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SOME RESULTS ON DYNAMICAL SYSTEMS AND THEIR GENERALIZATION

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The purpose of the present paper is to give a short presentation of some results on dynamical systems and their generalizations obtained recently in the Institute of Mathematics of the Jagellonian University in Kraków.

In order to exclude any misfits we shall establish precisely the terminology.

Let X be a nonempty set (called space in the sequel) and let $(G, +)$ be an abelian semi-group with the neutral element 0. Let π be a mapping $G \times X \to X$. We say that $(X, G; \pi)$ is a *semi-system* iff

$$\pi(0, x) = x \quad \text{for every } x \in X; \tag{0.1}$$

$$\pi(s, \pi(t, x)) = \pi(s+t, x) \quad \text{for } s, t \in G,\ x \in X. \tag{0.2}$$

If G is a group then $(X, G; \pi)$ is said to be a *system*. If X is a topological space, G is a topological semi-group (group) and π is continuous then $(X, G; \pi)$ is said to be a *dynamical semi-system* (*dynamical system*). If λ is a mapping from $G \times X$ into the family $\mathscr{P}(X)$ of all nonempty subsets of X, then $(X, G; \lambda)$ is said to be a *generalized semi-system* iff

$$\lambda(0, x) = \{x\} \quad \text{for every } x \in X; \tag{0.3}$$

$$\lambda(s, \lambda(t, x)) \subset \lambda(t+s, x) \quad \text{for } s, t \in G,\ x \in X, \tag{0.4}$$

where the natural notation

$$\lambda(s, A) := \bigcup \{\lambda(s, y) \colon y \in A\}, \quad s \in G,\ A \in \mathscr{P}(X) \tag{0.5}$$

is used.

Remarks. I. In most of cases we have in (0.4) the equality instead of the inclusion. Since however for our purpose the inclusion is enough, we shall consider this – formally more general – condition.

In order to give an example let us consider a system of ordinary differential equations

$$x' = g(x) \tag{0.6}$$

assuming that $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and that for every $x^0 \in \boldsymbol{R}^n$ all solutions of (0.6) starting at the time $t = 0$ from x^0, are defined at least in $[0, \infty)$. Denoting by $\lambda(t, x^0)$ the set

$$\{\varphi(t)\colon\ \varphi \text{ is a solution of (0.6) such that } \varphi(0) = x^0\},$$

we get a generalized semi-system $(\boldsymbol{R}^n, \boldsymbol{R}_+; \lambda)$. In this system, we have the equality instead of the inclusion (0.4).

II. If $\lambda(t, x)$ has for every $(t, x) \in G \times X$ exactly one element, then, denoting it by $\pi(t, x)$ (so $\lambda(t, x) = \{\pi(t, x)\}$), we obtain a semi-system $(X, G; \pi)$.

For bibliography concerning the theory of (dynamical) systems and semi-systems, we refer to [3], [14], [16]. Multivalued, generalized in our terminology, semi-systems were considered, for instance, in [15].

Sections 1–4 of the present paper deal with, roughly speaking, some special aspects of stability theory and limit sets in dynamical systems. Section 5 presents some results on dynamical systems on the plane. Section 6 gives certain applications of algebraic topology in dynamical systems. Extensions of the Ważewski topological method with respect to partial differential equations are mentioned in the last section.

Notice here that the stability theory founded by Lyapunov at the end of the XIX-th century is still in the centre of investigations of many authors. There were proposed several versions, generalizations and modifications of the classical Lyapunov stability and there appeared many results on connections between stability properties and other ones. Recently natural tendencies to some uniform presentation and general classification of notions and results have been observed. One such uniform presentation was proposed in [14]. This direction is now continued in Section 2 below. Connections between stability and limit sets appeared as the background of investigation mentioned in Section 3 and partially (together with certain natural motivation taken from the theory of oscillations) in Section 1.

Topological method of Ważewski ([20], [21]) was used, generalized and modified by several authors (the list of papers concerning directly this subject contains more than 150 items); this method and the main idea of applications of the retract theory give the background of the contents of Sections 6 and 7.

1. Asymptotically periodic, pseudoperiodic and asymptotically pseudoperiodic motions in dynamical systems on metric spaces

Let $(X, \boldsymbol{R}; \pi)$ be a dynamical system, with (X, ϱ) being a metric space. For every $x \in X$, we denote by π^x the motion of x, that is the mapping

$$\boldsymbol{R} \ni t \mapsto \pi^x(t) := \pi(t, x) \in X. \tag{1.1}$$

It is clear that if a dynamical system is induced in $\boldsymbol{R}^n$ by a system of autonomous ordinary differential equations

$$x' = f(x) \tag{$*$}$$

having the property of the existence and uniqueness of solutions (defined in the whole $\boldsymbol{R}$-axis and depending continuously on initial data), then the motion of a point $x^0 \in \boldsymbol{R}^n$ in the general sense defined above, coincides with the mapping associating with a time t, the position reached by the point x^0 after this time according to the rule of moving given by the system of equation $(*)$.

By $\Lambda^+(x)(\Lambda^-(x))$ we denote the positive (negative) limit set of x, that is the set of those y for which there is a sequence $\{t_m\}$ of real numbers such that

$$t_m \to \infty \ (t_m \to -\infty) \quad \text{and} \quad \pi(t_m, x) \to y$$

as $m \to \infty$.

The sets $\Lambda^+(x)$ and $\Lambda^-(x)$ are denoted also in several books and papers by $\omega(x)$ and $\alpha(x)$ respectively. Some examples of limit sets are presented at the end of the present section.

Let $\alpha \in \boldsymbol{R}$ and $\eta \in \boldsymbol{R}_+ = [0, \infty)$ be fixed. We say that the motion π^x is *(η, α)-pseudoperiodic* iff

$$\varrho\big(\pi(t, x), \pi(t+\alpha, x)\big) \leqslant \eta \quad \text{for } t \in \boldsymbol{R}. \tag{1.2}$$

We say that π^x is *positively (negatively) asymptotically (η, α)-pseudoperiodic* iff: for every $\varepsilon > 0$ there exists $s \geqslant 0$ $(s \leqslant 0)$ such that

$$\varrho\big(\pi(t, x), \pi(t+\alpha, x)\big) \leqslant \eta+\varepsilon \quad \text{for } t \geqslant s \ (t \leqslant s). \tag{1.3}$$

The motion π^x is said to be *positively (negatively) η-pseudostable* iff: for every $\varepsilon > 0$ there exist $\delta > 0$ and $s \geqslant 0$ $(s \leqslant 0)$ such that

$$\varrho(x, y) < \delta \Rightarrow \varrho\big(\pi(t, x), \pi(t, y)\big) \leqslant \eta+\varepsilon \quad \text{for } t \geqslant s \ (t \leqslant s). \tag{1.4}$$

Remarks. III. It is clear that $(0, \alpha)$-pseudoperiodicity of π^x is equivalent to the usual periodicity (α is a period in that case).

IV. A motion π^x which is positively (negatively) asymptotically $(0, \alpha)$-pseudoperiodic is positively (negatively) asymptotically periodic (in the sense of [9]), that is: for every $\varepsilon > 0$ there is $s \geqslant 0$ ($s \leqslant 0$) such that

$$\varrho(\pi(t, x), \pi(t+\alpha, x)) \leqslant \varepsilon \quad \text{for } t \geqslant s \ (t \leqslant s).$$

V. Recall that π^x is said to be *positively* (*negatively*) *Lyapunov stable* iff: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\varrho(x, y) < \delta \Rightarrow \varrho(\pi(t, x), \pi(t, y)) < \varepsilon \quad \text{for } t \geqslant 0 \ (t \leqslant 0). \tag{1.5}$$

So, every positively (negatively) Lyapunov stable motion is – trivially – positively (negatively) 0-pseudostable.

THEOREM 1.1 (see [10]). *Assume that π^x is positively (negatively) asymptotically (η, α)-pseudoperiodic. If $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$ then π^y is (η, α)-pseudoperiodic.*

THEOREM 1.2 ([10]). *Assume that $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, π^y is positively (negatively) asymptotically (η_1, α)-pseudoperiodic and positively (negatively) η_2-pseudostable. Then π^x is positively (negatively) asymptotically $(\eta_1 + 2\eta_2, \alpha)$-pseudoperiodic.*

COROLLARIES (see [9]). I. *If π^x is positively (negatively) asymptotically periodic and $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$ then π^y is periodic (π^y may be trivially periodic in the case of y being a stationary point).*

II. *If $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$ and π^y is periodic and positively (negatively) Lyapunov stable, then π^x is positively (negatively) asymptotically periodic.*

III. *If $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$ and the set $\{y\}$ is positively (negatively) Lyapunov stable [which is equivalent to saying that y is a stationary point and the – trivial – motion π^y is positively (negatively) Lyapunov stable], then π^x is positively (negatively) asymptotically periodic.*

THEOREM 1.3 (see [9]). *Assume that the space X is locally compact. If $x \in X$ is such that*

$$\Lambda^+(x) = \pi(y) := \pi^y(\boldsymbol{R}) \qquad (\Lambda^-(x) = \pi(y))$$

where π^y is periodic [it is not excluded that y is a stationary point] then the motion π^x is positively (negatively) asymptotically periodic.

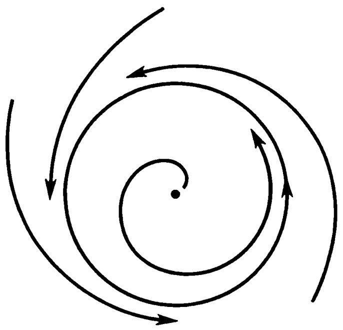
COROLLARY ([9]). *If X is locally compact, $\Lambda^+(x) = \pi(y)$ $(\Lambda^-(x) = \pi(y))$, then the following two conditions are equivalent*

(i) *π^y is periodic [possibly y is a rest point],*
(ii) *π^x is positively (negatively) asymptotically periodic.*

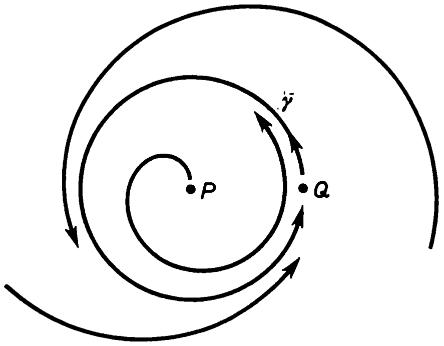
EXAMPLES (see [9], Sec. 8). I. Consider a dynamical system induced on $\boldsymbol{R}^2$ by the system written in polar coordinates in the form

$$r' = r(1-r), \qquad \theta' = 1.$$

The unit circle (being clearly a periodic trajectory) is the positive limit set for every point of $R^2 \setminus \{(0, 0)\}$, the set $\{(0, 0)\}$ is the negative limit set for every point belonging to the interior of the unit wheel. Thus the motion of every point of $R^2 \setminus \{(0, 0)\}$ is positively asymptotically periodic and the motion of every point belonging to the open unit wheel except the origin, is negatively asymptotically periodic. The trajectories are presented on the picture



II. Let us consider a dynamical system on R^2 having trajectories presented in the figure



There are two stationary points P and Q. Notice that the point Q is not Lyapunov stable (neither positively nor negatively). For any point $x \in R^2 \setminus \{\gamma \cup \{P\}\}$ (γ is equal to the unit circle without the point Q), the positive limit set $\Lambda^+(x)$ is equal to $\gamma \cup \{Q\}$. The motion of such a point is clearly not positively asymptotically periodic. Let us consider a new dynamical system on $R^2 \setminus \gamma$ obtained from the previous one by removing γ from the plane (and keeping all other trajectories as they were before). In this new system, we have $\Lambda^+(x) = \{Q\}$ for every $x \neq P$. However, for $x \neq P$ the motion π^x is still not positively asymptotically periodic. This example shows that the assumption of local compactness of X in Theorem 1.3 is essential.

2. Stability in generalized (multivalued) semi-systems

We shall consider generalized semi-systems $(X, G; \lambda)$. First of all however, in order to explain briefly the main idea of the present section, we shall discuss a special case; X being a metric space (with a metric ϱ) and $G := R_+$.

Let M be a nonempty subset of X. Put

$$B(M, \varepsilon) := \{y \in X \colon d(y, M) < \varepsilon\} \tag{2.1}$$

where $d(y, M) := \inf\{\varrho(x, y) \colon x \in M\}$, and

$$B(x, \varepsilon) := B(\{x\}, \varepsilon) = \{y \in X \colon \varrho(x, y) < \varepsilon\}. \tag{2.2}$$

Recall that M is said to be *Lyapunov stable* in a (classical) semi-system $(X, \boldsymbol{R}_+; \pi)$ where π satisfies the conditions (0.1) and (0.2) if and only if

$$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \left[\pi(B(x, \delta)) \subset B(M, \varepsilon)\right]$$

where clearly

$$\pi(B(x, \delta)) = \{\pi(t, y) \colon t \in \boldsymbol{R}_+, \, y \in B(x, \delta)\}.$$

This condition can be extended for $(X, \boldsymbol{R}_+; \lambda)$ in two natural ways:

$$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \left[\lambda(B(x, \delta)) \subset B(M, \varepsilon)\right] \tag{I}$$

and/or

$$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \ [\lambda(t, y) \cap B(M, \varepsilon) \neq \emptyset \text{ for } t \in \boldsymbol{R}_+, \, y \in B(x, \delta)]. \tag{II}$$

It is clear that (I) $\Rightarrow$ (II) but not conversely in general.

Fundamental results on stability conditions are stated with respect to so-called Lyapunov functions which are – in the classical form which requires usually differentiability – discussed in the theory of differential equations. A general definition of Lyapunov function in (general) semi-systems theory has been proposed in [14]; theorems giving sufficient and necessary conditions for a set M to be stable, by means of such Lyapunov functions are proved as well. These concepts and theorems can be extended for generalized (multivalued) semi-systems and stabilities of the type (I), which is, however, sometimes too strong. It seems to be impossible to get similar results with respect to stabilities of the type (II). This fact suggested some other stability-like condition, being in certain sense between (I) and (II).

We shall present it below following [11]. Let $(X, G; \lambda)$ be a generalized semi-system, and let $M \in \mathscr{P}(X)$ be given. Assume that there are: a non-empty subfamily Ω of $\mathscr{P}(X)$ and a mapping $\beta \colon M \to \mathscr{P}(\mathscr{P}(X))$ which is supposed to be "normal", that is, such that

$$x \in B \quad \text{for every } B \in \beta(x).$$

Definition A. Let $A \in \mathscr{P}(X)$ and $n \in N$ be fixed. We define $P(A, n) \subset X$ by the formula

$$y \in P(A, n) \Leftrightarrow \forall_{t_0 \in G} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1 \in G} \exists_{z_2 \in \lambda(t_1, z_1)} \ldots \\ \ldots \forall_{t_{n-1} \in G} \exists_{z_n \in \lambda(t_{n-1}, z_{n-1})} \forall_{t_n \in G} [\lambda(t_n, z_n) \cap A \neq \emptyset].$$

Definition B. For $A \in \mathscr{P}(X)$, we put

$$E(A) := \bigcap \{P(A, n) \colon n \in N\}.$$

It is not difficult to show that, for every $A \in \mathscr{P}(X)$,

$$E(A) \subset A \quad \text{and} \quad E(A) = E(E(A)).$$

Definition C. The set M is said to be $\{\Omega, \beta\}$-*stable* (shortly $M \in S\{\Omega, \beta\}$) iff

$$\forall_{Q \in \Omega} \forall_{x \in M} \exists_{B \in \beta(x)} B \subset E(Q).$$

Definition D. The set M satisfies the condition $L\{\Omega, \beta\}$ (shortly $M \in L\{\Omega, \beta\}$) iff there is a family $\mathscr{A} \in \mathscr{P}(\mathscr{P}(X))$ such that

(2.3) $$\beta(y) > \mathscr{A} \quad \text{for every } y \in M \text{ and } \mathscr{A} > \Omega,$$

and, moreover,

(2.4) $$A \subset E(A) \quad \text{for every } A \in \mathscr{A},$$

(here: $\mathscr{D} > \mathscr{C}$ means that for every $C \in \mathscr{C}$ there is $D \in \mathscr{D}$ such that $D \subset C$).

It is easy to prove the following

Theorem 2.1. $M \in S\{\Omega, \beta\} \Leftrightarrow M \in L\{\Omega, \beta\}$.

Suppose now that $N \in \mathscr{P}(X)$ is such that

(2.5) $$N \subset E(N).$$

Assume that $(T, \leqslant)$ is a partially ordered space and put

$$P := T \setminus \{\inf T\} \quad \text{if } \inf T \text{ exists}$$

and

$$P := T \quad \text{if } \inf T \text{ does not exist}.$$

Definition E. A function $V \colon N \to T$ is said to be a *Lyapunov function of the type* $\{N, T; \Omega, \beta\}$ *for* M iff the family

$$\mathscr{A} = \{A_\eta \colon \eta \in P\}$$

where

$$A_\eta := \{x \in N \colon V(x) < \eta\} \quad \text{for } \eta \in P$$

satisfies the conditions (2.3) and moreover

$$\forall_{x \in N} \forall_{\mu > V(x)} (x \in E(A_\mu)).$$

Theorem 2.2. *If there exists a Lyapunov function of the type* $\{N, T; \Omega, \beta\}$ *for* M, *then* $M \in S\{\Omega, \beta\}$.

THEOREM 2.3. *Assume that* $(T, \leqslant)$ *satisfies the following conditions*:

(i) *for every* $S \subset T$, $S \neq \emptyset$, *there exists* $\inf S \in T$:

(ii) *for every* $\eta \in P$ *there exists* $\mu \in P$ *such that* $\mu < \eta$;

(iii) *for every* $S \subset T$, $S \neq \emptyset$ *and every* $\eta \in P$, *such that* $\inf S < \eta$, *there is* $\sigma \in S$ *such that* $\sigma \leqslant \eta$.

Suppose that $\Omega = \{Q_\eta \colon \eta \in P\}$ *is such that*: $\eta < \mu \Rightarrow Q_\eta \subset Q_\mu$. *If* $M \in S\{\Omega, \beta\}$, *then there exists a subset* N *of* S *satisfying* (2.5) *and such that*

$$\forall_{x \in M} \exists_{B \in \beta(x)} (B \subset N),$$

and there exists a Lyapunov function $V \colon N \to T$ *of the type* $\{N, T; \Omega, \beta\}$ *for the set* M.

Those theorems generalize directly the author's results presented in [14]. The proof will be given in [11].

It is obvious that the space $T = \boldsymbol{R}_+$ of real non-negative numbers with the natural relation $\leqslant$ satisfies the conditions (i)–(iii); also the space $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ with the natural relation defined by the formula

$$(a, b) \leqslant (c, d) \Leftrightarrow a \leqslant c \quad \text{and } b \leqslant d,$$

fulfils (i)–(iii). Thus we can apply our results with respect to semi-systems induced by differential equations (compare Remark I in Sec. 0) obtaining corollaries concerning solutions of differential equations.

Details and examples will be published in [11].

Notice only that if some natural assumptions on the function g in (0.6) are satisfied, then the following, stability-like condition:

> for every $a > 0$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x^0| < \delta$ then there is a solution φ of (0.6) satisfying the initial condition $\varphi(0) = x^0$ and the estimation $|\varphi(t)| < \varepsilon$ for $0 \leqslant t \leqslant a$,

can be expressed as $\{\Omega, \beta\}$-stability with suitable family Ω and a natural mapping β associating with $x \in \boldsymbol{R}^n$ the family of open balls centered at x.

We may use also this concept of $\{\Omega, \beta\}$-stability with respect to generalized semi-systems of the type $(\boldsymbol{R}^n, \boldsymbol{R}_+ \times \boldsymbol{R}_+; \lambda)$, where λ is defined by values of solutions of some partial differential equations (of the type $z_{xy} = f(z)$ for instance). Details and examples will be published later.

Remark. Let us notice that in [11] generalized semi-systems in the sense of the definition introduced at the beginning of the present paper, are called simply generalized systems because there are considered only such systems; here we underline the parallelism of multivalued and classical semi-systems $(X, G; \pi)$ with G being essentially a semi-group in contrast to systems in which G has a group structure.

3. Limit sets in generalized semi-systems

Let (X, ϱ) be a metric space and let $(X, \boldsymbol{R}_+; \lambda)$ be a generalized semi-system.

Let $x \in X$ be fixed. We say that the motion

$$\lambda^x\colon\ \boldsymbol{R} \ni t \mapsto \lambda^x(t) := \lambda(t, x) \dot{\subset} X$$

in *semi-stable* (see [12]) iff for every $\varepsilon > 0$ there exist $\delta > 0$ and $t^0 \in \boldsymbol{R}_+$ such that

$$(y \in X,\ t \in \boldsymbol{R}_+,\ \varrho(x, y) < \delta) \Rightarrow \lambda(t^0+t, y) \subset B(\lambda(t^0+t, x), \varepsilon).$$

It is clear that this concept generalizes the notion of the (positive) Lyapunov stability of motions in classical dynamical systems (compare Remark V in Section 1).

Let us put (see [1], [12])

$\Lambda(x) := \{y \in X$: there are sequences $\{y_m\}$ of elements of X and $\{t_m\}$ of real nonnegative numbers, such that $y_m \in \lambda(t_m, x)$, $t_m \to \infty$, $y_m \to y$ as $m \to \infty\}$.

This set is called the *limit set of* x. We have the following theorem ([12]).

THEOREM 3.1. *For every x the limit set $\Lambda(x)$ is closed.*

THEOREM 3.2. *If X is locally compact, $x \in X$, λ^x is semistable, $\Lambda(x) \neq \emptyset$ then the mapping*

$$z \mapsto \Lambda(z) \tag{3.1}$$

is upper semi-continuous at the point x, (that is: if $x_m \to x$, $y_m \to y$, $y_m \in \Lambda(x_m)$ then $y \in \Lambda(x)$).

THEOREM 3.3. *If the mapping*

$$t \mapsto \lambda(t, x) \tag{3.2}$$

is upper semi-continuous at every point $t^0 \in \boldsymbol{R}_+$, then

$$\overline{\lambda(x)} = \lambda(x) \cup \Lambda(x). \tag{3.3}$$

THEOREM 3.4. *Assume that X is locally cpmpact and that for every $t, s \in \boldsymbol{R}_+$, $t < s$ the set*

$$[\lambda(t, x), \lambda(s, x)] := \bigcup \{\lambda(r, x)\colon\ r \in [t, s]\} \tag{3.4}$$

is connected. Suppose that $\Lambda(x) \neq \emptyset$ and $\Lambda(x)$ is compact.

Then $\Lambda(x)$ is connected.

The above theorems generalize clearly known results concerning classical dynamical semi-systems.

4. Negative stability in semi-systems

Let $(X, \boldsymbol{R}_+; \pi)$ be a semi-system. A mapping σ: $I \to X$, where I is an interval in $\boldsymbol{R}$ we shall call, following [2] (compare [19] for details) a *solution*, iff:

(4.1) $$\pi(t, \sigma(s)) = \sigma(t+s) \quad \text{for } s, s+t \in I, t \geqslant 0.$$

A solution σ is said to be a *left solution* (passing through x) if $0 = \max I$. Such a solution is said to be *maximal left solution* if for every left solution τ: $J \to X$ (passing through x) we have: $J \subset I$ and $\tau(t) = \sigma(t)$ for $t \in J$. We say (following [2]) that T is a *negative trajectory* passing through x iff there exists a maximal left solution σ: $I \to X$ passing through x, such that $\sigma(I) = T$. By $\mathscr{T}(x)$ we shall denote the family of all negative trajectories passing through x.

For $x \in X$, we put

(4.2) $$\pi_+(x) := \{\pi(t, x): t \geqslant 0\}$$

and call it the *positive trajectory* of x.

We put also

(4.3) $$F_x := \{y \in X: x \dot{\in} \pi_+(y)\}.$$

A subset M of X is said to be *negatively invariant* (see [2]) iff:

$$(x \in M, T \in \mathscr{T}(x)) \Rightarrow T \supset M,$$

and it is said to be *negatively semi-invariant* iff for every $x \in M$ there exists a negative trajectory T passing through x such that $T \subset M$.

Let now $M \in \mathscr{P}(X)$ be fixed, let $\Omega \in \mathscr{P}(\mathscr{P}(X))$ and let a normal mapping β: $X \to \mathscr{P}(\mathscr{P}(X))$ be given. We say (see [19]) that M is *negatively* (Ω, β)-*stable* iff

$$\forall_{x \in M} \forall_{Q \in \Omega} \exists_{B \in \beta(x)} [y \in B, T \in \mathscr{T}(y) \Rightarrow T \subset Q].$$

We say that (see [19]) M is *negatively* (Ω, β)-*semi-stable* iff

$$\forall_{x \in M} \forall_{Q \in \Omega} \exists_{B \in \beta(x)} \forall_{y \in B} \exists_{T \in \mathscr{T}(y)} (T \subset Q).$$

Let now W be a non-empty negatively invariant subset of X. We say that a function V: $W \to \boldsymbol{R}_+$ is an (Ω, β)-*Lyapunov function* for M if

(j) $$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{B \in \beta(x)} [y \in B \Rightarrow y \in W \text{ and } V(y) \leqslant \varepsilon];$$

(jj) $$\forall_{Q \in \Omega} \exists_{\delta > 0} [x \in W \backslash Q \Rightarrow V(x) \geqslant \delta];$$

(jjj) $$\forall_{y \in W} \forall_{T \in \mathscr{T}(y)} [V(z) \leqslant V(y) \text{ for } z \in T].$$

Remark. Conditions (jj) and (jjj) could be written as follows:

$$\inf\{V(z): z \in W \backslash Q\} > 0 \quad \text{for } Q \in \Omega$$

and

$$(\pi(t, z) \in W, t \geqslant 0) \Rightarrow V(\pi(t, z)) \leqslant V(z).$$

THEOREM 4.1 ([19]). *If there exists an (Ω, β)-Lyapunov function for M, then M is negatively (Ω, β)-stable.*

THEOREM 4.2 ([19]). *Assume that $\Omega = \{Q_\eta: \eta > 0\}$ and $Q_\eta \subset Q_\mu$ for $\eta \leqslant \mu$. If M is negatively (Ω, β)-stable, then there exists a negatively invariant subset W of X containing M and an (Ω, β) Lyapunov function V: $W \to \boldsymbol{R}_+$ for M.*

These theorems are analogous to the results concerning the (Ω, β)-stability presented in [14].

Suppose now that $W \neq \emptyset$ is negatively semi-invariant. We say that V: $W \to \boldsymbol{R}_+$ is a *semi-(Ω, β)-Lyapunov function for M* iff it satisfies the conditions (j) and (jj) and – instead of (jjj) – the following conditions

(kkk) $$\forall_{y \in W} \exists_{T \in \mathscr{T}(y)} [V(z) \leqslant V(y) \text{ for } z \in T].$$

THEOREM 4.3 (see [19]). *If there exists a semi-(Ω, β)-Lyapunov function for M, then M is negatively (Ω, β)-semi-stable.*

THEOREM 4.4 ([19]). *Assume that $\Omega = \{Q_\eta: \eta > 0\}$ is as in Theorem 4.2. Assume also that for every $y \in M$ the set*

$$\{\eta > 0: \textit{there is } T \in \mathscr{T}(y) \textit{ such that } T \subset Q\}$$

is an interval of the form $[a, \infty)$.

If M is negatively (Ω, β)-semi-stable and $M = \bigcap \{Q_\eta: \eta > 0\}$ then there exists a semi-(Ω, β)-Lyapunov function for M, defined in a negatively semi-invariant set W containing M.

Proofs and comments will be published in [19].

5. Some results on dynamical semi-systems on the plane

Let $(\boldsymbol{R}^2, \boldsymbol{R}_+; \pi)$ be a dynamical semi-system. It is known (see [2]) that in such a semi-system there are no start points (a point x is said to be a start point iff $F_x = \{x\}$ (see (4.3))). For $x \in \boldsymbol{R}^2$ we define the negative escape time N_x (see [5]) by the formula

$$N_x := \inf\{s \in (0, \infty): (-s, 0] \text{ is the domain of the definition of the maximal left solution passing through } x\}.$$

This definition is equivalent to that proposed in [8] but not equivalent to the definition stated in [2] (for details and further references, see [5]).

For $x \in \boldsymbol{R}^2$, $t \geqslant 0$, $a, b \geqslant 0$, $a \leqslant b$, $\emptyset \neq M \subset \boldsymbol{R}^2$, we put now

$$F(t, x) := \{y \in \boldsymbol{R}^2: \pi(t, y) = x\},$$

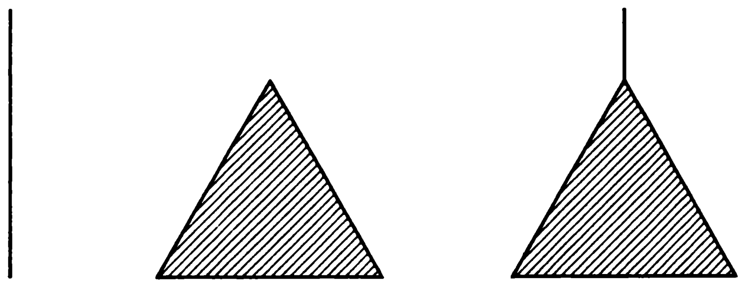
$$F([a, b], M) := \{y \in \boldsymbol{R}^2: \text{there exists } s \in [a, b] \text{ such that } \pi(s, y) \in M\}.$$

A point x is said to be *periodic* if it is not a rest point and there is $t > 0$ such that $\pi(t, x) = x$. A point x is called *regular* if it is neither a rest point nor periodic. Assume now that $N_x = \infty$ for every $x \in \boldsymbol{R}^2$.

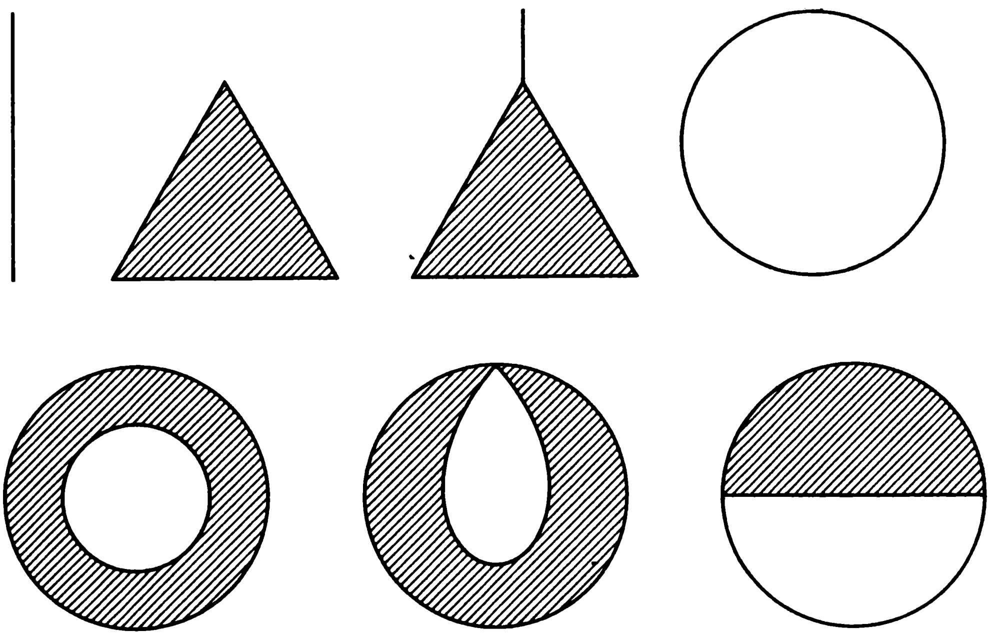
We have the following theorems (see [5]).

THEOREM 5.1. *Assume that x is not a stationary point. Then the set $F(t, x)$ is for every $t \geqslant 0$ a point or it is homeomorphic to an interval.*

THEOREM 5.2. *If x is regular then the set $F([s, t], x)$ is for every $t, s \geqslant 0$, $s < t$, homeomorphic to one of the three sets presented on the picture below.*



THEOREM 5.3. *If x is periodic then for every $s, t \geqslant 0$, $s < t$ the set $F([s, t], x)$ is homeomorphic to one of the seven sets presented on the picture below.*



THEOREM 5.4. *If M is a compact and connected (nonempty) subset of $\boldsymbol{R}^2$ and $0 \leqslant s \leqslant t$, then $F([s, t], M)$ is compact and connected.*

The next result is established without the assumption that $N_x = \infty$ as it was supposed above. So, let $(\boldsymbol{R}^2, \boldsymbol{R}_+; \pi)$ be a dynamical semi-system and let $x \in \boldsymbol{R}^2$ be fixed.

THEOREM 5.5. *There is satisfied exactly one condition: either*

(a) *$F(t, x)$ is a point for $t < N_x$ and $F(t, x) = \emptyset$ for $t \geqslant N_x$*

or

(b) *there exist α, β, $0 \leqslant \alpha < N_x \leqslant \beta \leqslant \infty$ such that $F(t, x)$ is a point for $t \in [0, \alpha]$, $F(t, x)$ is a compact arc for $t \in (\alpha, N_x)$, $F(t, x)$ has the form described below in (⁑) for $t \in [N_x, \beta)$ and $F(t, x) = \emptyset$ for $t \geqslant \beta$ (notice that the cases $[N_x, \beta) = \emptyset$ and/or $[\beta, \infty) = \emptyset$ are not excluded);*

(⁑) *$F(t, x)$ is a one-dimensional manifold having finitely or countably many connected closed components. At most two of them are homeomorphic to $\boldsymbol{R}_+$, all the other are homeomorphic to $\boldsymbol{R}$. If f is a parametrization of a component, then in the first case:*

$$\lim |f(r)| = \infty \quad \text{as } r \to \infty$$

while in the second one we have:

$$\lim |f(r)| = \lim |f(-r)| = \infty \quad \text{as } r \to \infty .$$

6. Some methods and results from algebraic topology applied to dynamical systems

6.1. We shall present here a summary of some results of paper [17].

Let $(X, \boldsymbol{R}; \pi)$ be a dynamical system with X being an ENR-space, that is Euclidean neighbourhood retract.

For an open and relatively compact subset U of X we put

$$I(\pi, U) := \lim_{t \to 0^+} \operatorname{ind}(\pi(t, \cdot), U).$$

Here $\operatorname{ind}(\pi(t, \cdot), U)$ denotes the fixed-point index of the mapping

$$U \ni x \mapsto \pi(t, x) \in X,$$

(for definition see [7]).

THEOREM 6.1 (see [17]). *If K is a compact subset of X such that for every x*

$$\{\pi(t, x) \colon t \geqslant 0\} \cap K \neq \emptyset \tag{6.1}$$

and there are no rest points in ∂K, then $I(\pi, \operatorname{int} K)$ is equal to the Euler characteristic $\chi(X)$ of X.

THEOREM 6.2 (see [17]). *Assume that X is a 2-dimensional topological manifold satisfying the second countability axiom. Suppose that all the Betti numbers of X are finite. Let K be a compact subset of X having no rest points of π in its boundary. If for every $x \in X$*

$$\{\pi(t, x) \colon t \in \boldsymbol{R}\} \cap K \neq \emptyset \tag{6.2}$$

then $I(\pi, \operatorname{int} K) \geqslant \chi(X)$.

COROLLARY. *Consider $X = \boldsymbol{R}^2$. If K is a compact set such that for every $x \in \boldsymbol{R}^2$ the condition* (6.2) *is satisfied then there exists in K a rest point.*

6.2. Remarks on periodic solutions of differential equations. We shall give an example of applications of general results on dynamical systems obtained by using topological methods having the background in Ważewski's papers ([20], [21]) developed and extended by introducing algebraic topology. Let us recall first of all the definition of an isolated invariant set in a given dynamical system $(X, \boldsymbol{R}; \pi)$. A compact set K is a *isolated invariant set for* π if there exists an open neighbourhood U of K, such that K is a maximal invariant set relative to π in U.

For an isolated invariant set K, it is introduced the notion of the index, noted by $\operatorname{ind} K$, by using the so-called isolated blocks and the Čech cohomology functor. The definition appeared firstly in [4]; a general and systematic investigation of this notion is presented in [6] (for details, see also [18]).

Consider now a system of autonomous differential equations

$$x' = f(x) \tag{6.3}$$

with $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ being continuous and so regular, that for every $x^0 \in \boldsymbol{R}^n$ there exists exactly one solution $\varphi(\cdot\,; x^0)$ of (6.3) passing through the initial point $(0, x^0)$, defined in $\boldsymbol{R}$; we assume also that solutions depend continuously on initial data.

It is well known that putting $\pi(t, x) := \varphi(t; x)$ we get a dynamical system $(\boldsymbol{R}^n, \boldsymbol{R}; \pi)$ (compare Section 1).

THEOREM 6.3 ([18]). *If K is an isolated invariant set in the above dynamical system and* $\operatorname{ind} K \neq 0$, *then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every C^1-function $h\colon \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ fulfilling the condition*

$$\|h(t, x)\| < \delta \quad \textit{for every } (t, x)$$

and such that for every fixed $x \in \boldsymbol{R}^n$ the function $h(\cdot, x)$ is T-periodic, the equation

$$x' = f(x) + h(t, x)$$

has a T-periodic solution φ such that $\operatorname{dist}(\varphi(\boldsymbol{R}), K) < \varepsilon$.

7. Remarks on the Ważewski method applied to partial differential equations

Fundamental idea of a topological Ważewski method (see [20], [21]) mentioned already above in Section 6 has been used recently in the theory of partial differential equations of the second order of the type

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) \tag{7.1}$$

where $t \geqslant 0$, x belongs to some (depending on t) interval in $\boldsymbol{R}$ with initial conditions of the type

$$u(0, x) = \varphi(x) \tag{7.2}$$

and – possibly – some further boundary conditions;

$$f = (f^1, \ldots, f^n), \quad u = (u^1, \ldots, u^n).$$

Let α and β be two real valued functions defined and continuous on $[0, \infty)$ such that

$$\alpha(t) < \beta(t) \quad \text{for } t \geqslant 0. \tag{7.3}$$

We consider (7.1) in the set

$$\{(t, x)\colon t \geqslant 0, \alpha(t) \leqslant x \leqslant \beta(t)\}, \tag{7.4}$$

or in an open set containing (7.4).

The problem is to establish conditions sufficient for the existence of a solution u of (7.1)–(7.2) (with possibly some further boundary conditions) for which the inequalities

$$\lambda^i(t) < u^i(t, x) < \mu^i(t), \quad i = 1, \ldots, n, \tag{7.5}$$

are satisfied in the domain of the existence of u intersected with the set (7.4), where λ^i and μ^i are given (sufficiently regular) functions. The main results are established by using conditions concerning egress and strict egress points introduced by T. Ważewski. These conditions are described by suitable inequalities required with respect to f at points (t, x, w, v, z) such that $x \in [\alpha(t), \beta(t)]$, $w^i = \lambda^i(t)$ or $w^i = \mu^i(t)$, $v = 0$ (or $v \geqslant 0$ or $v \leqslant 0$ in the case if $x \in \{\alpha(t), \beta(t)\}$) and $z \geqslant 0$ or $z \leqslant 0$.

These results will be given in [13].

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