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On boundary Hilbert differential complexes

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Abstract. Systems of partial differential equations with constant coefficients are studied in spaces of functions and distributions that are important for applications to tangential complexes.

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Introduction

The purpose of this paper is to develop the study of differential complexes with constant coefficients (Hilbert differential complexes) and of the tangential complexes they induce on hypersurfaces. These results generalize those obtained by several authors for the Dolbeault complex and for the tangential Cauchy–Riemann complex on hypersurfaces of $\boldsymbol{C}^n$, although some vanishing theorems on domains with a non-smooth boundary are seemingly new even in that case.

In this general setting, only geometrically convex or geometrically concave domains and boundaries of convex domains can be considered. Indeed, as in complex analysis, one should develop a convexity theory special to any given system of differential operators to obtain optimal vanishing theorems.

For reasons of invariance and conciseness, most of the results are stated in the language of homological algebra, i.e. homology and cohomology of differential complexes is expressed in terms of the Ext and Tor functors, but the reader will find no difficulty in the translation.

Also I preferred to study the tangential complexes in their "rough" form, i.e. on Whitney functions or on distributions concentrated on a hypersurface, the equivalence to more agreeable differential objects being assured then by the assumption that the hypersurface be "formally non-characteristic" for the given Hilbert differential complex, in the sense precised in [6].

This investigation is part of an effort to clarify the conditions for the validity of the Poincaré lemma for quite general differential complexes; so, while the example of Cauchy–Riemann differential complexes is a constant source of inspiration, I aimed here to a higer level of generality in the methods.

The main tool developed here is a theorem on differential systems with constant coefficients in the cathegory of C^∞ functions with compact support that solves a problem originally posed by Palamodov. From this several results are obtained for convex and concave sets that permit applications to hypersurfaces.

The paper is organized as follows: first I introduce the spaces of functions and distributions and the standard facts of homological algebra that will be needed in the sequel. In the second section I prove the afore mentioned result on solvability of systems of p.d.e. with constant coefficients by means of a lemma on the existence of certain plurisubharmonic functions. In the following section various applications are given to the computation of homology and cohomology for functions and distributions on convex and concave domains. Section 4 contains applications to the computation of the global homology and cohomology for "rough" boundary complexes, while in Section 5 the local case (Poincaré lemma) is treated. In Section 6 it is shown how these results apply to tangential boundary complexes.

1. Preliminaries

A. Spaces of functions and distributions. Given an open set Ω in $\mathbf{R}^n$, we denote by $\mathscr{E}(\Omega)$ the space of complex-valued C^∞ functions on Ω, with the Fréchet–Schwartz topology of uniform convergence with all derivatives on compact sets.

If K is a closed subset of Ω, then we denote by $\mathscr{E}_K(\Omega)$ the subspace of $f \in \mathscr{E}(\Omega)$ with support contained in K. This is a closed subspace of $\mathscr{E}(\Omega)$ and therefore inherits a Fréchet–Schwartz topology. We note that, if ω is an open neighborhood of K in Ω, then the restriction map $\mathscr{E}(\Omega) \to \mathscr{E}(\omega)$ induces an isomorphism $\mathscr{E}_K(\Omega) \cong \mathscr{E}_K(\omega)$. Therefore we shall write simply $\mathscr{E}_K$ instead of $\mathscr{E}_K(\Omega)$, as this space depends only on the locally closed set K and not on the neighborhood Ω. When K is compact, we write also $\mathscr{D}_K$ for $\mathscr{E}_K$.

Given any subset A of $\mathbf{R}^n$, we define

$$\mathscr{D}_A = \varinjlim_{K \text{compact} \subset A} \mathscr{D}_K$$

with the Schwartz direct limit topology. Notice that $\mathscr{D}_A$ can be identified with the subspace of $f \in \mathscr{E}(\boldsymbol{R}^n)$ having compact support contained in A. When A is open we write as customary $\mathscr{D}(A)$ for $\mathscr{D}_A$.

If Ω is an open set in $\boldsymbol{R}^n$, we denote by $\mathscr{D}'(\Omega)$ (resp. $\mathscr{E}'(\Omega)$) the space of distributions (resp. distributions with compact support) in Ω, with the topology of strong dual of $\mathscr{D}(\Omega)$ (resp. $\mathscr{E}(\Omega)$).

Given a subset A of Ω we set $\mathscr{D}'_A(\Omega)$ (resp. $\mathscr{E}'_A$) to denote distributions (distributions with compact support) with support contained in A. When A is locally closed (A closed in an open set Ω of $\boldsymbol{R}^n$), we write simply $\mathscr{D}'_A$ for $\mathscr{D}'_A(\Omega)$ as the remarks made above for functions apply also to this case.

Let now K be a closed subset of an open set Ω of $\boldsymbol{R}^n$ and denote by $\mathscr{F}_K(\Omega)$ (resp. $\mathscr{F}_K^{\text{comp}}(\Omega)$) the subset of $\mathscr{E}(\Omega)$ (resp. $\mathscr{D}(\Omega)$) of functions vanishing with all derivatives on K. Then we define the space $\mathscr{W}_K$ of (complex valued) Whitney functions on K by the exact sequence:

$$0 \to \mathscr{F}_K(\Omega) \to \mathscr{E}(\Omega) \to \mathscr{W}_K(\Omega) \to 0 \tag{1}$$

and the space $\mathscr{W}_K^{\text{comp}}$ of Whitney functions with compact support in K by the exact sequence:

$$0 \to \mathscr{F}_K^{\text{comp}}(\Omega) \to \mathscr{D}(\Omega) \to \mathscr{W}_K^{\text{comp}} \to 0 \tag{2}$$

endowing them with the natural quotient topologies. Notice that $\mathscr{W}_K$ has a topology of Fréchet–Schwartz.

We also define the space $\check{\mathscr{D}}'_K$ of "extendible distributions on K" by the exact sequence:

$$0 \to \mathscr{D}'_F \to \mathscr{D}'(\Omega) \to \check{\mathscr{D}}'_K \to 0, \tag{3}$$

where F is the closure of $\Omega - K$ in Ω, and the space $\check{\mathscr{E}}'_K$ of "extendible distributions with compact support in K" by the exact sequence:

$$0 \to \mathscr{E}'_F \to \mathscr{E}'(\Omega) \to \check{\mathscr{E}}'_K \to 0. \tag{4}$$

We note that there is a natural inclusion $\check{\mathscr{D}}'_K \subset \mathscr{D}'(\operatorname{Int} K)$: the extendible distributions on K are the distributions that can be continued beyond the points in $\partial K \cap K$.

We have an inclusion $\check{\mathscr{E}}'_K \subset \mathscr{D}'(\operatorname{int} K)$ and the elements of $\check{\mathscr{E}}'_K$ are those whose supports have a compact closure in K. On $\check{\mathscr{D}}'_K$ and $\check{\mathscr{E}}'_K$ we consider the natural quotient topologies.

If K is regular in the following sense:

for every $x \in K$ there is a neighborhood U of x and constants $C, d > 0$ such that for any $x_0, x_1 \in U \cap K$ there is a continuous rectifiable path γ: $[0, 1] \to K$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$ with

$$\text{lenght of } \gamma \leqslant C|x_1 - x_0|^d,$$

then $\check{\mathscr{E}}'_K$ is the strong dual of $\mathscr{W}_K$ and $\check{\mathscr{D}}'_K$ is the strong dual of $\mathscr{W}_K^{\text{comp}}$ (cf. [22]).

If K equals the closure in Ω of its interior, then $\mathcal{D}'_K$ is the strong dual of $\mathcal{D}_K$ and $\mathcal{E}'_K$ is the strong dual of $\mathcal{E}_K$ (cf. [18]).

We also notice that in this case, denoting by F the closure in Ω of $\Omega - K$, then

$$\mathcal{E}_F = \mathcal{F}_K(\Omega), \qquad \mathcal{D}_F = \mathcal{D}(\Omega) \cap \mathcal{F}_K(\Omega).$$

If also F equals the closure in Ω of its interior, we obtain from (1) and (2) by duality exact sequences:

$$(1)^* \qquad 0 \to [\mathcal{W}_K]' \to \mathcal{E}'(\Omega) \to \mathcal{E}'_F \to 0,$$

$$(2)^* \qquad 0 \to [\mathcal{W}_K^{\text{comp}}]' \to \mathcal{D}'(\Omega) \to \mathcal{E}'_F \to 0$$

from which we deduce that also in this case

$$[\mathcal{W}_K]' = \mathcal{E}'_K \quad \text{and} \quad [\mathcal{W}_K^{\text{comp}}]' = \mathcal{D}'_K.$$

We will consider all spaces of functions and distributions introduced above as left and right modules over the ring $\mathcal{P} = C[z_1, \ldots, z_n]$ of polynomials in n indeterminates by the action

$$f \cdot z_j = z_j \cdot f = i^{-1}\, \partial f/\partial x_j \quad \text{for } j = 1, \ldots, n.$$

B. Homological algebra. If M is a unitary left $\mathcal{P}$-module, a free resolution of M is an exact sequence

$$(*) \qquad 0 \leftarrow M \leftarrow F_0 \xrightarrow{A_0} F_1 \xrightarrow{A_1} F_2 \leftarrow \ldots$$

of unitary left $\mathcal{P}$-modules and $\mathcal{P}$-homomorphisms, where all the F_i are free.

Given another unitary left $\mathcal{P}$-module N, the groups $\text{Ext}^j_{\mathcal{P}}(M, N)$ are defined as the cohomology groups of the complex:

$$0 \to \text{Hom}(F_0, N) \xrightarrow{A_0^*} \text{Hom}(F_1, N) \xrightarrow{A_1^*} \text{Hom}(F_2, N) \to \ldots$$

and for a unitary right $\mathcal{P}$-module R the groups $\text{Tor}_j^{\mathcal{P}}(M, R)$ are defined as the cohomology groups of the complex

$$0 \leftarrow F_0 \otimes_{\mathcal{P}} R \xleftarrow{A_0^*} F_1 \otimes_{\mathcal{P}} R \xleftarrow{A_1^*} F_2 \otimes_{\mathcal{P}} R \leftarrow \ldots$$

If M is of finite type, then we can find a resolution $(*)$ by free $\mathcal{P}$-modules of finite type $F_i = \mathcal{P}^{p_i}$ and $p_i = 0$ for $i > n$. Then the $\mathcal{P}$-homomorphisms are represented by matrices $A_i = A_i(z)$ of size $p_i \times p_{i+1}$ and with polynomial entries. When N or R is one of the spaces of functions or distributions introduced in the previous subsection, with the $\mathcal{P}$-module structure precised above, then we have:

$$\text{Hom}_{\mathcal{P}}(\mathcal{P}^{p_i}, N) \cong N^{p_i} \quad \text{and} \quad \mathcal{P}^{p_i} \otimes_{\mathcal{P}} R \cong R^{p_i}$$

$$\text{and} \quad A_i^* = {}^tA_i(D), \quad A_{i^\circ} = A_i(D)$$

$$\text{for} \quad D = (i^{-1}\, \partial/\partial x_1, \ldots, i^{-1}\, \partial/\partial x_n),$$

while

$$\mathrm{Ext}^j_{\mathscr{P}}(M, N) = \frac{\mathrm{Ker}\,({}^t A_j(D)\colon\ N^{p_j} \to N^{p_{j+1}})}{\mathrm{Image}\,({}^t A_{j-1}(D)\colon\ N^{p_{j-1}} \to N^{p_j})}$$

and

$$\mathrm{Tor}^{\mathscr{P}}_j(M, R) = \frac{\mathrm{Ker}\,(A_j(D)\colon\ R^{p_j} \to R^{p_{j-1}})}{\mathrm{Image}\,(A_{j+1}(D)\colon\ R^{p_{j+1}} \to R^{p_j})}$$

are the cohomology groups of the complexes of linear spaces and linear partial differential operators with constant coefficients:

$$0 \to N^{p_0} \xrightarrow{{}^t A_0(D)} N^{p_1} \xrightarrow{{}^t A_1(D)} N^{p_2} \to \dots$$

and

$$0 \leftarrow R^{p_0} \xleftarrow{A_0(D)} R^{p_1} \xleftarrow{A_1(D)} R^{p_2} \leftarrow \dots$$

We will also use the following fact from homological algebra: if

$$0 \to E \to F \to G \to 0$$

is a short exact sequence of unitary left (resp. right) $\mathscr{P}$-modules, then for every unitary left $\mathscr{P}$-module M we obtain the long exact sequence for Ext (resp. for Tor):

$$\begin{aligned} 0 \to \mathrm{Ext}^0_{\mathscr{P}}(M, E) \to \mathrm{Ext}^0_{\mathscr{P}}(M, F) \to \mathrm{Ext}^0_{\mathscr{P}}(M, G) \to \\ \to \mathrm{Ext}^1_{\mathscr{P}}(M, E) \to \dots \qquad\qquad \\ \dots \to \mathrm{Ext}^j_{\mathscr{P}}(M, G) \to \\ \to \mathrm{Ext}^{j+1}_{\mathscr{P}}(M, E) \to \mathrm{Ext}^{j+1}_{\mathscr{P}}(M, F) \to \dots \end{aligned}$$

and respectively:

$$\begin{aligned} 0 \leftarrow \mathrm{Tor}^{\mathscr{P}}_0(M, G) \leftarrow \mathrm{Tor}^{\mathscr{P}}_0(M, F) \leftarrow \mathrm{Tor}^{\mathscr{P}}_0(M, E) \\ \leftarrow \mathrm{Tor}^{\mathscr{P}}_1(M, G) \leftarrow \dots \qquad\qquad \\ \dots \leftarrow \mathrm{Tor}^{\mathscr{P}}_j(M, E) \leftarrow \\ \leftarrow \mathrm{Tor}^{\mathscr{P}}_{j+1}(M, G) \leftarrow \mathrm{Tor}^{\mathscr{P}}_{j+1}(M, F) \leftarrow \mathrm{Tor}^{\mathscr{P}}_{j+1}(M, E) \leftarrow \dots \end{aligned}$$

2. Systems of differential equations on $\mathscr{D}_K$ and $\mathscr{W}_K$

Given a compact convex set K on $\boldsymbol{R}^n$ we denote by H_K its supporting function:

$$H_K(y) = \sup_K \langle y, x \rangle, \qquad x \in \boldsymbol{R}^n.$$

Then we have:

PROPOSITION 1. *Let K be a convex compact set in $\boldsymbol{R}^n$ with a non-empty interior.*

Let ψ be a real valued function on C^n with the property:
For every integer $m \geqslant 0$ there is a constant $\sigma_m > 0$ such that

(5) $$\psi(z) \leqslant \sigma_m (1+|z|)^{-m} \quad \text{on } C^n.$$

Then we can find a continuous plurisubharmonic function φ on C^n such that
For every integer $m \geqslant 0$ there is a constant $c_m > 0$ such that

(6) $$e^{\varphi(z)} \leqslant c_m (1+|z|)^{-m} e^{H_K(\operatorname{Im} z)} \quad \text{on } C^n,$$

(7) $$e^{\varphi(z)} \geqslant \psi(z) e^{H_K(\operatorname{Im} z)} \quad \text{on } C^n.$$

There is a constant $C > 0$ such that

(8) $$|\varphi(z) - \varphi(w)| \leqslant C, \quad \text{if } z, w \in C^n \text{ and } |z-w| \leqslant 1.$$

Proof. (A) First we consider the case $n = 1$ and $K = \{x \in \boldsymbol{R} | \ |x| \leqslant 1\}$. For a sequence of positive integers $\{h_n\}$ with

(*) $$h_1 \geqslant 1 \quad \text{and} \quad h_{n+1} \geqslant 4h_n \quad \text{for } n \geqslant 1$$

the infinite product

$$g(z) = z \prod_{n=1}^{\infty} (1 - (z/\pi h_n))$$

defines an entire function on C having only simple zeros at integral multiples of π. Then we can consider the entire function:

$$f(z) = \sin z / g(z).$$

Because $f(z)$ and $f(z - \frac{1}{2}\pi)$ have no common zeros, then

$$\varphi(z) = \tfrac{1}{2} \ln (|f(z)|^2 + |f(z - \tfrac{1}{2}\pi)|^2) + \ln(2\sigma_1)$$

is plurisubharmonic and smooth. One checks by an easy computation that, if we choose $\{h_n\}$ subject to the further condition that

$$h_1 \ldots h_n \geqslant 2^{n+4} \sigma_{n+1} \quad \text{for } n \geqslant 1,$$

then φ satisfies all the required conditions.

(B) Let us consider now the case $n > 1$ and K be the coordinate cube:

$$K = \{x \in \boldsymbol{R}^n | \ |x_j| \leqslant 1 \text{ for } j = 1, \ldots, n\}.$$

We define $g(t) = \sup\{0, \sup_{|z| \geqslant t} \psi(z)\}^{1/n}$. By point (A) we can find a continuous subharmonic function $h(w)$ on C such that conditions (6), (7), (8) are satisfied with respect to the function $\psi'(w) = g(|w|)$ ($w \in C$). Then the function $\varphi(z) = h(z_1) + \ldots + h(z_n)$ satisfies all requirements of the proposition. ▼

(C) By a linear change of coordinates we deduce from (B) that the proposition holds also in the case K is a parallelepiped:

$$K = \{x^1 v_1 + \dots + x^n v_n + v_0 \mid -1 \leqslant x^j \leqslant 1 \text{ for } j = 1, \dots, n\}$$

for $v_0, \dots, v_n \in \boldsymbol{R}^n$ and $v_1, \dots, v_n$ linearly independent.

(D) Let us consider now the general case. Let x_0 be an interior point of K and let $0 < r < R$ be such that

$$\{|x - x_0| \leqslant r\} \subset K \subset \{|x - x_0| \leqslant R\}.$$

Let $e_1, \dots, e_n$ be the vectors of the canonical basis of $\boldsymbol{R}^n$ and let us set

$$e = n^{-1/2}(e_1 + \dots + e_n)$$

and

$$v_j = \frac{r}{2n}\frac{re_j + Re}{r + R} \quad \text{for } j = 1, \dots, n.$$

We consider the parallelepiped

$$C = \{a^1 v_1 + \dots + a^n v_n \mid -1 \leqslant a^j \leqslant 1 \text{ for } j = 1, \dots, n\}$$

and we define

$$g(t) = \sup_{|z| \geqslant t} \psi(z) \quad \text{for } t \in \boldsymbol{R}.$$

Then by point (C) of the proof we can find a continuous plurisubharmonic function φ_1 on $\boldsymbol{C}^n$ such that

(i) $$e^{\varphi_1(z)} \geqslant g(|z|)\, e^{H_C(\operatorname{Im} z)} \quad \text{for } z \in \boldsymbol{C}^n$$

and for every integer $m \geqslant 0$ there is a constant c_m such that

(ii) $$e^{\varphi_1(z)} \leqslant c_m (1 + |z|)^{-m} e^{H_C(\operatorname{Im} z)} \quad \text{on } \boldsymbol{C}^n$$

and

(iii) $$|\varphi_1(z) - \varphi_1(z')| \leqslant C \quad \text{if } z, z' \in \boldsymbol{C}^n \text{ and } |z - z'| \leqslant 1.$$

Let F denote the subspace of $\partial K \times SO(n)$ of pairs (x, L), where $x \in \partial K$ and L is a rotation in $\boldsymbol{R}^n$ changing the vector e into the vector $(x - x_0)/|x - x_0|$. The set F is closed and therefore compact.

For $(x, L) \in F$ we define

$$v(x) = x - \tfrac{1}{2}r(x - x_0)/|x - x_0|$$

and

$$C(x, L) = v(x) + LC.$$

Then for every pair $(x, L) \in F$ we have

$$x \in C(x, L) \subset K.$$

Hence

$$H_K(y) = \sup_K \langle y, x \rangle = \sup_{(x,L)\in F} H_{C(x,L)}(y) \quad \text{for } y \in \mathbf{R}^n,$$

and

$$H_{C(x,L)}(y) = \langle y, v(x)\rangle + H_C({}^tLy).$$

Therefore, if we set

$$\varphi(x, L; z) = \langle v(x), \operatorname{Im} z\rangle + \varphi_1({}^tLz)$$

and we remark that $|{}^tLz| = |z|$ for $L \in SO(n)$, we obtain from (i) and (ii):

(i)′ $$e^{\varphi(x,L;z)} \geqslant g(|z|)\, e^{H_{C(x,L)}(\operatorname{Im} z)} \quad \text{on } \mathbf{C}^n$$

and

(ii)′ $$e^{\varphi(x,L;z)} \leqslant c_m (1+|z|)^{-m} e^{H_{C(x,L)}(\operatorname{Im} z)} \quad \text{on } \mathbf{C}^n$$

(where the constants c_m's are the same as in (ii)) and

(iii)′ $$|\varphi(x, L; z) - \varphi(x, L; z')| \leqslant C' \quad \text{if } z, z' \in \mathbf{C}^n \text{ and } |z - z'| \leqslant 1,$$

with a constant C' independent of $(x, L) \in F$.

Because F is compact,

$$\varphi(z) = \sup_{(x,L)\in F} \varphi(x, L; z)$$

is continuous and hence plurisubharmonic and satisfies all the conditions in the statement.

Then we obtain the following:

PROPOSITION 2. *Assume that K is a convex subset of $\mathbf{R}^n$. Then $\mathscr{D}_K$ is a flat $\mathscr{P}_n$-module and for every $q \times p$ matrix $A(z)$ with entries in $\mathscr{P}_n$ the linear map $A(D)$: $\mathscr{D}_K^p \to \mathscr{D}_K^q$ defined by the corresponding differential operator is a topological homomorphism.*

Proof. We have to prove that, if

(9) $$\mathscr{P}_n^{p_2} \xrightarrow{A_1(z)} \mathscr{P}_n^{p_1} \xrightarrow{A_0(z)} \mathscr{P}_n^{p_0}$$

is an exact sequence of free $\mathscr{P}_n$-modules of finite type and $\mathscr{P}_n$-homomorphisms, then the corresponding sequence

(10) $$\mathscr{D}_K^{p_2} \xrightarrow{A_1(D)} \mathscr{D}_K^{p_1} \xrightarrow{A_0(D)} \mathscr{D}_K^{p_0}$$

is an exact sequence of topological vector spaces and topological homomorphisms.

In the following we will assume that K is convex, compact and with a non-empty interior, as this case will imply the general case.

Let $f \in \mathscr{D}_K^{p_i}$ $(i = 0$ or $1)$ and let us denote by

$$\hat{f}(z) = (2\pi)^{-n/2} \int f(x)\, e^{-i\langle x,z\rangle}\, dx$$

its Fourier–Laplace transform. By the theorem of Paley Wiener $\hat{f}$ is entire on C^n and satisfies:

for every integer $m \geqslant 0$ *there is a constant* $\sigma_m > 0$ *such that*

$$\psi(z) = (1+|z|)^{2n}|\hat{f}(z)|\, e^{-H_K(\operatorname{Im} z)} \leqslant \sigma_m(1+|z|)^{-m}.$$

Then by Proposition 1 we can find a continuous plurisubharmonic function φ on C^n satisfying (6), (7), (8) with respect to ψ. Then

$$\int |\hat{f}(z)|^2 e^{-2\varphi(z)}\, d\lambda(z) < +\infty$$

and by Theorems 7.6.11 and 7.6.12 in [11] we deduce the following:

1° If $i = 1$ and $A_0(D)f = 0$, then $A_0(z)\hat{f}(z) = 0$ and we can find $\hat{v}(z) \in \mathcal{O}(C^n)$ such that

$$A_1(z)\hat{v}(z) = \hat{f}(z)$$

and

$$(*) \qquad \int |\hat{v}(z)|^2 (1+|z|^2)^{-N} e^{-2\varphi(z)}\, d\lambda(z) < +\infty$$

(where N is a positive integer only depending on A_0 and A_1).

Then, by Paley Wiener theorem, $\hat{v}$ is the Fourier–Laplace transform of a function $v \in \mathscr{D}_K^{p_2}$ such that

$$A_1(D)v = f.$$

2° If $i = 0$, and f is orthogonal to all exponential-polynomial solutions u of the homogeneous equation ${}^tA_0(D)u = 0$, then by the Nullstellensatz we have

$$\hat{f}(z) = A_0(z)h(z) \quad \text{for some } h \in \mathcal{O}^{p_1}(C^n)$$

and then we can find $\hat{v} \in \mathcal{O}^{p_1}(C^n)$ satisfying $(*)$ such that

$$\hat{f}(z) = A_0(z)\hat{v}(z).$$

But then $\hat{v}$ is the Fourier–Laplace transform of some $v \in \mathscr{D}_K^{p_1}$ such that

$$A_0(D)v = f.$$

Because the image of $A_0(D)$: $\mathscr{D}_K^{p_1} \to \mathscr{D}_K^{p_0}$ is contained in the orthogonal to the space of exponential polynomial solutions u of ${}^tA_0(D)u = 0$, we actually have equality. Thus $A_0(D)$: $\mathscr{D}_K^{p_1} \to \mathscr{D}_K^{p_0}$ having a closed image is a topological homomorphism because these spaces are Fréchet when K is compact. The general statement follows because, if F is any compact convex subset of the convex set K, then

$$\left(A_0(D)\mathscr{D}_K^{p_1}\right) \cap \mathscr{D}_F^{p_0} = A_0(D)\mathscr{D}_F^{p_1}$$

and therefore $A_0(D)\mathscr{D}_K^{p_1}$ is strictly bornologic.

By the duality theorem we obtain then the following:

PROPOSITION 3. *If K is a convex subset of $\mathbf{R}^n$ and $(\mathscr{D}_K)'$ denotes the strong dual of $\mathscr{D}_K$, then $(\mathscr{D}_K)'$ is an injective $\mathscr{P}_n$-module for the action of $\mathscr{P}_n$ on $(\mathscr{D}_K)'$ given by*

$$\langle z_j T, u\rangle = \langle T, -(1/i)\,\partial u/\partial x_j\rangle \quad \text{for } j = 1,\ldots,n;\ T\in(\mathscr{D}_K)',\ u\in\mathscr{D}_K.$$

Proof. Indeed, because when (9) is an exact sequence, then (10) is an exact sequence of topological vector spaces and topological homomorphisms, then by duality (cf. [9]) it follows that the exactness of (9) implies that of the sequence

$$(11)\qquad (\mathscr{D}_K)'^{p_0} \xrightarrow{{}^tA_0(D)} (\mathscr{D}_K)'^{p_1} \xrightarrow{{}^tA_1(D)} (\mathscr{D}_K)'^{p_2}.$$

Remark 1. If K is a closed convex set in $\mathbf{R}^n$, then, as we noted in Section 1, $(\mathscr{D}_K)' = \check{\mathscr{D}}'_K$ and when K is an open convex set, then $(\mathscr{D}_K)' = \mathscr{D}'(K)$: therefore Proposition 3 contains at the same time a statement about the solvability of systems of differential equations in the class of extensible distributions and in the class of distributions on a convex set.

Remark 2. Repeating the argument given in [5] for the case of an open K, we obtain that, if the interior of the convex set K is non-empty (i.e. excluding the trivial case where $\mathscr{D}_K = 0$), then $\mathscr{D}_K$ is faithfully flat over $\mathscr{P}_n$.

With the notations introduced in Section 1 we obtain the following statements that are easy consequences of [11]:

PROPOSITION 4. *If K is a convex set, then $\mathscr{E}'_K$ is a flat $\mathscr{P}_n$-module. Moreover, for every $q\times p$ matrix with polynomial entries the linear map defined by the corresponding differential operator:*

$$A(D)\colon\ \mathscr{E}'^p_K \to \mathscr{E}'^q_K$$

has a closed image.

Because $\mathscr{W}_K$ is of Fréchet–Schwartz, by duality (cf. [9]) we obtain:

PROPOSITION 5. *If K is a locally closed convex set, then $\mathscr{W}_K$ is an injective $\mathscr{P}_n$-module.*

Remark 3. One can show that, if K has a non-empty interior, then the exactness of the sequence

$$\mathscr{W}'^{p_0}_K \xrightarrow{{}^tA_0(D)} \mathscr{W}'^{p_1}_K \xrightarrow{{}^tA_1(D)} \mathscr{W}'^{p_2}_K$$

implies the exactness of (9).

3. Vanishing theorems on domains in $\mathbf{R}^n$

In this section we want to deduce, from the results of the previous one and some homological algebra, several vanishing theorems for the cohomology of differential complexes with constant coefficients on domains of $\mathbf{R}^n$.

In the following we will denote by Ω an open convex set in $\mathbf{R}^n$, by K

a convex closed subset of Ω and by G the closure in Ω of the complement of K.

From the exact sequence (1) in Section 1, we obtain for any given $\mathscr{P}_n$-module M corresponding long exact sequences for the derived functors, that yield the following statements:

PROPOSITION 6. *For any $\mathscr{P}_n$-module M we have:*

(a) $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{F}_K(\Omega)) = 0$ *for* $j \geq 2$.

(b) $\mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{F}_K(\Omega)) \cong \mathrm{Hom}_{\mathscr{P}_n}(M, \mathscr{W}_K)/\mathrm{Hom}_{\mathscr{P}_n}(M, \mathscr{E}(\Omega))$.

(c) *If* $j \geq 2$, *then* $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{F}_K(\Omega)) = 0$ *if* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

PROPOSITION 7. *For any $\mathscr{P}_n$-module M we have, under the additional assumption that K be compact and with a non-empty interior:*

(a) *If* $j \geq 1$, *then* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_G) = 0$ *if and only if* $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

(b) *If* $j \geq 2$, *then* $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{W}_G) = 0$ *if and only if* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

Indeed, when K is compact, then $\mathscr{F}_G(\Omega) = \mathscr{D}'_K$. The *only if* part in the statements follows from Remarks 2 and 3.

Let us consider now the consequences of the exact sequence (2). We obtain:

PROPOSITION 8. *The $\mathscr{P}_n$-module $\mathscr{W}_G^{\mathrm{comp}}$ is flat.*

Proof. It is sufficient to show that $\mathrm{Tor}_1^{\mathscr{P}_n}(M, \mathscr{W}_G^{\mathrm{comp}}) = 0$ for any $\mathscr{P}_n$-module M of finite type. We have an exact sequence:

$$0 \to \mathrm{Tor}_1^{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{comp}}) \to M \otimes_{\mathscr{P}_n} \mathscr{D}_K \to M \otimes_{\mathscr{P}_n} \mathscr{D}(\Omega). \tag{12}$$

If

$$\mathscr{P}_n^p \xrightarrow{A(z)} \mathscr{P}_n^q \to M \to 0$$

is a finite presentation of M, then (12) can be rewritten as

$$0 \to \mathrm{Tor}_1^{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{com}}) \to \mathrm{coker}\big(A(D)\colon \mathscr{D}_K^p \to \mathscr{D}_K^q)\big) \to \mathrm{coker}\big(A(D)\colon \mathscr{D}^p(\Omega) \to \mathscr{D}^q(\Omega)\big).$$

But the last map is injective by the argument in Part 2 of the proof of Proposition 2, and hence the statement follows.

COROLLARY. *We have* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_G) = 0$ *for* $j \geq 2$.

Then we obtain from the exact sequence (2):

PROPOSITION 9. *If K is compact and with a non-empty interior and M is any $\mathscr{P}_n$-module, then:*

(a) If $j \geq 1$, *then* $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{D}_G) = 0$ *if and only if* $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

(b) *If* $j \geq 2$, *then* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}_G) = 0$ *if and only if* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

(c) *If* $\mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$, *then*

$$\mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{D}_G) \cong \mathrm{Hom}_{\mathscr{P}_n}(M, \mathscr{W}_K)/\mathrm{Hom}_{\mathscr{P}_n}(M, \mathscr{D}(\Omega)).$$

We also have the following:

PROPOSITION 10. *Assume that K is closed and convex in $\boldsymbol{R}^n$ and has the following property*:

There exists a linear functional ξ: $\boldsymbol{R}^n \to \boldsymbol{R}$ such that for a constant $c \in \boldsymbol{R}$

(13) $$K \cap \{\langle \xi, x \rangle \leqslant c\} \quad \textit{is compact and non-empty.}$$

Then:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{comp}}) = 0 \quad \textit{for } j \geqslant 2,$$

$$\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{comp}}) = 0 \quad \textit{if} \quad \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

If $j \geqslant 3$, then

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}_G) = 0 \quad \textit{if} \quad \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

If $j \geqslant 1$

$$\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{D}_G) = 0 \quad \textit{if} \quad \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

Proof. Property (13) implies that

$$K_t = K \cap \{\langle \xi, x \rangle \leqslant t\}$$

is, for every $t \in \boldsymbol{R}$, compact. Let us set

$$F_t = K \cap \{\langle \xi, x \rangle \geqslant t\}, \qquad H_t = \{\langle \xi, x \rangle \leqslant t\}.$$

Then we consider the exact sequence:

$$0 \to E_t \to \mathscr{W}_K \to \mathscr{W}_{F_t} \to 0$$

where $E_t = \{f \in \mathscr{W}_K \mid f = 0 \text{ on } F_t\}$.

Then $\mathscr{W}_K^{\mathrm{comp}} = \lim\limits_{t \to \infty} E_t$, and hence the statement follows, because K and F_t are closed convex sets, from the long exact sequences:

$$\ldots \to \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{F_t}) \to \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, E_t) \to \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{W}_K) \to \ldots$$

and

$$\ldots \to \mathrm{Tor}_{j+1}^{\mathscr{P}_n}(M, \mathscr{W}_{F_t}) \to \mathrm{Tor}_j^{\mathscr{P}_n}(M, E_t) \to \mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{W}_K) \to \ldots$$

and from the long exact sequences for the derived functors that are deduced from (2).

COROLLARY. *Let us assume that condition* (13) *holds and that K is not compact. If M is a finitely generated torsion module, then*

$$\mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{D}_G) = 0 \quad \textit{if} \quad \mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

Proof. We have an exact sequence:

$$\mathrm{Ext}^0_{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{comp}}) \to \mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{D}_G) \to \mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{D}(\boldsymbol{R}^n)).$$

The last group is zero because we assumed that $\mathrm{Ext}^1_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$. Because M is torsion, the set of $\xi \in C^n$ such that $M \otimes_{\mathscr{P}_n} \mathscr{P}_n/m_\xi \neq 0$ is a proper algebraic subset of C^n. Then the set of real ξ not belonging to its asymptotic cone is open and dense in $\boldsymbol{R}^n$ and hence we can as well assume that property (13) holds for such a vector ξ. Then Holmgren's uniqueness theorem implies that $\mathrm{Ext}^0_{\mathscr{P}_n}(M, \mathscr{W}_K^{\mathrm{comp}}) = 0$.

COROLLARY. *Under the same assumptions of Proposition* 10:

(a) $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{D}'_K) = 0$ *for* $j \geqslant 2$.

(b) $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \check{\mathscr{D}}'_G) = 0$ *for* $j \geqslant 3$ *and* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$.

(c) *If* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$, *then* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_K) = 0$.

Let K be a closed convex set in R^n and let us denote by G the closure of $R^n - K$. Then the following statement is true:

PROPOSITION 11. *Assume that K has property* (13). *Then $\mathscr{F}_G(\boldsymbol{R}^n)$ is a flat $\mathscr{P}_n$-module.*

Proof. It is sufficient to show that $\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{F}_G(R^n)) = 0$ for $j \geqslant 1$ and M any $\mathscr{P}_n$-module of finite type.

We will use the notations of the proof of Proposition 10. Let

$$0 \leftarrow M \leftarrow \mathscr{P}_n^{p_0} \xleftarrow{A_0(z)} \mathscr{P}_n^{p_1} \xleftarrow{A_1(z)} \mathscr{P}_n^{p_2} \leftarrow \ldots$$

be a Hilbert resolution of M and let $f \in \mathscr{F}_G^{p_j}(\boldsymbol{R}^n)$ satisfy

$$A_{j-1}(D) f = 0 \quad \text{on } K_t \text{ for some } t \in \boldsymbol{R}.$$

Let $s \in R$, $s < t$ and let us choose a real valued, C^∞ function of a real variable r, $\chi(r)$, with the property that

$$\chi(r) = 1 \quad \text{for } r \leqslant s \quad \text{and} \quad \chi(r) = 0 \quad \text{for } r \geqslant t.$$

Then $f \cdot \chi(\langle \xi, x \rangle) \in \mathscr{D}'^{p_j}_{K_t}$ and

$$A_{j-1}(D)(f \cdot \chi(\langle \xi, x \rangle)) = g \in \mathscr{D}'^{p_{j-1}}_{K_t \cap F_s}.$$

Because $A_{j-1}(D)$: $\mathscr{D}'^{p_j}_{K_t \cap F_s} \to \mathscr{D}'^{p_{j-1}}_{K_t \cap F_s}$ has a closed image, by the second part of the proof of Proposition 2 we can find $w \in \mathscr{D}'^{p_j}_{K_t \cap F_s}$ such that

$$g = A_{j-1}(D) w.$$

Therefore

$$f' = f \cdot \chi(\langle \xi, x \rangle) - w \in \mathscr{D}'^{p_j}_{K_t}$$

satisfies

$$A_{j-1}(D) f' = 0$$

and

$$f' | K_s = f.$$

Then by Proposition 2 we can find $u \in \mathscr{D}_{K_t}^{pj+1}$ such that

$$A_j(D)u = f'.$$

Let us assume now that $f \in \mathscr{F}_G^{pj}(\boldsymbol{R}^n)$ satisfy $A_{j-1}(D)f = 0$ on $\boldsymbol{R}^n$. Then, by the construction above with $t = m+1$, $s = m$ for m varying in the non-negative integers, we obtain a sequence $\{u_m\}$ with

$$u_m \in \mathscr{D}_{K_{m+1}}^{pj+1} \quad \text{satisfying} \quad A_{j+1}(D)u_m = f \quad \text{on } K_m.$$

We claim that we can construct a sequence $\{w_m\}$ with the properties:

$$\text{(i)} \qquad w_m \in \mathscr{D}_{K_{m+1}}^{pj+1},$$

$$\text{(ii)} \qquad w_{m+1} = w_m \quad \text{on } K_{m-1/2},$$

$$\text{(iii)} \qquad A_{j+1}(D)w_m = f \quad \text{on } K_m.$$

Indeed, set $w_0 = u_0$ and assume that $w_0, \ldots, w_m$ have been defined. Then

$$A_{j+1}(D)(w_m - u_{m+1}) = 0 \quad \text{on } K_m$$

and, being $w_m - u_{m+1} \in \mathscr{F}_G^{pj+1}(\boldsymbol{R}^n)$, by the first part of the proof we can find $h_{m+1} \in \mathscr{D}_{K_m}^{pj+1}$ such that

$$w_{m+1} = u_{m+1} + A_{j+2}(D)h_{m+1} \quad \text{be equal to } w_m \text{ on } K_{m-1/2}.$$

Clearly w_{m+1} satisfies conditions (i) and (iii). Setting then $u = w_m$ on $K_{m-1/2}$ we obtain an element of $\mathscr{F}_G(\boldsymbol{R}^n)$ satisfying $A_{j+1}(D)u = f$.

From the exact sequence (1) we deduce the following

Corollary. *Assume that K has property* (13). *Then, if M is any $\mathscr{P}_n$-module, we have for $j \geq 1$*

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_G) = 0 \quad \textit{if} \quad \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

4. Rough Hilbert complexes at the boundary of convex sets (global results)

We assume in the first part of this section that K is a compact convex set in $\boldsymbol{R}^n$. Then we have an exact sequence:

$$(14) \qquad 0 \to \mathscr{E}'_{\partial K} \to \mathscr{E}'_K \to \check{\mathscr{D}}'_K \to 0.$$

Let M be any $\mathscr{P}_n$-module. Because $\mathscr{E}'_K$ is flat and $\check{\mathscr{D}}'_K$ is injective, then the long exact sequences for the derived functors deduced from (14) yield the following isomorphisms:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{E}'_{\partial K}) \cong \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{E}'_K) \quad \text{if } j \geq 2,$$

$$\mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{E}'_{\partial K}) \cong \mathrm{Tor}_j^{\mathscr{P}_n}(M, \check{\mathscr{D}}'_K) \quad \text{if } j \geq 2.$$

Thus from the results obtained in the previous section we obtain:

Proposition 12. *If K is a compact convex set with a non-empty interior, then for any $\mathscr{P}_n$-module M we have for $j \geq 2$:*

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{E}'_{\partial K}) \cong \mathrm{Tor}_j^{\mathscr{P}_n}(M, \mathscr{E}'_{\partial K}) = 0 \quad \textit{if and only if} \quad \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

We have also the exact sequence:

$$0 \to \mathscr{F}_{\partial K}(\boldsymbol{R}^n) \to \mathscr{E}(\boldsymbol{R}^n) \to \mathscr{W}_{\partial K} \to 0. \tag{15}$$

Because $\mathscr{E}(\boldsymbol{R}^n)$ is an injective $\mathscr{P}_n$-module and $\mathscr{F}_{\partial K}(\boldsymbol{R}^n) = \mathscr{D}_K \oplus \mathscr{F}_K(\boldsymbol{R}^n)$, by point (a) in Proposition 6 we obtain for $j \geq 1$ isomorphisms:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K}) \cong \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{D}_K)$$

for any $\mathscr{P}_n$-module M. Therefore:

PROPOSITION 13. *If K is a compact convex set with a non-empty interior in $\boldsymbol{R}^n$ and M is any $\mathscr{P}_n$-module, then we have*

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K}) = 0 \quad \textit{if and only if} \quad \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

Let us drop now the assumption that K be compact. We assume only that K be a closed convex set in $\boldsymbol{R}^n$. Then we have an exact sequence:

$$0 \to \mathscr{D}'_{\partial K} \to \mathscr{D}'_K \to \check{\mathscr{D}}'_K \to 0. \tag{16}$$

Because we showed that $\check{\mathscr{D}}'_K$ is an injective $\mathscr{P}_n$-module, we obtain for $j \geq 2$ and any $\mathscr{P}_n$-module M isomorphisms:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{\partial K}) \cong \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_K).$$

Thus, by the second corollary to Proposition 10 we obtain:

PROPOSITION 14. *Assume that K is a closed convex set in $\boldsymbol{R}^n$ satisfying condition* (13). *Then, if M is a $\mathscr{P}_n$-module satisfying* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$ *and $j \geq 2$, we have* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{\partial K}) = 0$.

From the exact sequence:

$$0 \to \mathscr{F}_{\partial K}(\boldsymbol{R}^n) \to \mathscr{E}(\boldsymbol{R}^n) \to \mathscr{W}_{\partial K} \to 0 \tag{17}$$

because $\mathscr{E}(\boldsymbol{R}^n)$ is an injective $\mathscr{P}_n$-module and by Proposition 6 we have $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_K(\boldsymbol{R}^n)) = 0$ for every $\mathscr{P}_n$-module M, we deduce that for every $\mathscr{P}_n$-module M we have isomorphisms:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K}) \cong \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_G(\boldsymbol{R}^n)).$$

Therefore, by Proposition 11, we obtain

PROPOSITION 15. *If K is a closed convex set in $\boldsymbol{R}^n$ having property* (13), *if $j \geq 1$ and M is a $\mathscr{P}_n$-module such that* $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$, *then* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K}) = 0$.

5. Rough Hilbert complexes at the boundary of convex sets (local results)

We recall that a closed set K in $\boldsymbol{R}^n$ is said to be *convex* (resp. *strictly convex*) at a point $x \in \partial K$ if we can find a neighborhood U of x in $\boldsymbol{R}^n$ such that $\bar{U} \cap K$ is convex (resp. strictly convex).

The closed set $G = \overline{\boldsymbol{R}^n - K}$ is said to be *concave* (resp. *strictly concave*) at $x \in \partial G$ if the closed set K is convex (resp. strictly convex) at x.

Given any closed subset C in $\mathbf{R}^n$, we denote by $\mathscr{F}_{C_x}$ the space of germs at x of complex valued C^∞ functions in $\mathbf{R}^n$ vanishing with all derivatives on C and by $\mathscr{W}_{C_x}$ the space of germs of Whitney functions on C at x.

Denoting by $\mathscr{E}_x$ the space of germs of C^∞ functions in $\mathbf{R}^n$ at x, we have an exact sequence

$$(18)\qquad 0 \to \mathscr{F}_{C_x} \to \mathscr{E}_x \to \mathscr{W}_{C_x} \to 0,$$

that can be also thought as an exact sequence of $\mathscr{P}_n$-modules (for the action of polynomials as partial differential operators with constant coefficients).

Repeating the first part of the proof of Proposition 11 we obtain:

PROPOSITION 16. *If G is a closed set in $\mathbf{R}^n$, strictly concave at $x_0 \in \partial G$, then $\mathscr{F}_{G_{x_0}}$ is a flat $\mathscr{P}_n$-module.*

From the long exact sequence for the Ext functor deduced from (18), we obtain

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{G_{x_0}}) \cong \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_{G_{x_0}}).$$

Therefore we have

PROPOSITION 17. *If G is a closed subset of $\mathbf{R}^n$, strictly concave at $x_0 \in \partial G$, then, if $j \geq 1$ and the $\mathscr{P}_n$-module M satisfies $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$, we have $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{G_{x_0}}) = 0$.*

Let K be a closed subset of $\mathbf{R}^n$ and let G be the closure in $\mathbf{R}^n$ of $\mathbf{R}^n - K$. Then the two closed sets K and G are regularly situated (cf. [23]) and therefore for every $x_0 \in K \cap G = \partial K$ we obtain an exact sequence:

$$(19)\qquad 0 \to \mathscr{E}_{x_0} \to \mathscr{W}_{K_{x_0}} \otimes \mathscr{W}_{G_{x_0}} \to \mathscr{W}_{\partial K_{x_0}} \to 0.$$

Considering this sequence as an exact sequence of $\mathscr{P}_n$-modules, because $\mathscr{E}_{x_0}$ and $\mathscr{W}_{K_{x_0}}$ are injective, we obtain for every $j \geq 1$ and every $\mathscr{P}_n$-module M an isomorphism:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K_{x_0}}) \cong \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{G_{x_0}}).$$

Thus we have:

PROPOSITION 18. *Assume that K is strictly convex at $x_0 \in \partial K$. Then, if $j \geq 1$ and M is a $\mathscr{P}_n$-module satisfying $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0$, we have $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K_{x_0}}) = 0$.*

Remark. Nothing can be said in general about the local cohomology groups $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K_{x_0}})$, when $\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \neq 0$, as the two examples of the De Rham (where all groups are zero for $j \geq 1$) and of the Dolbeault complex in $\mathbf{C}^n$ (where the group corresponding to $j = n-1$ is infinite dimensional by the Hans Lewy example) show.

If M is a $\mathscr{P}_n$-module of finite type, then the groups $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n)$ have a natural structure of $\mathscr{P}_n$-modules of finite type.

The following proposition, together with the results of the next sections, gives a generalization of the Hans Lewy example (cf. [2]):

PROPOSITION 19. *We assume that K is strictly convex at $x_0 \in \partial K$ and that y_0 is a vector in $\mathbf{R}^n$ such that we can find an open neighbourhood Ω of x_0 in $\mathbf{R}^n$ such that*

$$\langle y_0, x-x_0\rangle < 0 \, \forall x \in K \cap \Omega - \{x_0\}. \tag{20}$$

Then, if M is a $\mathscr{P}_n$-module of finite type such that the asymptotic cone V^0 of the algebraic variety

$$V = \{z \in \mathbf{C}^n \mid \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \otimes_{\mathscr{P}_n} \mathscr{P}_n/\mathbf{m}_z \neq 0\}$$

(where $\mathbf{m}_z$ is the maximal ideal of the point z) contains a point z_0 with $\operatorname{Im} z_0 = y_0$, *and $j \geqslant 1$, then*

$$\operatorname{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K_{x_0}}) \neq 0.$$

Proof. We have

$$\operatorname{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{\partial K_{x_0}}) \cong \operatorname{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{G_{x_0}}) \cong \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_{G_{x_0}})$$

and, because $\mathscr{F}_{G_{x_0}}$ is a flat $\mathscr{P}_n$-module,

$$\operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_{G_{x_0}}) \cong \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \otimes_{\mathscr{P}_n} \mathscr{F}_{G_{x_0}}.$$

Let

$$\mathscr{P}_n^p \xrightarrow{A(z)} \mathscr{P}_n^q \to \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \to 0 \tag{21}$$

be a presentation of the $\mathscr{P}_n$-module of finite type $\operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n)$.

Then we obtain:

$$\operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{F}_{G_{x_0}}) \cong \mathscr{F}^q_{G_{x_0}}/A(D)\mathscr{F}^p_{G_{x_0}}$$

and therefore we are reduced to prove that the map

$$A(D)\colon \mathscr{F}^p_{G_{x_0}} \to \mathscr{F}^q_{G_{x_0}}$$

is not onto.

We prove this fact by contradiction.

If $A(D)\mathscr{F}^p_{G_{x_0}} = \mathscr{F}^q_{G_{x_0}}$, then by a standard argument of functional analysis (cf. [12] or [3], § 5) we obtain:

If $A(D)\mathscr{F}^p_{G_{x_0}} = \mathscr{F}^q_{G_{x_0}}$, then, given any open neighborhood U of x_0 in $\mathbf{R}^n$ we can find an open neighborhood ω of x_0 in U, a constant $C > 0$, an integer $m \geqslant 0$, such that:

$$\Big|\int_{K\cap\omega} \varphi f\,dx\Big| \leqslant C \sup_{K\cap\omega} |{}^t A(D)\varphi| \sup_{U} \sup_{|\alpha|\leqslant m} |D^\alpha f| \, \forall \varphi \in \mathscr{D}^q(\omega), \quad f \in \mathscr{F}_G(U). \tag{22}$$

We assume to simplify the notations that $x_0 = 0$. Let us choose an open neighborhood U_0 of $x_0 = 0$ such that $\bar{U}_0 \cap K$ is strictly convex. Let x_1 be an interior point of $U_0 \cap K$ and let $0 < r < R$ be real numbers with

$$\bar{B}(x_1, R) = \{|x - x_1| \leqslant R\} \subset U_0 \cap K.$$

Then we can choose a real valued, C^∞ function g on $\boldsymbol{R}^n$ with the properties:

$$0 \leqslant g \leqslant 1 \quad \text{on } \boldsymbol{R}^n,$$
$$g(x) = 1 \quad \text{for} \quad x \in \bigcup_{t \geqslant 1} t\bar{B}(x_1, r),$$
$$g(x) = 0 \quad \text{for} \quad x \notin \bigcup_{t \geqslant 1} t\bar{B}(x_1, R),$$
$$g(tx) = g(x) \quad \text{if} \quad |x| \geqslant 2|x_1| \quad \text{and } t \geqslant 1.$$

Let $U = U_0 \cap \{x \in \boldsymbol{R}^n | \ \langle x, x_1 \rangle < |x_1|^2/2\}$. Then for every $\nu > 0$ the function $g(\nu^2 x)$ belongs to $\mathscr{F}_G(U)$: indeed its support is contained in the cone

$$\bigcup_{t > 0} t\bar{B}(x_1, R)$$

that intersects the boundary of U in a set whose points are interior to $U_0 \cap K$.

Let ω be an open neighborhood of 0 in U such that (22) holds for some C and m relative to U and ω. Let $\varrho > 0$ be such that

$$\{|x| \leqslant \varrho\} \subset \omega,$$

and let us assume as we can that U has compact closure in Ω.

Then we can find a constant $\varepsilon > 0$ such that

$$\langle y_0, x \rangle \leqslant -\varepsilon \quad \text{if } x \in U \cap K \text{ and } |x| \geqslant \varrho.$$

Let $\{z_\nu\}$ be a sequence of points in V such that

$$|z_\nu/\nu - z_0| < L/\nu$$

for a constant $L > 0$ (cf. [4], p. 63).

By the definition of V, for every ν we can choose $X_\nu \in \boldsymbol{C}^q$ with

$$|X_\nu| = 1 \quad \text{and} \quad {}^tA(D)(X_\nu e^{-i\langle z_\nu, x\rangle}) = 0.$$

Then, with a function $\chi \in \mathscr{D}(\omega)$ with

$$0 \leqslant \chi \leqslant 1 \quad \text{on } \boldsymbol{R}^n, \quad \chi(x) = 1 \quad \text{if } |x| \leqslant \varrho,$$

we set:

$$\varphi_\nu(x) = \nu^n \chi(x) e^{-i\langle z_\nu, x\rangle} X_\nu, \quad f_\nu(x) = g(\nu^2 x) e^{-i\langle \operatorname{Re} z_\nu, x\rangle} \bar{X}_\nu.$$

Then $\varphi_\nu \in \mathscr{D}^q(\omega)$, $f_\nu \in \mathscr{F}_G^q(U)$, so that we must have

$$(*) \qquad \left|\int \varphi_\nu f_\nu \, dx\right| \leqslant C \sup_{K \cap \omega} |{}^tA(D)\varphi_\nu| \cdot \sup_U \sup_{|\alpha| \leqslant m} |D^\alpha f_\nu| \quad \text{for every } \nu.$$

But we have

$$\int \varphi_\nu f_\nu \, dx = \int \chi(x/\nu)\, g(\nu x)\, e^{\langle \frac{\operatorname{Im} z_\nu}{\nu}, x\rangle} dx$$

and therefore the integral is real for every ν and

$$\liminf_{\nu \to +\infty} \int \varphi_\nu f_\nu \, dx \geqslant \int_{\bigcup_{t>0} t\bar{B}(x_1, r)} e^{\langle y_0, x\rangle} dx > 0.$$

On the other hand we have

$$\sup_U \sup_{|\alpha| \leqslant m} |D^\alpha f_\nu| \leqslant \text{const} \cdot \nu^{2m}$$

and

$$\sup_{K\cap\omega} |{}^t A(D)\varphi_\nu| \leqslant \text{const} \cdot \nu^n \sup_{\substack{K\cap\omega \\ |x|\geqslant \varrho}} e^{\langle \operatorname{Im} z_\nu, x\rangle} \leqslant \text{const} \cdot \nu^n e^{-\varepsilon\nu},$$

that implies that the right-hand side of (*) tends to zero, while the left-hand side is bounded from below by a positive constant as $\nu \to +\infty$. This gives a contradiction and therefore our statement is proved.

Let K be a closed set in $\boldsymbol{R}^n$, which is the closure of its interior. For $x_0 \in \partial K$ we define:

$\mathscr{D}'_{x_0}$ = space of germs of distributions at x_0;

$\mathscr{D}'_{K_{x_0}}$ = space of germs at x_0 of distributions in $\boldsymbol{R}^n$ having support contained in K;

$\check{\mathscr{D}}'_{K_{x_0}}$ = space of germs at x_0 of extensible distributions defined in the interior of K;

and we consider them as $\mathscr{P}_n$-modules by the action of polynomials as partial differential operators with constant coefficients.

By Proposition 3 we have:

PROPOSITION 20. *If K is convex at x_0, then $\check{\mathscr{D}}'_{K_{x_0}}$ is an injective $\mathscr{P}_n$-module.*

By an argument similar to that in the proof of Proposition 10 we obtain:

PROPOSITION 21. *If K is strictly convex at x_0, then $\check{\mathscr{D}}'_{K_{x_0}}$ is a flat $\mathscr{P}_n$-module.*

Let G denote the closure in $\boldsymbol{R}^n$ of the complement of K. Then for any fixed $x_0 \in \partial K$ we have exact sequences:

(a) $$0 \to \mathscr{D}'_{\partial K_{x_0}} \to \mathscr{D}'_{K_{x_0}} \to \check{\mathscr{D}}'_{K_{x_0}} \to 0,$$

(b) $$0 \to \mathscr{D}'_{G_{x_0}} \to \mathscr{D}'_{x_0} \to \check{\mathscr{D}}'_{K_{x_0}} \to 0,$$

(c) $$0 \to D'_{K_{x_0}} \to \mathscr{D}'_{x_0} \to \check{\mathscr{D}}'_{G_{x_0}} \to 0.$$

If K is convex at x_0, then $\check{\mathscr{D}}'_{K_{x_0}}$ and $\mathscr{D}'_{x_0}$ are injective and we have for every $\mathscr{P}_n$-module M isomorphisms:

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{\partial K_{x_0}}) \cong \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{K_{x_0}}) \quad \text{for } j \geq 2,$$
$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \check{\mathscr{D}}'_{G_{x_0}}) = \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{D}'_{K_{x_0}}) \quad \text{for } j \geq 1.$$

Thus we deduce the following statement:

PROPOSITION 22.

(i) *If K is convex at x_0, then* $\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{G_{x_0}}) = 0$ *for* $j \geq 2$.

(ii) *If K is strictly convex at x_0, then*

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{D}'_{\partial K_{x_0}}) = 0 \quad \textit{if} \quad \mathrm{Ext}^j_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0 \quad \textit{and } j \geq 2.$$

(iii) *If K is strictly convex at x_0, then*

$$\mathrm{Ext}^j_{\mathscr{P}_n}(M, \check{\mathscr{D}}'_{G_{x_0}}) = 0 \quad \textit{if} \quad \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0 \quad \textit{and } j \geq 1.$$

(iv) *With the same assumptions of Proposition* 19, *we have*:

$$\mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{D}'_{\partial K_{x_0}}) \neq 0.$$

6. Tangential complexes

Let Ω be an open set in $\mathbf{R}^n$ and let ϱ be a real valued C^∞ function on Ω such that

$$d\varrho(x) \neq 0 \quad \text{on} \quad S = \{x \in \Omega |\ \varrho(x) = 0\}.$$

Then S is an oriented smooth hypersurface in Ω and we set

$$\Omega^+ = \{x \in \Omega |\ \varrho(x) \geq 0\}, \quad \Omega^- = \{x \in \Omega |\ \varrho(x) \leq 0\}.$$

Given a complex of linear partial differential operators

$$\text{(23)} \qquad \mathscr{E}^{p_0}(\Omega) \xrightarrow{A_0(x,D)} \mathscr{E}^{p_1}(\Omega) \xrightarrow{A_1(x,D)} \mathscr{E}^{p_2}(\Omega) \to \dots$$

we define $\mathscr{I}_{A_j}(S, \Omega)$ as the subspace of functions $u \in \mathscr{E}^{p_j}(\Omega)$ such that

$$\int_{\Omega^+} A_j(x, D) u \cdot v \, dx = \int_{\Omega^+} u \cdot {}^t A_j(x, D) v \, dx \ \forall v \in \mathscr{D}^{p_{j+1}}(\Omega).$$

Then one verifies that

$$A_j(x, D) \mathscr{I}_{A_j}(S, \Omega) \subset \mathscr{I}_{A_{j+1}}(S, \Omega)$$

so that we can consider the subcomplex of (23):

$$\text{(24)} \qquad \mathscr{I}_{A_0}(S, \Omega) \xrightarrow{A_0(x,D)} \mathscr{I}_{A_1}(S, \Omega) \xrightarrow{A_1(x,D)} \mathscr{I}_{A_2}(S, \Omega) \to \dots$$

and the quotient complex

$$\text{(25)} \qquad \mathscr{Q}^{(0)}(S) \xrightarrow{A_0|S} \mathscr{Q}^{(1)}(S) \xrightarrow{A_2|S} \mathscr{Q}^{(2)}(S) \to \dots,$$

where $\mathscr{Q}^{(j)}(S) = \mathscr{E}^{p_j}(\Omega) / \mathscr{I}_{A_j}(S, \Omega)$.

We say that (25) is the tangential complex on S canonically associated to (23).

Let us consider now a $\mathscr{P}_n$-module M of finite type.

We denote by V^0 the asymptotic cone at ∞ of the algebraic variety

$$V = \{z \in \mathbf{C}^n | \ M \otimes_{\mathscr{P}_n} \mathscr{P}_n / \mathbf{m}_z \neq 0\}.$$

We say that S is *non-characteristic at* $x_0 \in S$ for M if

$$d\varrho(x_0) \notin V^0.$$

In [6] it was proved that, if S is non-characteristic for M at x_0, then we can choose a Hilbert resolution

$$0 \to \mathscr{P}_n^{p_d} \xrightarrow{{}^tA_{d-1}(z)} \mathscr{P}_n^{p_{d-1}} \to \ldots \to \mathscr{P}_n^{p_1} \xrightarrow{{}^tA_0(z)} \mathscr{P}_n^{p_0} \to M \to 0 \tag{26}$$

of M and an open neighbourhood ω of x_0 in Ω in such a way that the canonical tangential complex (25) associated to the complex of differential operators (with constant coefficients)

$$\mathscr{E}^{p_0}(\omega) \xrightarrow{A_0(D)} \mathscr{E}^{p_1}(\omega) \xrightarrow{A_1(D)} \mathscr{E}^{p_2}(\omega) \to \ldots \to \mathscr{E}^{p_d}(\omega) \to 0 \tag{27}$$

is a complex of linear differential operators on trivial vector bundles over $S \cap \omega$.

Let us denote by $H^j(Q^*, A|\ S \cap \omega^*)$ the cohomology groups of this complex and let

$$H^j(Q^*, A|\ S^*, x_0) = \lim_{\omega \text{ open} \ni x_0} H^j(Q^*, A|\ S \cap \omega^*).$$

As in [6] it was proved (Formal Cauchy Kowalewska Theorem) that

$$H^j(Q^*, A|\ S^*, x_0) = \operatorname{Ext}^j_{\mathscr{P}_n}(M, \mathscr{W}_{S_{x_0}})$$

we obtain from Propositions 18 and 19:

PROPOSITION 23. *Assume that* Ω^+ *is strictly convex at* $x_0 \in S$ *and assume that* S *is non-characteristic at* x_0. *Then for* $j \geqslant 1$

$$H^j(Q^*, A|\ S^*, x_0) = 0 \quad \textit{if} \quad \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

If $\operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \neq 0$ *and the asymptotic cone of the algebraic variety*

$$W = \{z \in \mathbf{C}^n | \ \operatorname{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) \otimes_{\mathscr{P}_n} \mathscr{P}_n / \mathbf{m}_z \neq 0\}$$

contains a point z_0 *with* $\operatorname{Im} z_0 = d\varrho(x_0)$, *then*

$$H^j(Q^*, A|\ S^*, x_0) \neq 0.$$

Remark. The condition of the last part of the statement is always fulfilled in the case of the tangential Cauchy Riemann complex induced by the Dolbeault complex in $\mathbf{C}^n$ on the boundary of a strictly convex domain. In this way we recover the original example of Hans Lewy of equations not locally solvable (cf. [2]). However, we obtain here a non-trivial extension of those results, as we do not need to make any assumption about the order of contact of the analytic tangent space to S at x_0.

Assume now that we can choose integers $m_0, m_1, \ldots, m_{d-1}$ in such a way that all entries of the matrices $A_j(z)$ in (26) have degree $\leqslant m_j$ and, denoting by $A_j^0(z)$ the matrix obtained taking only homogeneous parts of degree m_j of the entries of $A_j(z)$, the sequence

$$(26)^0 \qquad 0 \to \mathscr{P}_n^{p_d} \xrightarrow{{}^tA^0_{d-1}(z)} \mathscr{P}_n^{p_{d-1}} \to \ldots \to \mathscr{P}_n^{p_1} \xrightarrow{{}^tA^0_0(z)} \mathscr{P}_n^{p_0}$$

stays exact.

This situation corresponds to considering complexes with "classical gradings" for the terminology of [6].

Then, if M is elliptic, i.e. $V^0 \cap \boldsymbol{R}^n \subset \{0\}$, the canonical tangential complex is a well defined complex of linear partial differential operators on vector bundles over S for any given hypersurface S.

In this case Proposition 11 yields the following result:

Proposition 24. *Let M be elliptic and assume that* (22) *and* $(22)^0$ *are exact. Then, if K is any convex compact set with a smooth boundary, we have for $j \geqslant 1$:*

$$H^j(Q^*, A|\ \partial K^*) = 0 \quad \textit{if and only if} \quad \mathrm{Ext}^{j+1}_{\mathscr{P}_n}(M, \mathscr{P}_n) = 0.$$

Example. Let K be a convex domain in $R^k \times C^m$ and let us consider the ideal M in $\mathscr{P}_n = \mathscr{P}_{k+m} = C[u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_m]$ generated by $v_1 + iw_1, \ldots, v_m + iw_m$. If ∂K is a smooth hypersurface at a point x_0, it is formally non-characteristic for the Hilbert resolution of the ideal M if the tangent hyperplane to K at x_0 intersects C^m in a hypersurface. If this is the case and if, moreover, K is strictly convex at x_0, then the boundary complex, that in this case is equivalent to a tangential Cauchy–Riemann complex induced on a submanifold of real codimension $k+1$ of a complex manifold of dimension n, admits by Proposition 23 the Poincaré Lemma at the places $1, 2, \ldots, m-1$.

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