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SOME ESTIMATIONS OF TWO-TERMINAL SERIES-PARALLEL SYSTEM RELIABILITY

To determine the optimum structure of a system of elements with respect to the characteristics of system reliability the following two criteria may be used: (I) the structure is called *optimum* if it maximizes the mean working time of the system or (II) the structure is called *optimum* if it maximizes the system reliability in the initial working period of the system. In practice, Criterion I may be used where the designed system works till it breaks down and where the damaged systems are exchanged by new ones. Criterion II is better in situations where the designed systems do not have to work till its breakdown but only in a short time interval and where breakdown of the system means the failure of the whole project. Criterion II may be used also to evaluate the reliability of constructions the elements of which break down owing to pressure (cf. Kopociński and Kowal [4]). Criteria I and II are not equivalent in complex reliability systems (cf. Kopociński [3]).

The object of our study is the class of two-terminal series-parallel structures under assumption that two types of failure described bellow are possible. The structures of this class were discussed in detail by Łomnicki in [5] where they were called *simple* structures. In section 3 of that paper *essentially parallel* and *essentially series* structures were defined and these two classes will be later used in treating our problem. The optimization of these structures in the sense of Criterion I presented in [6] where the practical importance of two kinds of breakdown was additionally stressed. In this note we intend to analyze the optimality of structures in the considered class with respect to Criterion II.

Let us suppose that the structure elements are identical and independent with exponential reliability with parameter λ, and that breakdowns may be of twofold kind: with conditional probability p — there are ***breakdowns to operate*** — and with conditional probability $1-p$ — there are ***breakdowns to idle*** (cf. Łomnicki [5]). In the terminology borrowed from electrical problems, p is the conditional probability of open-circuit failure and $1-p$ is the conditional probability of closed-circuit failure (cf. Barlow and Proschan [1]). The probabilities of breakdown to operate and breakdown to idle in the time interval $(0, t)$ are equal to

(1) $$a = p(1-e^{-\lambda t}) \quad \text{and} \quad b = (1-p)(1-e^{-\lambda t}),$$

respectively.

Table 1 shows all possible two-terminal series-parallel structures for $n = 2, 3, 4$ and 5 elements (cf. [5], Table 1).

Let us number the structures elements by ordering series substructures before parallel substructures. Let X_i ($i = 1, 2, \ldots, n$) denote the Boolean representation of the i-th element: it equals 1 if the element is able to operate and it equals 0 if it is damaged. We define a minimal critical path as a set of those elements whose breakdown makes the structure break down even though other elements operate, but ceases to be a path when any of the elements of this set operates.

If we have a structure and the Boolean representations of its elements, then it is easy to find the Boolean representation of system reliability with respect to failure to operate and with respect to failure to idle. The Boolean representations of reliability of the essentially series-type structures with respect to failure to operate are given in Table 2 (cf. [5], Table 1). The expressions of multiplication in the Boolean representation of a structure of essentially series-type form a minimal critical path with respect to breakdown to operate. The expressions of summation in the Boolean representation of a dual structure of essentially parallel-type form a minimal critical path with respect to breakdown to idle. The mentioned duality is as follows: by changing multiplication into addition, and vice versa, we obtain from the essentially series-type structure the dual essentially parallel-type structure, and vice versa.

If we have the Boolean representation of system reliability, then it is easy to find the probability $u(a)$ of breakdown to operate; however, with the duality argument we find the probability $v(b)$ of breakdown to idle (cf. [6], formula (1)): $v(b) = 1-u(1-b)$. The probability of structure breakdown in the time interval $(0, t)$, e. g. the distribution function of working time of the structure, is equal to $F = u(a)+v(b)$.

Expanding the function F in a power series with respect to t, we have

(2) $$F = w_k(p)t^k + w_{k+1}(p)t^{k+1} + \ldots$$

The power index in the first expression of this expansion is called the *range* of the considered structure. For any n, the class of all possible structures built from n elements divides into subclasses of structures with the same range. Because the structure reliability for $t \to 0$ depends mainly upon the first expression of the asymptotical expansion (2), therefore optimum structures with respect to Criterion II have range as great as possible and among all structures of this range have the smallest coefficient in the first expression (cf. Gnedenko et al. [2]).

TABLE 1. Series-parallel two-terminal networks

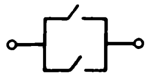
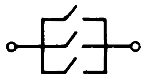
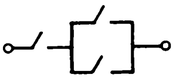
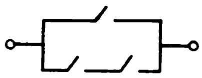
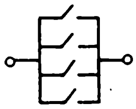
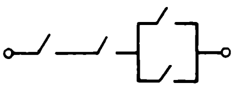
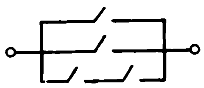
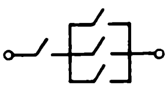
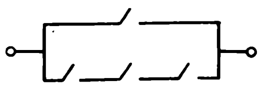
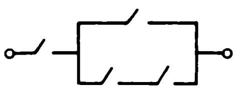
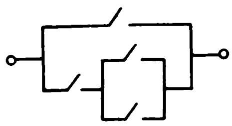
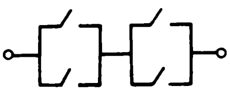
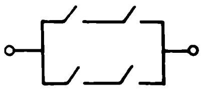
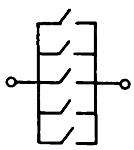
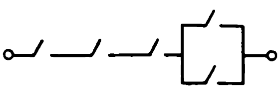
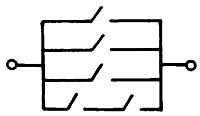
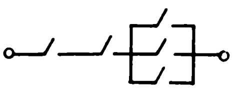
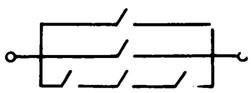
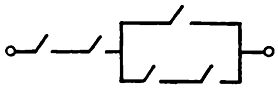
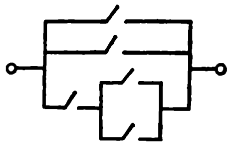
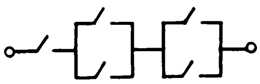
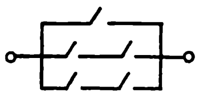
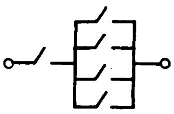
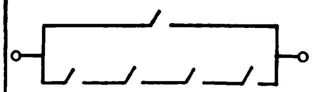
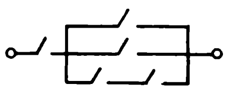
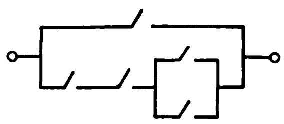
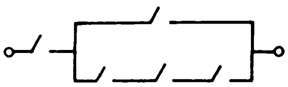
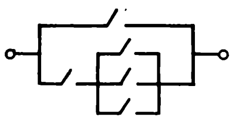
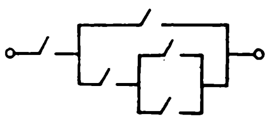
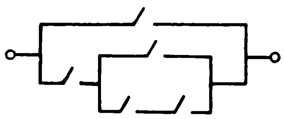
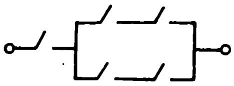
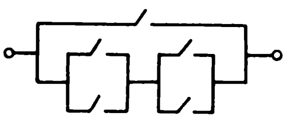
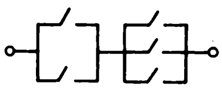
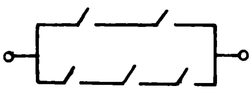
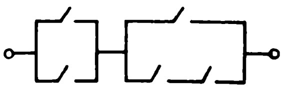
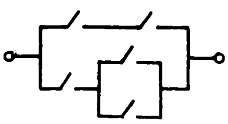
Number of elements	Essentially series-type		Essentially parallel-type	
	Number of structure	Network	Number of structure	Network
1	2	3	4	5
2	1		1'	
3	1		1'	
	2		2'	
4	1		1'	
	2		2'	
	3		3'	
	4		4'	
	5		5'	
5	1		1'	
	2		2'	
	3		3'	

TABLE 1 (continuation)

1	2	3	4	5
	4		4'	
	5		5'	
	6		6'	
	7		7'	
5	8		8'	
	9		9'	
	10		10'	
	11		11'	
	12		12'	

The range of the considered structures for $n = 2, 3, 4$ and 5 may be read from Table 2 for $0 < p < 1$. In extremal cases for $p = 0$, the optimum one is the essentially parallel-type structure no. $1'$ and for $p = 1$ the optimum one is the essentially series-type structure no. 1. They are structures of range n with working time distribution function $F = (1-e^{-\lambda t})^n$.

TABLE 2. Reliability of two-terminal series-parallel structures

Numer of elements	Number of structure	Boolean representation	Distribution function
2	1	X_1X_2	$2px+(-p-p^2+(1-p)^2)x^2+o(x^2)$
3	1	$X_1X_2X_3$	$3px+\left(-\frac{3}{2}p-3p^2\right)x^2+o(x^2)$
	2	$X_1(X_2+X_3)$	$px+\left(-\frac{1}{2}p+p^2+2(1-p)^2\right)x^2+o(x^2)$
4	1	$X_1X_2X_3X_4$	$4px+(-2p-6p^2)x^2+o(x^2)$
	2	$X_1X_2(X_3+X_4)$	$2px-px^2+o(x^2)$
	3	$X_1(X_2+X_3+X_4)$	$px+\left(-\frac{1}{2}p+3(1-p)^2\right)x^2+o(x^2)$
	4	$X_1(X_2+X_3X_4)$	$px+\left(-\frac{1}{2}p+2p^2+(1-p)^2\right)x^2+o(x^2)$
	5	$(X_1+X_2)(X_3+X_4)$	$(2p^2+4(1-p)^2)x^2+o(x^2)$
5	1	$X_1X_2X_3X_4X_5$	$5px+\left(-\frac{5}{2}p+10p^2\right)x^2+o(x^2)$
	2	$X_1X_2X_3(X_4+X_5)$	$3px+\left(-\frac{3}{2}p-2p^2\right)x^2+o(x^2)$
	3	$X_1X_2(X_3+X_4+X_5)$	$2px+(-p-p^2)x^2+o(x^2)$
	4	$X_1X_2(X_3+X_4X_5)$	$2px+(-p+p^2)x^2+o(x^2)$
	5	$X_1(X_2+X_3)(X_4+X_5)$	$px+\left(-\frac{1}{2}p+2p^2\right)x^2+o(x^2)$
	6	$X_1(X_2+X_3+X_4+X_5)$	$px+\left(-\frac{1}{2}p+4(1-p)^2\right)x^2+o(x^2)$
	7	$X_1(X_2+X_3+X_4X_5)$	$px+\left(-\frac{1}{2}p+2(1-p)^2\right)x^2+o(x^2)$
	8	$X_1(X_2+X_3X_4X_5)$	$px+\left(-\frac{1}{2}p+3p^2+(1-p)^2\right)x^2+o(x^2)$
	9	$X_1(X_2+X_3(X_4+X_5))$	$px+\left(-\frac{1}{2}p+p^2+(1-p)^2\right)x^2+o(x^2)$
	10	$X_1(X_2X_3+X_4X_5)$	$px+\left(-\frac{1}{2}p+4p^2\right)x^2+o(x^2)$
	11	$(X_1+X_2)(X_3+X_4+X_5)$	$(p^2+6(1-p)^2)x^2+o(x^2)$
	12	$(X_1+X_2)(X_3+X_4X_5)$	$(3p^2+2(1-p)^2)x^2+o(x^2)$

The evaluation of the system reliability we describe now on the example of structure no. 8 for $n = 5$. The Boolean representation of the system reliability with respect to failure to idle has the form

$$X_1(X_2+X_3X_4X_5) = X_1X_2+X_1X_3X_4X_5,$$

therefore the Boolean representation of reliability of this structure with respect to failure to operate has the form

$$X_1+X_2(X_3+X_4+X_5) = X_1+X_2X_3+X_2X_4+X_2X_5.$$

In the considered structure we have 4 minimal critical paths with respect to failure to operate, namely the sets of elements $\{1\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, and 2 minimal critical paths with respect to failure to idle, namely the sets of elements $\{1, 2\}$, $\{1, 3, 4, 5\}$. Hence and from the Inclusion-Exclusion Theorem we have

$$u(a) = a+3a^2-6a^3+4a^4-a^5,$$
$$v(b) = b^2+b^4-b^5.$$

If

(3) $$u(a) = c_1 a+c_2 a^2+c_3 a^3+\ldots$$

and if $x = \lambda t$, then

(4) $$u(a) = c_1 p\sum_{k=1}^{\infty} x^k(-1)^{k+1}/k!+2c_2p^2\sum_{k=2}^{\infty}(2^{k-1}-1)x^k(-1)^k/k!+ +3c_3p^3\sum_{k=3}^{\infty}(3^{k-1}-2^k+1)x^k(-1)^{k+1}/k!+\ldots,$$

so that in our case we have

$$u(a) = px-\tfrac{1}{2}px^2+3p^2x^2+o(x^2),$$
$$v(b) = (1-p)^2x^2+o(x^2),$$

therefore

$$F = px+\big(-\tfrac{1}{2}p+3p^2+(1-p)^2\big)x^2+o(x^2).$$

The initial terms of the asymptotical expansion (2) of the distribution function of the system working time for the essentially series-type structures considered in this note (also the above-mentioned estimation) are given in the last column of Table 2. The estimations of the distribution functions of the system working time of the essentially parallel-type structures may be obtained by substitution of $1-p$ in place of p. These estimations may be used to compare the structures with respect to reliability and determine the optimum ones.

The conclusions from Table 2 are as follows. For $n = 2$ we have 2 structures, and structure 1 of the essentially series-type is optimum for small p ($0 < p < 0.5$) while structure 1′ of the essentially parallel-type is optimum for large p ($0.5 < p < 1$). For $n = 3$ we have 4 structures:

structure 2 is optimum for small p and structure $2'$ is optimum for large p. For $n = 4$ we have 10 structures with 2 structures of range 2; structure 5 is optimum for small p and structure $5'$ is optimum for large p. For $n = 5$ we have 24 structures with 4 structures of range 2. In this case each structure of range 2 is optimum in some interval of the parameter p. These intervals and the corresponding optimum structures may be found in Fig. 1 which presents the graphs of the functions $w = w_2(p)\lambda^{-2}$, e. g. the characterization of the first expression in (2). Notice that for $0 < p < 1$ the structures of range 3 arise in systems built from $n = 9$ elements.

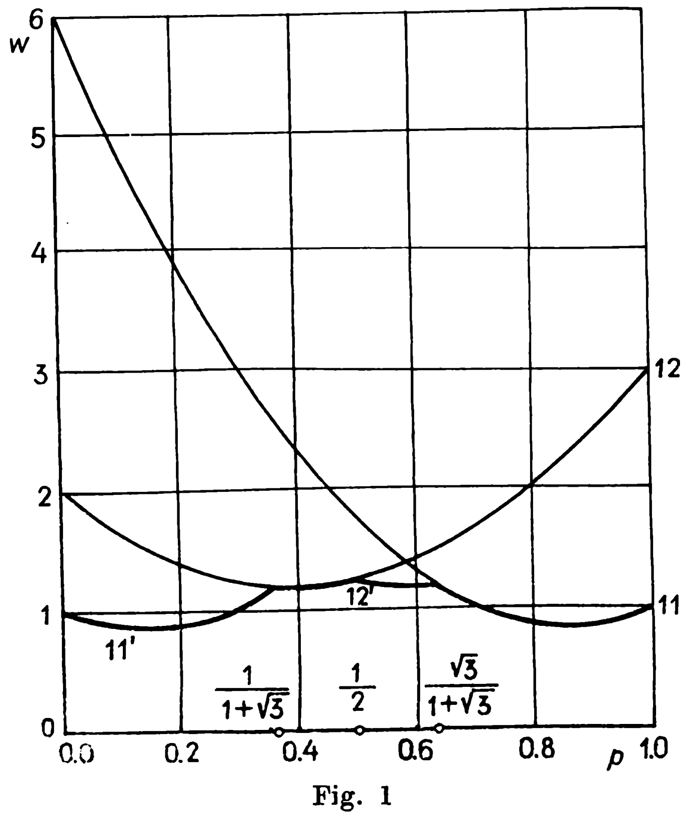


Fig. 1

The above-mentioned structures optimum with respect to Criterion II differ from the structures optimum with respect to Criterion I obtained in [6]. Criteria I and II are thus not equivalent in the considered model of the systems reliability.

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[1] R. E. Barlow and F. Proschan, *Mathematical theory of reliability*, New York 1965.

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[4] — and Z. Kowal, *Losowa nośność graniczna konstrukcji o dwóch minimalnych krytycznych zbiorach elementów mających elementy wspólne*, Archiwum Inżynierii Lądowej 18 (1972), p. 103-115.

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PEWNE OSZACOWANIA NIEZAWODNOŚCI SYSTEMÓW O UKŁADZIE SZEREGOWO-RÓWNOLEGŁYM

STRESZCZENIE

Rozważmy systemy elementów o układzie szeregowo-równoległym, o jednostkowym, wykładniczym rozkładzie czasu pracy elementów, przy założeniu, że awarie elementów mogą być dwojakie: z prawdopodobieństwem p — są to awarie polegające na błędnym działaniu elementu — i z prawdopodobieństwem $1-p$ — są to awarie polegające na niespodziewanym działaniu elementu, gdy nie powinien on pracować (por. Łomnicki [6]). Mogą być brane pod uwagę dwa kryteria optymalności systemu: (I) maksymizowanie średniego czasu pracy systemu albo (II) maksymizowanie niezawodności systemu w początkowym okresie pracy systemu. Systemy optymalne w sensie kryterium I rozważał Łomnicki w pracy [5]; systemy optymalne w sensie kryterium II są rozważane w tej pracy.