $w \in b$ and $z \notin b$; hence $x \in b$ and $y \notin b$. The set b satisfies the condition exactly as in 1.

3. Suppose $[x, y]$ is a chain, but $d(y) \neq d(x) \cup [x, y]$. That is, $\exists z \leqslant y$ with $z \| x$. Pick clopen decreasing subsets p, q of X_L such that $x \in p$, $z \notin p$ and $x \notin q$, $z \in q$. Let $b = (p \to q) \to (q \leftarrow p)$; then $x \in b$ and $z \notin b$, so $y \notin b$. Again, the set b satisfies the condition exactly as in 1.

4. Suppose $[x, y]$ is a chain, but $i(x) \neq i(y) \cup [x, y]$. That is, $\exists w \geqslant x$ with $w \| y$. Pick clopen decreasing subsets p, q of X_L such that $w \in p$, $y \notin p$ and $w \notin q$, $y \in q$. Let $b = (p \to q) \to (q \leftarrow p)$; then $w \in b$ and $y \notin b$, so $x \in b$. Again, the set b satisfies the condition exactly as in 1.

Thus S_A is a separating set for a sublattice of L.

Let A be the sublattice of L defined by S_A.

We are now in a position to prove the main theorem of this section.

THEOREM 5.5. *Let L be a double Heyting algebra; then the sublattice A defined by S_A is the exocenter of L.*

Proof. Suppose that Q is any sublattice of L that is properly contained in A; then the separating set S_Q for Q properly contains S_A. Thus there exists $(x, y) \in S_Q \setminus S_A$; $(x, y) \notin S_A \cup \geqslant$, and exactly as in the proof of Theorem 5.4, we can always construct a generator b for the exocenter such that $x \in b$ and $y \notin b$. That is, there is a generator b for E_L that is not compatible with S_Q. Hence $b \notin Q$ and Q does not contain the exocenter of L. But by Lemma 5.3, every generator for E_L is compatible with S_A and so A contains E_L. Thus $A = E_L$.

Our original motivation for defining the set S_A was to find the separating set for the exocenter of a double Heyting algebra of order n; we have been rewarded with the answer for any double Heyting algebra. However, a part of our motivation was directed towards the following theorem.

THEOREM 5.6. *Let $(L, e_0, \ldots, e_{n-1})$ be a double Heyting algebra of order n; then A is the smallest sublattice with respect to which L is chain based.*

Proof. From the above it is clear that if Q is any proper sublattice of A, then L is not chain based with respect to Q. However, by Theorem 5.5, $A = E_L$ and by Theorem 4.6, L is chain based with respect to E_L. Hence A is the smallest sublattice with respect to which L is chain based.

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MATHEMATICAL INSTITUTE
ST. GILES, OXFORD

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