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Accessible sets for a control system

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Abstract. Given a Lipschitz-continuous control system the set accessible from a certain domain in a fixed time is considered. The set accessible from a Lipschitz manifold is proved to be a Lipschitz manifold, provided the time is sufficiently short.

In this note we deal with Lipschitz-continuous control systems. Sets accessible by trajectories from a compact domain in a given (short) time are considered. The boundary of the domain is assumed to be locally (a graph of) a Lipschitz-continuous function in a certain ortogonal coordinate system. We prove that the boundary of the accessible set has the same property. Similar results for certain unbounded domains have been obtained in [2]. The boundary of an accessible set plays an essential role in optimal control problems.

Notations and definitions. By x, y, p we denote points of R^n, $n \geqslant 2$; $|x| = (x_1^2+\ldots+x_n^2)^{\frac{1}{2}}$. For sets $A, B \subset R^n$ we use Hausdorff's distance

$$r(A, B) = \max\big(q(A, B), q(B, A)\big),$$

where

$$q(A, B) = \sup_{x\in A} \inf_{y\in B} |x-y|, \qquad |A| = \sup_{x\in A} |x|.$$

The control system is written in the form with eliminated control variables (orientor form)

$$x' \in F(t, x), \tag{1}$$

where $F(t, x)$ is a set-valued function. The control system in the form $x' = f(t, x, u)$, $u \in U$, corresponds to the form (1) with $F(t, x) = f(t, x, U)$, see [3]. We assume that the sets $F(t, x)$ are (orientors i.e.) convex compact and non-empty sets in R^n. Under very general assumptions the convexity of sets F is necessary for the set of trajectories to be locally compact.

We say that a vector function $x(t)$ defined on an open interval J is a trajectory of (1) if it is absolutely continuous on every compact subinterval of J and satisfies the condition $x'(t) \in F\big(t, x(t)\big)$ for almost every $t \in J$.

Let D be a given set $D \subset R^n$. The union of all trajectories of (1) having at least one common point with $D \times \{0\}$ is called *emission zone* *of* D with respect to (1). The intersection of an emission zone with the hyperplane $t = a$ (projected on R^n) is called an *accessible set* for $t = a$.

Now we define a subclass Lip of compact Lipschitz manifolds with boundary. We say that a set $D \subset R^n$ *belongs to* Lip if and only if D is compact and for every point p of the boundary of D there exists a neighbourhood N and a straight line s that

(2) the set $N \cap D$ is determined by the relations $x \in N, x_s \leqslant g(x_S)$,

where x_s is the (ortogonal) projection of x in s, x_S is the projection of x in the hyperplane S perpendicular to s, and $g(x_S)$ satisfies uniform Lipschitz condition.

THEOREM. *Let the function $F(t, x)$, mapping R^{n+1} into the family of convex, compact and non-empty sets, be measurable in t, bounded i.e. satisfying the inequality*

(3) $$|F(t, x)| \leqslant C, \quad \text{on } R^{n+1}, \quad \text{for a certain constant } C,$$

and uniformly Lipschitz-continuous in x i.e. satisfying the inequality

(4) $$r\big(F(t, x), F(t, y)\big) \leqslant K|x-y|, \quad \text{on } R \times R^n \times R^n,$$
$$\text{for a certain constant } K.$$

Under these assumptions sets accessible from a set D, $D \subset R^n$, $D \in$ Lip, *belong to* Lip *for sufficiently small positive t.*

Proof. From the definition of class Lip it follows that for every point p of the boundary of D, there exists a neighbourhood N satisfying (2). Denote by $N(p) = N_s(p) \times N_S(p)$, $N_s(p) \subset s, N_S(p) \subset S$ such an open subset of N that $p_S \in N_S$ and $g\big(N_S(p)\big) \subset N_s(p)$. Let $M(p)$ be a neighbourhood of p contained with its closure in $N(p)$. Let the neighbourhoods $M(p_1), \ldots, M(p_k)$ cover the boundary of D. Denote by $A(t, Z)$ the set accessible from set Z in time t. It follows from [1] for $l(t, s) = Ks$ that

(5) $$A\big(t, M(p_j)\big), 1 \leqslant j \leqslant k, \text{ are open}$$
$$\text{and} \quad B\big(A(t, D)\big) \subset \bigcup_{1 \leqslant j \leqslant k} A\big(t, M(p_j)\big),$$

where $B(Z)$ denotes the boundary of set Z. In the following number t will be supposed to be positive and sufficiently small. Let j be an integer $1 \leqslant j \leqslant k$. Let $g_e(x_S)$ be the uniformly Lipschitz-continuous extension of function $g(x_S)$ from $N_S(p_j)$ on the whole space R^{n-1}. From the theorem in [2] it follows easily that for the set $V_j = \{x\colon x_s \leqslant g_e(x_S)\}$ there exists such a uniformly Lipschitz-continuous function $h(t, x)$ that

(6) $$A(t, V_j) = \{x\colon x_s \leqslant h(t, x_S)\}.$$

In virtue of (3) it is evident that $A\big(t, R^n \dot{-} N(p_j)\big) \cap A\big(t, M(p_j)\big)$ is empty for small t. Therefore $A(t, D) \cap A(t, M(p_j)) = A(t, V_j) \cap A(t, M(p_j))$ and by (5), (6) the proof is complete.

Remark. The theorem can be easily reformulated for $F(t, x)$ defined on an open subset of R^{n+1}.

References

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