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The regularity of convolution and restriction of distributions

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Abstract. We considered in [4] local order functions of distributions and we proved a theorem on the regularity of the product of distributions in terms of local order functions, which corresponds to the well-known inclusion: $WF(UV) \subset WF(U) \cup WF(V) \cup (WF(U) \oplus WF(V))$. In this paper we apply this result to prove a theorem on the regularity of the restriction (trace) of a distribution $U \in D'(M')$ to M, where M is a C^∞ submanifold of a C^∞ manifold M'. We also give a bound for the value of the local order function of the convolution of two distributions on $\boldsymbol{R}^n$ in terms of the values of the local order functions of its factors.

We first define the local order function of order p: $1 \leqslant p \leqslant +\infty$ for $U \in D'(\boldsymbol{R}^n)$. The function, denoted by O_U^p, is defined on the set $\boldsymbol{R}^n \times S^{n-1}$ and its values belong to the set $\boldsymbol{R}^*$ consisting of the elements

$$t^- = \{r \in \boldsymbol{R} |\ r < t\}, \quad t^+ = \{r \in \boldsymbol{R} |\ r \leqslant t\} \quad \text{for } t \in \boldsymbol{R}$$

and the element $\infty = \boldsymbol{R}$. The ordering in $\boldsymbol{R}^*$ is given by the relation of inclusion. If $\{\tau_a\}_{a \in A} \subset \boldsymbol{R}^*$ and, for some $\tau \in \boldsymbol{R}^*$, $\tau \leqslant \tau_a$ for $a \in A$, then we can naturally define the minimum of the set $\{\tau_a\}_{a \in A}$ (denoted $\min\limits_{a \in A} \tau_a$) as the unique element $\tau_0 \in \boldsymbol{R}^*$ with the following property: $\tau_0 \leqslant \tau_a$ for $a \in A$ and if the same is true for some $\tau_0' \in \boldsymbol{R}^*$, then $\tau_0' \leqslant \tau_0$. The operation of addition is defined in $\boldsymbol{R}^*$ by

$$s^- + t^- = s^- + t^+ = s^+ + t^- = (s+t)^-,$$

$$s^+ + t^+ = (s+t)^+, \quad t^- + \infty = t^+ + \infty = \infty.$$

Definition 1. $O_U^p(x, l) = \{\alpha \in \boldsymbol{R} |$ there exist neighbourhoods Q of x and L of l such that $(\omega U)\hat{}(y)(1+y^2)^{\alpha/2} \in L^p(\Gamma_L)$ for $\omega \in D(Q)\}$, where $\Gamma_L = \{r\lambda |\ \lambda \in L,\ r \in \boldsymbol{R}^+\}$ and $\wedge$ denotes the Fourier transform.

Sometimes it is useful to define O_U^p on $\boldsymbol{R}^n \times (\boldsymbol{R}^n \setminus \{0\})$ by setting $O_U^p(x, l) = O_U^p(x, l/|l|)$ for $(x, l) \in \boldsymbol{R}^n \times (\boldsymbol{R}^n \setminus \{0\})$.

In the case of a distribution U on a C^∞ manifold M, one can localize the problem and define O_U^p in a natural way on $T^* M \setminus 0$ (cotangent bundle minus the zero section) or on the cotangent sphere $S^* M$, since the notion of O_U^p is invariant under diffeomorphisms.

1. Restriction. Given p, q with $1 \leqslant p, q \leqslant +\infty$ and $1/p+1/q \geqslant 1$, we write:

$$M_{p,q}(n) = \{(U, V) \in D'(\mathbf{R}^n) \times D'(\mathbf{R}^n) |\ O_U^p(x, l) + O_V^q(x, -l) \geqslant 0^+ \quad \text{for all } (x, 1) \in \mathbf{R}^n \times S^{n-1}\}$$

and

$$M(n) = \{(U, V) \in D'(\mathbf{R}^n) \times D'(\mathbf{R}^n) |\ \text{for every } x \in \mathbf{R}^n \text{ there exists}$$
$$\text{a neighbourhood } Q_x \text{ such that, for all } \omega, \psi \in D(Q_x), (\omega U)\hat{}\,(\psi V)\check{} \in L^1(\mathbf{R}^n)\},$$

where $\vee$ is the inverse Fourier transform. We showed in [4] that $M_{p,q}(n) \subset M(n)$. We define the product UV for $(U, V) \in M(n)$ as follows: for every $\omega \in D(Q_x)$ we choose $\psi \in D(Q_x)$ with $\psi = 1$ on $\operatorname{supp} \omega$ and define

$$(UV)[\omega] = \int (\omega U)\hat{}\,(\psi V)\check{}.$$

Then $UV \in D'(Q_x)$; piecing together (over $x \in \mathbf{R}^n$) we obtain a distribution $UV \in D'(\mathbf{R}^n)$ The correctness of the definition was shown in [1].

We proved in [4] an analogue to the well-known inclusion

$$WF(UV) \subset WF(U) \cup WF(V) \cup (WF(U) \oplus WF(V))$$

(see e.g. [3], Theorem 8.2.10 or [5], Proposition D1.2, p. 237).

THEOREM 1. *Let p, q, r be such that $1 \leqslant p, q, r \leqslant +\infty$ and $1/p+1/q = 1+1/r$. Let $(U, V) \in M_{p,q}(n)$. Then:*

$$O_{UV}^r(x, k) \geqslant \min\Big(\min_{(1,m) \in [k]} (O_U^p(x, l) + O_V^q(x, m)), O_U^p(x, k), O_V^q(x, k)\Big)$$

for $(x, k) \in \mathbf{R}^n \times S^{n-1}$, where $[k]$ is the closure of the set:

$$k^+ = \{(l, m) \in S^{n-1} \times S^{n-1} |\ \textit{there exist } a, b \in \mathbf{R}^+ \textit{ such that } al+bm = k\}.$$

We will use this theorem to get an estimate for the local order function for the restriction of a distribution on $\mathbf{R}^n$ to a subspace H of $\mathbf{R}^n$ of codimension n_0, $n_0 < n$. We take $H = \{x \in \mathbf{R}^n |\ x_1 = 0, \ldots, x_{n_0} = 0\}$ and write

$$\perp H = \{l \in S^{n-1} |\ l = (l_1, \ldots, l_{n_0}, 0, \ldots, 0)\},$$
$$\| H = \{l \in S^{n-1} |\ l = (0, \ldots, 0, l_{n_0+1}, \ldots, l_n)\}.$$

We consider the distribution $\delta_H \in D'(\mathbf{R}^n)$ defined by

$$\delta_H[\Psi] = \int_H \Psi|_H \quad \text{for } \Psi \in D(\mathbf{R}^n).$$

Let $U \in D'(\mathbf{R}^n)$. If $(U, \delta_H) \in M(n)$, then there exists the product $U\delta_H$ and the restriction $U|_H$ of U to H given by the formula

$$U|_H[\varphi] = U\delta_H[\Phi] \quad \text{for } \varphi \in D(H),$$

where $\Phi \in D(\boldsymbol{R}^n)$ is such that $\Phi|_H = \varphi$. It is obvious that the expression does not depend on the choice of Φ.

A simple calculation shows that

$$O^q_{\delta_H}(x, l) = \infty \quad \text{iff } x \notin H \quad \text{or} \quad l \notin \perp H \quad \text{for } q\colon 1 \leqslant q \leqslant +\infty,$$

$$O^q_{\delta_H}(x, l) = (-n_0/q)^- \quad \text{for } x \in H,\ l \in \perp H,\ q\colon 1 \leqslant q < +\infty,$$

$$O^\infty_{\delta_H}(x, l) = 0^+ \quad \text{for } x \in H,\ l \in \perp H.$$

Thus we have: If $1 \leqslant p, q \leqslant +\infty$, $1/p + 1/q \geqslant 1$ and $O^p_U(x, l) + (-n_0/q)^- \geqslant 0^+$ for $x \in H$, $l \in \perp H$, then $(U, \delta_H) \in M(n)$ and the restriction $U|_H$ exists. It is not difficult to prove the following:

Remark 1. If $1 \leqslant p \leqslant +\infty$, $U \in D'(\boldsymbol{R}^n)$ and $O^p_U(x, l) + (n_0(1/p-1))^- \geqslant 0^+$ for $x \in H$, $l \in \perp H$, then $U|_H$ exists and

$$O^p_{U|H}(x, k) = O^p_{U\delta_H}(x, k) + (n_0/p)^+ \quad \text{for } x \in H,\ k \in \| H,$$

where the function $O^p_{U|H}$ is considered as a function on $\boldsymbol{R}^{n-n_0} \times S^{n-n_0-1}$.

THEOREM 2. *Assume the conditions of Remark* 1. *Then $U|_H$ exists and*

$$O^p_{U|H}(x, k) \geqslant \min_{l \in k + \perp H} O^p_U(x, l) + (n_0(1/p-1))^-$$

for $x \in H$, $k \in \| H$, where $k + \perp H$ is the set of all $l \in S^{n-1}$ which can be represented in the form $l = ak + bn$ for some $a, b \geqslant 0$, $n \in \perp H$.

Proof. The theorem is a simple consequence of Theorem 1 and Remark 1 in the case where $O^p_U(x, l) + (-n_0)^- \geqslant 0^+$ for $x \in H$, $l \in \perp H$. In fact: taking $r = p$, $q = 1$, we have from Theorem 1:

$$O^p_{U\delta_H}(x, k) \geqslant \min\Big(\min_{(l,m) \in [k]} \big(O^p_U(x, l) + O^1_{\delta_H}(x, m)\big), O^p_U(x, k), O^1_{\delta_H}(x, k)\Big)$$

$$\geqslant \min_{l \in k + \perp H} O^p_U(x, l) + (-n_0)^- \quad \text{for } x \in H,\ k \in \| H;$$

so, applying Remark 1 we obtain

$$O^p_{U|H}(x, k) \geqslant \min_{l \in k + \perp H} O^p_U(x, l) + (n_0(1/p-1))^- \quad \text{for } x \in H,\ k \in \| H.$$

In the general case where $O^p_U(x, l) + (n_0(1/p-1))^- \geqslant 0^+$ for $x \in H$, $l \in \perp H$, the assumptions of Theorem 1 are not satisfied. In this case Theorem 2 can be proved directly. Since the proof is rather long, we omit it.

Theorem 2 can be reformulated for the case of C^∞ manifolds. Let M be a C^∞ submanifold of a C^∞ manifold M', of codimension n_0, $n_0 < n$, n

$= \dim M'$ (this means that M can be defined by the equations $x_1 = \ldots = x_{n_0} = 0$ in local coordinates). The natural injection i of M into M' defines linear maps $i_x^*\colon T_x^* M' \to T_x^* M$ for $x \in M$. The kernel of i_x^* is the conormal space $N_x^* M$ to M at x. The set-theoretical union of the linear spaces $N_x^* M$ for $x \in M$ is the conormal bundle $N^* M$.

THEOREM 2'. *Let M be a C^∞ submanifold of a C^∞ manifold M', of codimension n_0, $n_0 < n$, $n = \dim M'$. Let $1 \leqslant p \leqslant +\infty$, $U \in D'(M')$ and $O\mathcal{E}_U^p(x, \xi) + (n_0(1/p-1))^- \geqslant 0^+$ for $(x, \xi) \in N^* M \setminus 0$. Then $U|_M$ exists and*

$$O\mathcal{E}_{U|M}^p(x, \eta) \geqslant \min_{\xi \in i_x^{*-1}(\eta)} O\mathcal{E}_U^p(x, \xi) + (n_0(1/p-1))^-$$

for $(x, \eta) \in T^* M \setminus 0$.

As a consequence of Theorem 2' we obtain the well-known property of Sobolev spaces (see [5], Proposition 7.1, p. 72):

PROPERTY 1. *Let M, M' be as in Theorem 2'. If $U \in H_{\mathrm{loc}}^s(M')$ for some $s > n_0/2$, then $U|_M$ exists and belongs to $H_{\mathrm{loc}}^{s-n_0/2}(M)$.*

We note that Theorem 2' corresponds to the following:

PROPERTY 2. *Let M, M' be as in Theorem 2'. If $U \in D'(M')$ is such that $WF(U) \cap N^* M = \emptyset$, then $U|_M$ exists and*

$$WF(U|_M) \subset \{(x, \eta) \in T^* M \mid \exists (y, \xi) \in WF(U),\ y = x,\ i_x^*(\xi) = \eta\}$$

(see [2], Theorem 7 and Theorem 9 Chapter IV, or [3] Theorem 8.2.4 and Corollary 8.2.7).

2. Convolution. Now we recall a sufficient condition for the existence of the convolution of two distributions. To this purpose we introduce some useful definitions.

For $A \subset \boldsymbol{R}^n$ and $\delta > 0$ we denote the set $\{\xi \in \boldsymbol{R}^n \mid \operatorname{dist}(\xi, A) \leqslant \delta\}$ by A_δ.

We say that two sets $A, B \subset \boldsymbol{R}^n$ are *compatible* if for every compact set $H \subset \boldsymbol{R}^n$ the set

$$\boldsymbol{H} = \boldsymbol{H}(A, B, H) = \{(a, b) \mid a \in A,\ b \in B,\ a+b \in H\}$$

is bounded.

We denote, for $A, B \subset \boldsymbol{R}^n$, $x \in \boldsymbol{R}^n$, $\delta > 0$:

$$Z_x(A, B) = A \cap \tilde{B}_{\tau(x)} \quad \text{and} \quad Z_x^\delta(A, B) = A \cap (\tilde{B}_{\tau(x)})_\delta,$$

where $\tilde{B}_{\tau(x)} = \{\xi \in \boldsymbol{R}^n \mid x - \xi \in B\}$.

It is not difficult to show the following

LEMMA 1. *The sets $A, B \subset \boldsymbol{R}^n$ are compatible iff, for every $x \in \boldsymbol{R}^n$, $\delta > 0$, the set $Z_x^\delta(A, B)$ is bounded.*

Remark 3 (see [6], Theorem 1, § 15, p. 123). If the supports of distributions $U, V \in D'(\mathbf{R}^n)$ are compatible, then there exists the convolution $U * V$ of U, V.

Let $U, V \in D'(\mathbf{R}^n)$. We write, for simplicity,

$$Z_x(U, V) = Z_x(\operatorname{supp} U, \operatorname{supp} V) = \operatorname{supp} U \cap \operatorname{supp} \tilde{V}_{\tau(x)},$$

and

$$Z_x^\delta(U, V) = Z_x^\delta(\operatorname{supp} U, \operatorname{supp} V) = \operatorname{supp} U \cap (\operatorname{supp} \tilde{V}_{\tau(x)})_\delta$$

for $x \in \mathbf{R}^n$, $\delta > 0$, where $\tilde{V}_{\tau(x)}$ denotes the distribution $V(x-\cdot)$. Since the supports of U, V are closed sets, we see that if the supports are compatible, $Z_x(U, V)$ and $Z_x^\delta(U, V)$ are compact and $Z_x(U, V) = \bigcap_{\delta>0} Z_x^\delta(U, V)$.

Theorem 3. *Assume that $U, V \in D'(\mathbf{R}^n)$ and the supports of U, V are compatible. Let $1 \leqslant p, q, s \leqslant +\infty$ and $1/p+1/q = 1/s$. Then*

$$O_{U*V}^s(x, l) \geqslant \min_{t \in Z_x(U,V)} \left(O_U^p(t, l) + O_V^q(x-t, l)\right)$$

for $x \in \mathbf{R}^n$, $l \in S^{n-1}$.

Proof. Let us fix $x \in \mathbf{R}^n$, $l \in S^{n-1}$. We first show that

$$O_{U*V}^s(x, l) \geqslant \min_{t \in Z_x^\delta(U,V)} \left(O_U^p(t, l) + O_V^q(x-t, l)\right) \quad \text{for } \delta > 0.$$

Fix $\delta > 0$ and assume that

$$\alpha^+ \leqslant \min_{t \in Z_x^\delta(U,V)} \left(O_U^p(t, l) + O_V^q(x-t, l)\right) \quad \text{for some } \alpha \in \mathbf{R}.$$

For every $t \in Z_x^\delta(U, V)$ there exist numbers $u_t, v_t, u_t^+ \leqslant O_U^p(t, l)$, $v_t^+ \leqslant O_V^q(x-t, l)$ and open sets Q_t, L_t such that

$$t \in Q_t \subset \mathbf{R}^n, \quad l \in L_t \subset S^{n-1}, \quad u_t + v_t = \alpha$$

and

$$(\varphi U)\hat{}(y)(1+y^2)^{u_t/2} \in L^p(\Gamma_{L_t}) \quad \text{for } \varphi \in D(Q_t),$$

$$(\psi V)\hat{}(y)(1+y^2)^{v_t/2} \in L^p(\Gamma_{L_t}) \quad \text{for } \psi \in D\left((Q_t)_{\tau(x)}\right).$$

The family $\{Q_t\}_{t \in Z_x^\delta(U,V)}$ is an open covering of the compact set $Z_x^\delta(U, V)$, so we can choose a finite subcover $Q_1, \ldots, Q_m$.

Let $L = \bigcap_{j=1}^m L_j$, $Q = \bigcup_{j=1}^m Q_j$. Let P be a compact set such that $Z_x^\delta(U, V) \subset P \subset Q$ and $\operatorname{dist}\left(Z_x^\delta(U, V), CP\right) = \varepsilon_1 > 0$. We take a partition of unity $\{\lambda_j\}_{j=1,\ldots,m}$ such that the function $\lambda = \sum_{j=1}^m \lambda_j$ equals 1 on P and

$\operatorname{supp}\lambda_j \subset Q_j$ for $j = 1, \ldots, m$. It is obvious that the number $\varepsilon_2 = \min_j \operatorname{dist}(\operatorname{supp}\lambda_j, CQ_j)$ is positive. Let $\varepsilon = \frac{1}{2}\min(\varepsilon_1, \varepsilon_2, \delta)$.

For $j = 1, \ldots, m$ we can find functions $\eta_j \in D(Q_j)$ such that $\operatorname{dist}(\operatorname{supp}\lambda_j, \operatorname{supp}(1-\eta_j)) > \varepsilon$. Let

$$\lambda_j'(\cdot) = \lambda_j(x-\cdot), \quad \eta_j'(\cdot) = \eta_j(x-\cdot) \quad \text{for } j = 1, \ldots, m$$

and

$$\lambda'(\cdot) = \lambda(x-\cdot).$$

Let $\omega \in D(B(x, \varepsilon))$. We show that

$$\tilde{V}*\omega(z) = (\tilde{\lambda}'\tilde{V})*\omega(z) \quad \text{for } z \in (Z_x^\delta(U, V))_\varepsilon.$$

In fact:

$$(\tilde{\lambda}'\tilde{V})*\omega(z) = \lambda' V[\omega(\cdot - z)] = \lambda(x-\cdot)V[\omega(\cdot - z)] = V[\lambda(x-\cdot)\omega(\cdot - z)].$$

Notice that $\operatorname{supp}\omega(\cdot - z) \subset B(x-z, \varepsilon)$. If $\xi \in B(x-z, \varepsilon)$, then $\lambda(x-\xi) = \lambda(z+\varrho)$, where $|\varrho| < \varepsilon$. Since $z \in (Z_x^\delta(U, V))_\varepsilon$, then $z+\varrho \in (Z_x^\delta(U, V))_{2\varepsilon} \subset (Z_x^\delta(U, V))_{\varepsilon_1} \subset P$, so $\lambda(z+\varrho) = 1$. Hence $\lambda(x-\cdot)\omega(\cdot - z) = \omega(\cdot - z)$. Thus

$$(\tilde{\lambda}'\tilde{V})*\omega(z) = \tilde{V}*\omega(z) \quad \text{for } z \in (Z_x^\delta(U, V))_\varepsilon.$$

This and the property: $\operatorname{supp} U \cap \operatorname{supp}(\tilde{V}*(\omega\sigma)) \subset Z_x^\delta(U, V)$ for $\sigma \in D(\boldsymbol{R}^n)$ imply:

$$\begin{aligned}\omega(U * V)[\sigma] &= U[\tilde{V}*(\omega\sigma)] = U[(\tilde{\lambda}'\tilde{U})*(\omega\sigma)] = (\lambda U)[(\tilde{\lambda}'\tilde{V})*(\omega\sigma)]\\ &= (\lambda U)*(\lambda' V)[\omega\sigma] = \omega((\lambda U)*(\lambda' V))[\sigma].\end{aligned}$$

So,

$$\omega(U * V) = \omega((\lambda U)*(\lambda' V)).$$

Hence,

$$\begin{aligned}\omega(U * V) &= \omega\Big(\Big(\sum_{i=1}^{m} \lambda_i U\Big)*\Big(\sum_{j=1}^{m} \lambda_j' V\Big)\Big) = \omega \sum_{i,j=1}^{m} (\lambda_i U)*(\lambda_j' V)\\ &= \omega \sum_{i,j=1}^{m} (\lambda_i U)*(\eta_i'\lambda_j' V) = \sum_{i,j=1}^{m} \omega((\lambda_i U)*(\eta_i'\lambda_j' V)),\end{aligned}$$

and thus

$$(\omega(U * V))^\wedge = \sum_{i,j=1}^{m} \hat{\omega}*((\lambda_i U)^\wedge (\eta_j'\lambda_j' V)^\wedge).$$

Since $\lambda_i \in D(Q_i)$ and $\eta_i'\lambda_j' \in D((\tilde{Q}_i)_{\tau(x)})$, we have

$$(\lambda_i U)^\wedge(y)(1+y^2)^{u_i/2} \in L^p(\Gamma_L)$$

and

$$(\eta_i'\lambda_j' V)^\wedge(y)(1+y^2)^{v_i/2} \in L^q(\Gamma_L),$$

and this implies that

$$(\lambda_i U)^{\wedge}(y)(\eta_i' \lambda_j' V)^{\wedge}(y)(1+y^2)^{\alpha/2} \in L^s(\Gamma_L) \quad \text{for } i, j = 1, \ldots, m.$$

We write for simplicity $f_{ij}(y) = (\lambda_i U)^{\wedge}(y)(\eta_i' \lambda_j' V)^{\wedge}(y)$ for $i, j = 1, \ldots, m$. Let L' be a neighbourhood of l in S^{n-1} such that $\overline{L'} \subset L$. We show that

$$(\hat{\omega} * f_{ij})(y)(1+y^2)^{\alpha/2} \in L^s(\Gamma_{L'}).$$

Observe that $\hat{\omega} \in S(\boldsymbol{R}^n)$, $f_{ij} \in O_M(\boldsymbol{R}^n)$. Denoting by $\chi_{\Gamma_{L'}}$ the characteristic function of the set $\Gamma_{L'}$ and by s' the number satisfying the condition $1/s + 1/s' = 1$, we have

$$\begin{aligned}
&\int_{\Gamma_{L'}} [|\hat{\omega} * f_{ij}|(y)(1+y^2)^{\alpha/2}]^s dy \\
&\leqslant \int \Big[\int |f_{ij}(z)|\,|\hat{\omega}(y-z)|(1+y^2)^{\alpha/2} \chi_{\Gamma_{L'}}(y)\, dz\Big]^s dy \\
&\leqslant C_k \int \Big[\int |f_{ij}(z)|(1+y^2)^{\alpha/2} \chi_{\Gamma_{L'}}(y)\big(1+(y-z)^2\big)^{-2k} dz\Big]^s dy \\
&\leqslant C_k \int \Big[\int \Big(|f_{ij}(z)|(1+y^2)^{\alpha/2} \chi_{\Gamma_{L'}}(y)\big(1+(y-z)^2\big)^{-k}\Big)^s dz \Big(\int \big(1+(y-z)^2\big)^{-ks'} dz\Big)^{s/s'}\Big] dy \\
&\leqslant C_k M_k \int |f_{ij}(z)|^s \int (1+y^2)^{s\alpha/2} \chi_{\Gamma_{L'}}(y)\big(1+(y-z)^2\big)^{-ks} dy\, dz
\end{aligned}$$

for sufficiently large $k \in N_0$.

It is not difficult to show that there exist $k \in N_0$ and $C > 0$ such that

$$\int (1+y^2)^{s\alpha/2} \chi_{L'}(y)\big(1+(y-z)^2\big)^{-ks} dy \leqslant C(1+z^2)^{s\alpha/2} \quad \text{for } z \in \Gamma_L$$

and

$$|f_{ij}(z)|^s \int (1+y^2)^{s\alpha/2} \chi_{\Gamma_{L'}}(y)\big(1+(y-z)^2\big)^{-ks} dy \leqslant C(1+z^2)^{-n} \quad \text{for } z \notin \Gamma_L.$$

Then

$$\begin{aligned}
&\int_{\Gamma_{L'}} [|\hat{\omega} * f_{ij}|(y)(1+y^2)^{\alpha/2}]^s dy \\
&\qquad\qquad \leqslant M\Big[\int_{\Gamma_L} |f_{ij}(z)|^s (1+z^2)^{s\alpha/2} dz + \int_{C\Gamma_L} (1+z^2)^{-n} dz\Big] < +\infty.
\end{aligned}$$

Hence

$$\big(\omega(U * V)\big)^{\wedge}(y)(1+y^2)^{\alpha/2} \in L^s(\Gamma_{L'})$$

and since $\alpha^+ \leqslant O^s_{U*V}(x, l)$, we obtain

$$O^s_{U*V}(x, l) \geqslant \min_{t \in Z^\delta_x(U,V)} \big(O^p_U(t, l) + O^q_V(x-t, l)\big) \quad \text{for } \delta > 0.$$

Now we can finish the proof of the theorem. Let $\alpha \in \boldsymbol{R}$ and

$$\alpha^+ \leqslant \min_{t \in Z_x(U,V)} \big(O^p_U(t, l) + O^q_V(x-t, l)\big).$$

Assume that $\alpha^+ > O^s_{U*V}(x, l)$. Using the inequality just proved we infer that $\alpha^+ > \min_{t \in Z^\delta_x(U,V)} (O^p_U(t, l) + O^q_V(x-t, l))$ for $\delta > 0$. So there exists a sequence (t_i) such that $t_i \in Z^{1/i}_x(U, V) - Z_x(U, V)$ and $\alpha^+ > O^p_U(t_i, l) + O^q_V(x-t_i, l)$ for $i = 1, 2, \ldots$ Since all the sets $Z^{1/i}_x(U, V)$ are compact for $i = 1, 2, \ldots$, the sequence (t_i) (or a subsequence, denoted for simplicity also by (t_i)) has a limit $t_0 \in \bigcap_{i=1}^{\infty} Z^{1/i}_x(U, V) = Z_x(U, V)$. At the same time, $\alpha^+ \leqslant O^p_U(t_0, l) + O^q_V(x-t_0, l)$. So there exist numbers $\alpha_1, \alpha_2, \alpha_1^+ \leqslant O^p_U(t_0, l)$, $\alpha_2^+ \leqslant O^q_V(x-t_0, l)$ with $\alpha_1 + \alpha_2 = \alpha$. The definition of the local order function implies: $\alpha_1^+ \leqslant O^p_U(t_i, l)$, $\alpha_2^+ \leqslant O^q_V(x-t_i, l)$ for sufficiently large i. Thus $\alpha^+ \leqslant O^p_U(t_i, l) + O^q_V(x-t_i, l)$ for large i. This contradiction finishes the proof.

Remark 4. Assume the conditions of Theorem 3. Because of symmetry we can write:

$$O^s_{U*V}(x, l) \geqslant \min_{t \in Z_x(V,U)} \big(O^p_U(x-t, l) + O^q_V(t, l)\big).$$

COROLLARY 2. *Let* $U \in E'(\boldsymbol{R}^n)$, $V \in D'(\boldsymbol{R}^n)$, $1 \leqslant p, q, s \leqslant +\infty$ *and* $1/p + 1/q = 1/s$. *Then*

$$O^s_{U*V}(x, l) \geqslant \min_{t \in \operatorname{supp} U} \big(O^p_U(t, l) + O^q_V(x-t, l)\big)$$

for $x \in \boldsymbol{R}^n$, $l \in S^{n-1}$.

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