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Boundary value problem for differential and difference second order systems

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Abstract. The aim of this paper is to present certain theorems concerning the existence and uniqueness of solutions of the problem

(a) $x'' = f(t, x, x')$, $x(0) - Ax'(0) = 0$, $x(1) + Bx'(1) = 0$,

to give analogous theorems for the discrete problem

(b) $\nabla \Delta x_i = g(i, x_i, \Delta x_i)$, $i = 1, \ldots, m-1$, $x_0 - \bar{A}\Delta x_0 = 0$, $x_m + \bar{B}\nabla x_m = 0$,

and to prove a theorem concerning the convergence of solutions of appropriately defined discrete problems of the form (b) to the solution of problem (a).

0. Introduction. The subject of the present paper is the problem of the existence and uniqueness of solutions to the equation

$$x'' = f(t, x, x'), \tag{0.1}$$

$f\colon X = [0, 1] \times R^d \times R^d \to R^d$, satisfying the boundary conditions

$$x(0) = 0, \quad x(1) = 0 \tag{0.2}$$

or, more generally,

$$x(0) - Ax'(0) = 0, \quad x(1) + Bx'(1) = 0. \tag{0.3}$$

A and B are $d \times d$ constant matrices satisfying the condition $A \geqslant 0$, $B \geqslant 0$ (see Notations).

Together with the differential problem we shall be considering the analogous problem for the system of difference equations, i.e., the question of the existence and uniqueness of solutions to difference equations of the form

$$\nabla \Delta x_i = g(i, x_i, \Delta x_i), \quad i = 1, \ldots, m-1, \tag{0.4}$$

$g\colon Y_m = \{0, 1, \ldots, m\} \times R^d \times R^d \to R^d$, satisfying the boundary conditions

$$x_0 = 0, \quad x_m = 0 \tag{0.5}$$

or, more generally,

$$x_0 - \bar{A}\Delta x_0 = 0, \quad x_m + \bar{B}\nabla x_m = 0. \tag{0.6}$$

$\bar{A}$ and $\bar{B}$ are constant $d \times d$ matrices, $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$.

The aim of this paper is to present certain theorems concerning the existence and uniqueness of solutions of problem (0.1), (0.3), to give analogous theorems for the discrete problem (0.4), (0.6) and to prove a theorem concerning the convergence of solutions of appropriately defined discrete problems of the form (0.4), (0.6) to the solution of problem (0.1), (0.3).

Throughout the paper we shall assume that the function f is continuous and there exist constants $K \geqslant 0$, $c \geqslant 0$, satisfying the condition

$$c < (1+\tfrac{1}{2}|A|)^{-1} \tag{0.7}$$

and such that the inequality

$$|v|^2 + \langle f(t, u, v), u\rangle \geqslant -K(|u|+1) - \tfrac{1}{2}c|u|^2 \tag{0.8}$$

holds for every $(t, u, v) \in X$. Further, we shall assume that for every $(t, u, v) \in X$

$$|f(t, u, v)| \leqslant \varrho(t, u)\,(1+|v|^{2-s}), \tag{0.9}$$

where $0 < s \leqslant 2$ and $\varrho\colon [0, 1]\times R^d \to [0, \infty)$ is a function bounded on bounded sets. It can easily be seen that condition (0.8) is fulfilled if the function f satisfies the condition

$$|v-\bar{v}|^2 + \langle f(t, u, v) - f(t, \bar{u}, \bar{v}),\ u-\bar{u}\rangle \geqslant -\tfrac{1}{2}c|u-\bar{u}|^2. \tag{0.10}$$

In addition, we shall assume that the function g is continuous and there exist constants $K_m \geqslant 0$ and $c_m \geqslant 0$ satisfying the condition

$$c_m \leqslant (m+\tfrac{1}{2}|\bar{A}|)^{-1}m^{-1} \tag{0.11}$$

and such that

$$|v|^2 + \langle g(i, u, v), u\rangle \geqslant -K_m(|u|+1) - \tfrac{1}{2}c_m|u|^2 \tag{0.12}$$

holds for $u, v \in R^d$, $i = 1, \ldots, m-1$. As in the case of the function f, the condition

$$|v-\bar{v}|^2 + \langle g(i, u, v) - g(i, \bar{u}, \bar{v}), u-\bar{u}\rangle \geqslant -\tfrac{1}{2}c_m|u-\bar{u}|^2 \tag{0.13}$$

implies inequality (0.12).

Problem (0.1), (0.3) and assumptions of the type (0.8), (0.9) have been studied for the case $d = 1$ by Keller [9], Bebernes and Gaines [1] and others. The case $d > 1$, with the assumption $A = B = 0$, has been the subject of a series of papers by Scorza-Dragoni [14], Ph. Hartman [6], [7], Heimes [8]. The most recent paper in this field was that of A. Lasota and J. A. York [10]. The analogous problems for difference equations have been the subject of far fewer papers (chiefly concerning (0.4), (0.5)) [5], [11], [12], [3], [13].

1. Definitions and notations. We shall denote the Euclidean norm and the scalar product in the d-dimensional space R^d by $|\cdot|$, $\langle\cdot,\cdot\rangle$, respectively. If $b \in (R^d)^{m+1}$, $b = \{b_i\}$, $b_i \in R^d$, then by definition $\|b\| = \max\{|b_i|: i = 0, 1, \ldots, m\}$. As usual, C^d will denote the linear space of all real d-dimensional vector functions defined and continuous in the interval $[0, 1]$, endowed with the norm of uniform convergence $\|x\| = \max\{|x(t)|: t \in [0, 1]\}$. The set $C_1^d \subset C^d$ consists of all continuously differentiable functions. For a $d\times d$ matrix A we will say that A is positive definite and write $A > 0$ if $\langle x, Ax\rangle > 0$ for all x in R^d. We will say $A \geqslant 0$ if either $A > 0$ or A is identically 0. $|A|$ is the norm of the linear operator A from R^d into R^d. The derivative matrices of the vector functions f and g will be denoted accordingly by $f_x, f_{x'}, g_x, g_{\Delta x}$. For the transpose matrix, we shall write f_x^T, $g_{\Delta x}^T$.

Let $x = (x_0, \ldots, x_m) \in (R^d)^{m+1}$. The symbols Δ and ∇ denote the difference operators from $(R^d)^{m+1}$ to $(R^d)^{m+1}$ defined by

$$\Delta x_i = \begin{cases} x_{i+1} - x_i, & i = 0, \ldots, m-1, \\ 0, & i = m, \end{cases}$$

$$\nabla x_i = \begin{cases} x_i - x_{i-1}, & i = 1, \ldots, m, \\ 0, & i = 0. \end{cases}$$

We shall write $\Delta x = (\Delta x_0, \ldots, \Delta x_{m-1}, 0)$, $\nabla x = (0, \nabla x_1, \ldots, \nabla x_n)$.

2. Existence and uniqueness theorems for the differential system. The following theorem will be proved:

THEOREM 2.1. *Let A and B be $d\times d$ constant matrices, $A \geqslant 0$, $B \geqslant 0$, and let K and c be non-negative constants satisfying inequality* (0.7). *If a function $f: X \to R^d$ is continuous and satisfies conditions* (0.8) *and if there are a constant ε, $0 < \varepsilon \leqslant 2$, and a function ϱ: $[0, 1]\times R^d \to [0, \infty)$ bounded on bounded sets such that* (0.9) *holds, then equation* (0.1) *has at least one solution satisfying condition* (0.3).

Remark. The boundary problem

$$y'' = f(t, y, y'), \quad y(0) - Ay'(0) = r_0, \quad y(1) + By'(1) = r_1,$$

$r_0, r_1 \in R^d$, $A \geqslant 0$, $B \geqslant 0$, can be reduced to the homogeneous problem (0.1), (0.3) by the linear substitution $x(t) = y(t) + v_0 + v_1 t$, $t \in [0, 1]$. According to the assumption $A \geqslant 0$, $B \geqslant 0$, the vectors v_0 and v_1 are determined uniquely by the conditions

$$(2.1) \qquad v_0 - Av_1 = -r_0, \quad v_0 + v_1 + Bv_1 = -r_1.$$

Before proceeding to the proof of Theorem 2.1, which is based on the Leray–Schauder alternative, we shall prove two lemmas.

Let $\lambda \in [0, 1]$. Denote by $S(\lambda)$ the set of functions x: $[0, 1] \to R^d$ of class C_2 satisfying the equation

$$x'' = \lambda f(t, x, x') \tag{2.2}$$

and the boundary conditions (0.3) ($A \geqslant 0, B \geqslant 0$).

LEMMA 2.1. *If there is a constant P such that for every $\lambda \in [0, 1]$ and $x \in S(\lambda)$ we have $\|x\| + \|x'\| \leqslant P$, then the set $S(1)$ is non-empty.*

Proof. First, the function $x = 0$ is the unique solution of the equation $x'' = 0$, satisfying the boundary condition (0.3). The solution of this problem is of the form $x(t) = v_0 + v_1 t$, where v_0 and v_1 satisfy conditions (2.1) with $r_0 = r_1 = 0$. The assumption $A \geqslant 0$, $B \geqslant 0$ implies $v_0 = v_1 = 0$. Therefore, in accordance with the general theory of differential equations (see [2]), there exists Green's function G: $[0, 1] \times [0, 1] \to R^{d^2}$, such that equation (2.2) with the boundary conditions (0.3) is equivalent to the integral equation

$$x(t) = \lambda \int_0^1 G(t, s) f\big(s, x(s), x'(s)\big)\, ds. \tag{2.3}$$

According to our assumptions concerning the function f, the mapping

$$x(\cdot) \to \int_0^1 G(\cdot, s) f\big(s, x(s), x'(s)\big)\, ds$$

defined in the space C_1^d (with the norm $\|x\| + \|x'\|$) and taking values in this space, is completely continuous. This fact, as well as the assumptions of the lemma, allow us to apply the Leray–Schauder alternative to complete the proof [4].

The following lemma will be used to prove the boundedness of the set S.

LEMMA 2.2. *Let a, K, L be non-negative constants satisfying the condition*

$$L_0 = 1 - (1 + a)L > 0. \tag{2.4}$$

If a function u: $[0, 1] \to R$ of class C_2, is non-negative and fulfils the inequality

$$u''(t) \geqslant -K\big(\sqrt{2u(t)} + 1\big) - Lu(t), \quad t \in [0, 1] \tag{2.5}$$

and the conditions

$$u'(0) \geqslant 0, \quad u'(1) \leqslant 0, \tag{2.6}$$

$$u(0) \leqslant au'(0), \tag{2.7}$$

then

$$\|u\| \leqslant 2L_0^{-2}(1+a)^2K^2+2L_0^{-1}(1+a)K, \quad (2.8)$$
$$\|u'\| \leqslant 2L_0^{-2}(1+a)K^2+2L_0^{-1}K.$$

Proof. From inequality (2.5) and conditions (2.6) we get

$$(2.9)\qquad |u'(t)| \leqslant \int_0^1 K\left(\sqrt{2u(s)}+1\right)+Lu(s)\,ds, \quad t\in[0,1].$$

Assumption (2.7) implies the inequality

$$(2.10)\qquad u(t)\leqslant \int_0^1 |u'(s)|\,ds+au'(0), \quad t\in[0,1].$$

Approaching the maxima in (2.9) and (2.10) from both sides, we obtain

$$\|u'\|\leqslant K\left(\sqrt{2\|u\|}+1\right)+L\|u\|, \quad \|u\|\leqslant \|u'\|\,(1+a),$$

whence, by (2.4), we can easily deduce the assertion of the lemma.

Proof of Theorem 2.1. Let $x\in S$. We define a function u by

$$(2.11)\qquad u(t)=\tfrac{1}{2}|x(t)|^2, \quad t\in[0,1].$$

The following identities:

$$(2.12)\qquad u'(t)=\langle x(t),x'(t)\rangle,$$
$$(2.13)\qquad u''(t)=\langle x(t),x''(t)\rangle+|x'(t)|^2, \quad t\in[0,1],$$

hold. As a consequence of definition (2.13), equation (2.2) and assumption (0.8), we obtain

$$u''(t)=\langle x(t),\lambda f(t,x(t),x'(t))\rangle+|x'(t)|^2$$
$$\geqslant -\lambda K\left(|x(t)|+1\right)-\lambda|x'(t)|^2-\tfrac{1}{2}\lambda c\,|x(t)|^2+|x'(t)|^2.$$

Since $\lambda\in[0,1]$, we get in view of (2.11)

$$(2.14)\qquad u''(t)\geqslant -K\left(\sqrt{2u(t)}+1\right)-cu(t), \quad t\in[0,1].$$

Since x fulfils the boundary conditions (0.3) and from the assumptions $A\geqslant 0$, $B\geqslant 0$, we have

$$(2.15)\qquad \begin{aligned} u'(0)&=\langle x(0),x'(0)\rangle=\langle Ax'(0),x'(0)\rangle\geqslant 0,\\ u'(1)&=\langle x(1),x'(1)\rangle=\langle -Bx'(1),x'(1)\rangle\leqslant 0.\end{aligned}$$

Further, from (0.3), the assumption $A\geqslant 0$ and by the generalized from of Schwarz's inequality, we obtain

$$u^2(0)=\tfrac{1}{4}\langle x(0),Ax'(0)\rangle^2\leqslant \tfrac{1}{4}\langle Ax(0),x(0)\rangle\,\langle Ax'(0),x'(0)\rangle$$
$$\leqslant \tfrac{1}{4}|A|\,|x(0)|^2\langle x(0),x'(0)\rangle=\tfrac{1}{2}|A|\,u(0)u'(0);$$

thus, writing $a = \frac{1}{2}|A|$, we can rewrite this inequality as

$$u(0) \leqslant a u'(0). \tag{2.16}$$

The definition of a and assumption (0.7) imply condition (2.4) with $L = c$. Hence, and from relations (2.14), (2.15), (2.16), it is apparent that the function u fulfils the assumptions of Lemma 2.2. The assertion of this lemma implies, on account of (2.11) and (2.12), the existence of constans M_1 and M_2 such that

$$\|x\| \leqslant M_1, \quad \|\langle x(\cdot), x'(\cdot)\rangle\| \leqslant M_2 \tag{2.17}$$

holds for every $x \in S$.

We shall now show the common boundedness of the functions x', $x \in S$. Forming the scalar product between (2.2) and $x(t)$ and integrating over the interval $[0, 1]$ we obtain the inequality

$$\left|\int_0^1 \langle x(t), x''(t)\rangle dt\right| \leqslant \int_0^1 |x(t)|\,\big|f\big(t, x(t), x'(t)\big)\big| dt.$$

Integrating by parts and using (2.17), we get

$$\int_0^1 |x'(t)|^2 dt \leqslant 2M_2 + M_1 \int_0^1 \big|f\big(t, x(t), x'(t)\big)\big| dt. \tag{2.18}$$

The properties of ϱ from assumption (0.9) and the first estimate of (2.17) imply the existence of a constant M_3 such that $\varrho\big(t, x(t)\big) \leqslant M_3$, $t \in [0, 1]$, $x \in S$. Therefore by assumption (0.9) we can write

$$\int_0^1 \big|f\big(t, x(t), x'(t)\big)\big| dt \leqslant M_3 \int_0^1 \big(1 + |x'(t)|^{2-\varepsilon}\big) dt. \tag{2.19}$$

Now by (2.18)

$$\int_0^1 |x'(t)|^2 dt \leqslant M_4 + M_5 \int_0^1 |x'(t)|^{2-\varepsilon} dt, \tag{2.20}$$

where M_4, M_5 are suitable constants. Applying Hölder's inequality we obtain

$$\int_0^1 |x'(t)|^2 dt \leqslant M_4 + M_5 \Big[\int_0^1 |x'(t)|^2 dt\Big]^{1-\varepsilon/2}, \quad 0 < \varepsilon \leqslant 2.$$

From this inequality we can deduce the existence of a constant M such that

$$\int_0^1 |x'(t)|^2 dt \leqslant M \quad \text{for } x \in S. \tag{2.21}$$

We will show that a similar condition is also fulfilled in the case of the maximum norm. Consider two cases: firstly $A > 0$ or $B > 0$, secondly $A = 0$ and $B = 0$. If $A > 0$, then there exists the matrix A^{-1} and by the boundary condition (0.3), the identity

$$x'(t) = \int_0^1 x''(s)\,ds + x'(0),$$

equation (2.2) and inequality (2.19), we have

$$|x'(t)| \leqslant M_3 \int_0^1 \left(1 + |x'(t)|^{2-\varepsilon}\right) dt + |A^{-1}|\,|x(0)|.$$

Again making use of Hölder's inequality and definitions (2.17) and (2.21) of the constants M_1 and M, respectively, we obtain the estimate

$$|x'(t)| \leqslant M_3 + M_3\, M^{2-\varepsilon} + |A^{-1}|\, M_1 \quad \text{for } t \in [0,1],\ x \in S.$$

The case $B > 0$ can be treated analogously. Now suppose that $A = 0$, $B = 0$. Then from the boundary conditions (0.3) and the mean value theorem we get

$$|x'(t)| \leqslant d \int_0^1 |x''(t)|\,dt.$$

Proceeding as in the case $A > 0$, we obtain the following inequality: $|x'(t)| \leqslant dM_3 + dM_3\, M^{2-\varepsilon}$ for $t \in [0, 1]$, $x \in S$. Thus, there exists in both cases a constant $\bar{M}$ such that $\|x'\| \leqslant \bar{M}$ for every $x \in S$. This inequality and the first of the relations in (2.17) allow as to apply Lemma 2.1, and this completes the proof of the theorem.

The next theorem concerns the existence and uniquenes of solutions of the boundary value problem (0.1), (0.3).

THEOREM 2.2. *Let A and B be constant matrices, $A \geqslant 0$, $B \geqslant 0$ and let c be a non-negative constant such that condition* (0.7) *holds. If the function f: $X \to R^d$ is continuous and satisfies condition* (0.10) *and condition* (0.9) *with ε and ϱ as in Theorem* 2.1, *then the boundary value problem* (0.1), (0.3) *has exactly one solution.*

Proof. Put $K = \max\{|f(t, 0, 0)|:\ t \in [0, 1]\}$.

From assumption (0.10), putting $\bar{u} = \bar{v} = 0$, we obtain the inequality $|v|^2 + \langle f(t, u, v), u\rangle \geqslant -\frac{1}{2}c|u|^2 - K|u|$ for $(t, u, v) \in X$. Thus we see that f satisfies condition (0.8) of Theorem 2.1. The remaining conditions of Theorem 2.1 are a direct result of our present assumptions. Thus there exists at least one solution of (0.1), (0.3).

To prove the uniqueness, let x and y be two different solutions of (0.1), (0.3). By (0.10) the function $z = x - y$ satisfies the inequality

$$|z'(t)|^2 + \langle z''(t), z(t)\rangle \geqslant -\tfrac{1}{2}c|z(t)|^2, \quad t \in [0, 1]. \tag{2.22}$$

As in the proof of Theorem 2.1, let $\bar{u}(t) = \frac{1}{2}|z(t)|^2$, $t \in [0, 1]$. Using identities analogous to (2.12), (2.13) and inequality (2.22), we obtain the relation

$$\bar{u}''(t) \geqslant -c\bar{u}(t), \qquad t \in [0, 1].$$

Thus $\bar{u}$ satisfies condition (2.5) of Lemma 2.2 with $K = 0$, $L = c$. Since x and y satisfy (0.3), z must also satisfy this condition. Hence, proceeding identically as in the proof of Theorem 2.1, we can verify that $\bar{u}$ satisfies the remaining assumptions of Lemma 2.2. Since $K = 0$, the first inequality in the assertion of Lemma 2.2 gives $\|\bar{u}\| = 0$, which completes the proof of the theorem.

Now let us suppose, additionally, that f has continuous derivatives f_x and $f_{x'}$ satisfying

$$(2.23) \qquad \langle [f_x(t, u, v) - \tfrac{1}{4} f_{x'}(t, u, v) f_{x'}^T(t, u, v)] z, z\rangle \geqslant -\tfrac{1}{2}c|z|^2$$

for $t \in [0, 1]$, $u, v, z \in R^d$, where c is a non-negative constant.

THEOREM 2.3. *Let A, B be constant matrices, $A \geqslant 0$, $B \geqslant 0$, and let c be a non-negative constant satisfying* (0.7). *If the function $f\colon X \to R^d$ is continuous and has continuous derivatives f_x, $f_{x'}$ satisfying* (2.23) *and if there exists a function ϱ and a constant ε with properties as in Theorem* 2.1 *and such that inequality* (0.9) *holds, then the boundary problem* (0.1), (0.3) *has exactly one solution.*

Proof. By Taylor's formula, we have

$$(2.24) \qquad f(t, u, v) - f(t, \bar{u}, \bar{v}) = F(t)(u - \bar{u}) + H(t)(v - \bar{v}),$$

where the matrices $F(t)$ and $H(t)$ are given by the identities

$$F(t) = F(t, u, \bar{u}, v, \bar{v}) = \int_0^1 f_x(t, \cdot, \cdot)\, ds,$$

$$H(t) = H(t, u, \bar{u}, v, \bar{v}) = \int_0^1 f_{x'}(t, \cdot, \cdot)\, ds,$$

and the argument of the matrices of the derivatives f_x, $f_{x'}$ is the expression $\big(t, su + (1-s)\bar{u}, sv + (1-s)\bar{v}\big)$. It is easily checked that for any square matrices F and H and for any vectors a, b the following identity holds:

$$(2.25) \qquad \langle Fa + Hb, a\rangle + |b|^2 = |b + \tfrac{1}{2}H^T a|^2 + \langle [F - \tfrac{1}{4}HH^T]a, a\rangle.$$

Applying this identity to the matrices F and H and to the vectors $u - \bar{u}$, $v - \bar{v}$ and using assumption (2.23), we obtain the following inequality

$$\langle F(t)(u - \bar{u}) + H(t)(v - \bar{v}), u - \bar{u}\rangle \geqslant -|v - \bar{v}|^2 - \tfrac{1}{2}c|u - \bar{u}|^2.$$

Hence and from equation (2.24) we see that assumption (0.10) is satisfied. Since the remaining assumptions of Theorem 2.2 are results of the assumptions of this theorem, the proof is complete.

3. Existence and uniqueness theorems for difference systems. In this chapter we shall derive the analogues of the theorems and lemmas of the previous section for difference systems. Although the methods applied will be similar to those used for the continuous case, in view of the different situation and of the requirements of the next section we shall give the proofs of the theorems and lemmas in their entirely.

The equivalent of Theorem 2.1 for the discrete case is the following theorem.

THEOREM 3.1. *Let $\bar{A}$, $\bar{B}$ be constant square matrices satisfying the condition $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$, and let K_m, c_m be non-negative constants obeying* (0.11). *If the function $g\colon Y_m \to R^d$ is continuous and fulfils condition* (0.12), *then the boundary problem* (0.4), (0.6) *has at least one solution.*

Remark. The boundary problem

$$\begin{aligned} \nabla\Delta y_i &= g(i, y_i, \Delta y_i), \quad i = 1, \ldots, m-1, \\ y_0 - \bar{A}\Delta y_0 &= r_0, \\ y_m + \bar{B}\nabla y_m &= r_1, \quad r_0, r_1 \in R^d, \end{aligned}$$

can be reduced to a problem with homogeneous conditions (0.4), (0.6) by substitution $x_i = y_i + v_0 + iv_1$, $i = 0, \ldots, m$. The vectors v_0 and v_1 are given by the following conditions

$$\begin{aligned} v_0 - \bar{A}v_1 &= -r_0, \\ v_0 + mv_1 + \bar{B}v_1 &= -r_1. \end{aligned} \tag{3.1}$$

The assumption $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$ determines mutual correspondence between vectors r_0, r_1 and vectors v_0, v_1.

Similarly to the proof of Theorem 2.1, the proof of Theorem 3.1 will be based on the Leray–Schauder alternative. Again, this proof will be preceded by some lemmas.

Denote by $S(\lambda)$, $\lambda \in [0, 1]$, the set of all solutions to the equation

$$\nabla\Delta x_i = \lambda g(i, x_i, \Delta x_i), \quad i = 1, \ldots, m-1, \tag{3.2}$$

satisfying the boundary conditions (0.6), and let $S = \bigcup_{\lambda\in[0,1]} S(\lambda)$.

LEMMA 3.1. *Let $\bar{A}$, $\bar{B}$ be matrices, $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$. If there exists a constant P such that*

$$\|x\| \leqslant P \quad \textit{for every } x = (x_0, \ldots, x_m) \in S, \tag{3.3}$$

then the set $S(1)$ is non-empty.

Proof. The assumption $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$ guarantees that the unique solution of the equation $\nabla \Delta x_i = 0$, $i = 1, \ldots, m-1$, satisfying the boundary conditions (0.6) is the vector $x_i = 0$, $i = 0, \ldots, m$. Indeed, the solution of this problem has the form

$$x_i = v_0 + iv_1, \quad i = 0, \ldots, m,$$

where the vectors $v_0, v_1 \in R^d$ satisfy system (3.1). The conditions $r_0 = 0$, $r_1 = 0$ and $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$, determine the relations $v_0 = 0$, $v_1 = 0$. Thus there exists Green's matrix G_{ij}, $i, j = 0, \ldots, m$, for problem (3.1), (0.6). This problem is equivalent to the equation

$$x_i = \lambda \sum_{j=0}^{m-1} G_{ij} g(j, x_j, \Delta x_j), \quad i = 0, \ldots, m. \tag{3.4}$$

We will simplify this equation in the following way: $x = \lambda G(x)$, where G is a continuous and therefore also a completely continuous transformation of the space $(R^d)^{m+1}$ into itself. Assumption (3.1) states that all solutions of this equation for $\lambda \in [0, 1]$ are bounded by a common constant. We can now apply the Leray–Schauder alternative. Its assertion, in view of the fact that equation (3.4) with $\lambda = 1$ is equivalent to problem (0.4), (0.6), completes the proof of the lemma.

Let $x \in (R^d)^{m+1}$, $x = (x_0, \ldots, x_m)$. We define a transformation of $(R^d)^{m+1}$ into R^{m+1} in the following way:

$$u_i = \tfrac{1}{2}|x_i|^2, \quad i = 0, \ldots, m. \tag{3.5}$$

We obtain the identities

$$\Delta u_i = \tfrac{1}{2}\langle \Delta x_i, x_{i+1} + x_i \rangle, \quad i = 0, \ldots, m-1, \tag{3.6}$$

$$\nabla \Delta u_i = \langle \nabla \Delta x_i, x_i \rangle + |\Delta x_i|^2 - \tfrac{1}{2}(|\Delta x_i|^2 - |\nabla x_i|^2), \tag{3.7}$$

$i = 1, \ldots, m-1$. Definitions (3.5), (3.6) also allow us to use the following identities:

$$\Delta u_i = \tfrac{1}{2}|\Delta x_i|^2 + \langle \Delta x_i, x_i \rangle, \quad i = 0, \ldots, m-1, \tag{3.8}$$

$$\nabla u_i = -\tfrac{1}{2}|\nabla x_i|^2 + \langle \nabla x_i x_i \rangle, \quad i = 1, \ldots, m. \tag{3.9}$$

Lemma 3.2. *Let $\alpha, \beta, \gamma, K, L$ be non-negative constants satisfying the condition*

$$L_0 = 1 - (m + \alpha)mL > 0. \tag{3.10}$$

If the vector $u = (u_0, \ldots, u_m) \in R^{m+1}$ given by equation (3.5) satisfies the inequality

$$\nabla \Delta u_i \geqslant -K\left(\sqrt{2u_i} + 1\right) - Lu_i - \tfrac{1}{2}(|\Delta x_i|^2 - |\nabla x_i|^2), \tag{3.11}$$

$i = 1, \ldots, m-1$, *and the relations*

(3.12) $$\nabla u_m + \tfrac{1}{2}|\nabla x_m|^2 \leqslant \gamma, \qquad \Delta u_0 \geqslant -\gamma,$$

(3.13) $$u_0 \leqslant a\Delta u_0 + \beta,$$

then

(3.14) $$\|u\| \leqslant 2[L_0^{-1}(m+a)mK]^2 + 2L_0^{-1}[(m+a)mK + (m+a)\gamma + \gamma + \beta].$$

Proof. Summing (3.11) on both sides over $j = i+1, \ldots, m-1$ we obtain the inequality

$$\Delta u_{m-1} - \Delta u_i \geqslant -\sum_{j=i+1}^{m-1} \left(K\sqrt{2u_j} + K + Lu_j\right) - \tfrac{1}{2}|\Delta x_{m-1}|^2 + \tfrac{1}{2}|\Delta x_i|^2$$

for $i = 0, \ldots, m-2$. Thus, in virtue of assumption (3.12), estimating u_j by $\|u\|$ and omitting the expression $\frac{1}{2}|\Delta x_i|^2$, we obtain the relation

(3.15) $$\Delta u_i \leqslant m(K\sqrt{2\|u\|} + K + L\|u\|) + \gamma, \qquad i = 0, \ldots, m-2.$$

The formula

$$u_i = \sum_{j=0}^{i-1} \Delta u_j + u_0, \qquad i = 1, \ldots, m-1$$

gives, on account of assumption (3.13) and relation (3.15), the inequality

(3.16) $$u_i \leqslant (m+a)m(K\sqrt{2\|u\|} + K + L\|u\|) + (m+a)\gamma + \beta$$

for $i = 1, \ldots, m-1$. From assumptions (3.12) we obtain

$$u_0 \leqslant u_1 + \gamma, \qquad u_m \leqslant u_{m-1} + \gamma,$$

which together with (3.16) justifies the estimate

$$\|u\| \leqslant (m+a)m(K\sqrt{2\|u\|} + K + L\|u\|) + (m+a)\gamma + \beta + \gamma.$$

Further, by assumption (3.10), we can reduce this inequality to the form

$$\|u\| \leqslant L_0^{-1}(m+a)mK\sqrt{2\|u\|} + L_0^{-1}[(m+a)mK + (m+a)\gamma + \gamma + \beta]$$

which, after simple calculation, gives the assertion of the lemma.

Proof of Theorem 3.1. Now let $x = (x_0, \ldots, x_m) \in S$. Define a vector $u = (u_0, \ldots, u_m) \in R^{m+1}$ by (3.5), (3.6), (3.7). We shall show that it satisfies all the assumptions of Lemma 3.2. Identity (3.7) and equation (3.2) give the relation

$$\nabla\Delta u_i = \lambda\langle g(i, x_i, \Delta x_i), x_i\rangle + |\Delta x_i|^2 - \tfrac{1}{2}(|\Delta x_i|^2 - |\nabla x_i|^2)$$

for $i = 1, \ldots, m-1$ from which, by assumption (0.12) and the definition of vector u, follows the inequality

(3.17) $$\nabla\Delta u_i \geqslant -K_m\left(\sqrt{2u_i} + 1\right) - c_m u_i - \tfrac{1}{2}(|\Delta x_i|^2 - |\nabla x_i|^2)$$

for $i = 1, \ldots, m-1$. Since $x \in S$, the vector x satisfies the boundary condition (0.6), and since also by definition $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$, then

$$\langle x_0, \Delta x_0\rangle = \langle \bar{A}\Delta x_0, \Delta x_0\rangle \geqslant 0,$$
$$\langle x_m, \nabla x_m\rangle = \langle -\bar{B}\nabla x_m, \nabla x_m\rangle \leqslant 0.$$

From this and from identities (3.8) and (3.9) we obtain the relations

(3.18) $$\Delta u_0 \geqslant \langle \Delta x_0, x_0\rangle \geqslant 0, \qquad \nabla u_m + \tfrac{1}{2}|\nabla x_m|^2 = \langle \nabla x_m x_m\rangle \leqslant 0.$$

Also the assumption $\bar{A} \geqslant 0$ and Schwarz's inequality yield the relations

$$\begin{aligned} u_0^2 &= \tfrac{1}{4}\langle x_0, x_0\rangle^2 = \tfrac{1}{4}\langle x_0, \bar{A}\Delta x_0\rangle^2 \\ &\leqslant \tfrac{1}{4}\langle \bar{A}x_0, x_0\rangle \langle \bar{A}\Delta x_0, \Delta x_0\rangle \leqslant \tfrac{1}{4}|\bar{A}|\, |x_0|^2\langle x_0, \Delta x_0\rangle \\ &= \tfrac{1}{2}|\bar{A}|\langle x_0, \Delta x_0\rangle u_0 \end{aligned}$$

which, together with identity (3.8), give the estimate

(3.19) $$u_0 \leqslant \tfrac{1}{2}|\bar{A}|\langle x_0, \Delta x_0\rangle \leqslant \tfrac{1}{2}|\bar{A}|\,\Delta u_0 = \alpha u_0 + \beta,$$

where $\alpha = \frac{1}{2}|\bar{A}|$, $\beta = 0$. Note, however, that assumption (0.11) and the definition of α imply condition (3.10) with the constant $L = c_m$. This fact, as well as the relations deduced (3.17), (3.18) and (3.19), allow us to apply Lemma 3.2, which asserts the boundedness of the set S. Applying now Lemma 3.1, we obtain the assertion of the theorem.

The question of the existence and uniqueness of solutions to the boundary problem (0.4), (0.6) is the subject of the following theorem:

THEOREM 3.2. *Let $\bar{A}$ and $\bar{B}$ be constant $d \times d$ matrices, $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$, and let c_m be a non-negative constant fulfilling condition* (0.11). *If the function $g\colon Y_m \to R^d$ is continuous and satisfies condition* (0.13), *then the boundary problem* (0.4), (0.6) *has exactly one solution.*

Proof. Let K_m be a constant such that $|g(i, 0, 0)| \leqslant K_m$ for $i = 1, \ldots$ $\ldots, m-1$. By assumption (0.13), putting $\bar{u} = \bar{v} = 0$, it follows that

$$|v|^2 + \langle g(i, u, v), u\rangle \geqslant -\tfrac{1}{2}c_m|u|^2 - K_m|u|, \qquad i = 1, \ldots, m-1,$$

$u, v \in R^d$. This inequality and assumption (0.11) allow us to apply Theorem 3.1, which completes the proof of the existence of a solution to problem (0.4), (0.6).

To prove the uniqueness, let $x = (x_0, \ldots, x_m)$, $y = (y_0, \ldots, y_m)$ be two different solutions of the boundary problem (0.4), (0.6). The vector $z_i = x_i - y_i$, $i = 0, \ldots, m$, fulfils the boundary condition (0.6) and the equations

$$\nabla\Delta z_i = g(i, x_i, \Delta x_i) - g(i, y_i, \Delta y_i), \qquad i = 1, \ldots, m-1.$$

Hence and from (0.13)

(3.20) $$\langle \nabla\Delta z_i, z_i\rangle \geqslant -\tfrac{1}{2}c_m|z_i|^2 - |\Delta z_i|^2, \qquad i = 1, \ldots, m-1.$$

Define a vector $v = (v_0, \ldots, v_m) \in R^{m+1}$ by

$$v_i = \tfrac{1}{2}|z_i|^2, \quad i = 0, \ldots, m.$$

A relation analogous to (3.7) and inequality (3.20) produce the following:

$$\nabla\Delta v_i \geqslant -c_m v_i - \tfrac{1}{2}(|\Delta z_i|^2 - |\nabla z_i|^2), \quad i = 1, \ldots, m-1.$$

Thus the vector v satisfies condition (3.11) of Lemma 3.2 with the constants $K = 0$, $L = c_m$. By definition, c_m obeys (0.11). From the fact that the vector z fulfils the boundary condition (0.6) we can deduce (identically as in the proof of Theorem 3.1, as regards vector u) that v satisfies the remaining assumptions of Lemma 3.2 with $\alpha = \frac{1}{2}|\bar{A}|$, $\beta = 0$, $\gamma = 0$. Since $K_m = 0$, applying Lemma 3.2, we obtain $\|v\| = 0$ which, together with the definitions of vectors v and z, completes the proof of uniqueness, and thus also the proof of Theorem 3.2.

In addition, let us assume, for use in the next theorem, that the function g has continuous derivatives g_x, $g_{\Delta x}$ with respect to the second and third variables, satisfying the condition

$$(3.21) \qquad \langle [g_x(i, u, v) - \tfrac{1}{4} g_{\Delta x}(i, u, v) g_{\Delta x}^T(i, u, v)]z, z\rangle \geqslant -\tfrac{1}{2} c_m |z|^2,$$

$i = 1, \ldots, m-1$, where c_m is a non-negative constant, u, v, $z \in R^d$.

The following theorem is an analogue of Theorem 2.3 for the discrete case.

THEOREM 3.3. *Let $\bar{A}$, $\bar{B}$ be constant $d \times d$ matrices, $\bar{A} \geqslant 0$, $\bar{B} \geqslant 0$. If the function $g\colon Y_m \to R^d$ is continuous and has continuous derivatives g_x, $g_{\Delta x}$ for $i = 1, \ldots, m-1$, satisfying* (3.21), *in which the constant c_m obeys* (0.11), *then the boundary problem* (0.4), (0.6) *has exactly one solution.*

Proof. We shall verify that the assumptions of Theorem 3.2 are fulfilled. From Taylor's formula

$$(3.22) \qquad g(i, u, v) - g(i, \bar{u}, \bar{v}) = F_i(u - \bar{u}) + H_i(v - \bar{v}),$$

where the matrices F_i and H_i are given by

$$F_i = F_i(u, \bar{u}, v, \bar{v}) = \int_0^1 g_x(i, \cdot, \cdot)\, ds,$$

$$H_i = H_i(u, \bar{u}, v, \bar{v}) = \int_0^1 g_{\Delta x}(i, \cdot, \cdot)\, ds, \quad i = 1, \ldots, m-1,$$

and the argument of integrands is $\big(i, su + (1-s)\bar{u}, sv + (1-s)\bar{v}\big)$. From identity (2.25) and assumption (3.21) it follows that

$$\langle F_i(u - \bar{u}) + H_i(v - \bar{v}), u - \bar{u}\rangle \geqslant -|v - \bar{v}|^2 - \tfrac{1}{2} c_m |u - u|^2.$$

Hence and from (3.22) we obtain (0.13). Since, by definition, the constant c_m obeys condition (0.11), then by Theorem 3.2 the proof is complete.

4. An approximation theorem. In parallel with the differential boundary problem (0.1), (0.3), let us consider a sequence of difference boundary problems of the form

$$(4.1) \qquad \nabla\Delta z_i = h_m^2 f\left(t_i^m, z_i, \frac{\Delta z_i}{h_m}\right), \quad i = 1, \ldots, m-1,$$

$$(4.2) \qquad z_0 - A\frac{\Delta z_0}{h_m} = 0, \quad z_m + B\frac{z_m}{h_m} = 0,$$

where m runs over natural numbers and h_m and t_i^m are given by

$$h_m = \frac{1}{m}, \quad t_i^m = ih_m, \quad i = 0, \ldots, m.$$

We shall now prove the following theorem:

THEOREM 4.1. *Let A, B be constant $d\times d$ matrices, $A \geqslant 0$, $B \geqslant 0$, and let c be a non-negative constant satisfying* (0.7). *If the function $f\colon X\to R^d$ is continuous and fulfils condition* (0.10) *with the constant c, and if there exist a constant ε, $0 < \varepsilon \leqslant 2$, and a function $\varrho\colon [0,1]\times R^d \to [0,\infty)$ bounded on bounded set, such that* (0.9) *holds, then*

1° *for each natural m there exists exactly one solution $z^m = (z_0^m, \ldots, z_m^m) \in (R^d)^{m+1}$ of problem* (4.1), (4.2),

2° $\lim_{m\to\infty} |z_i^m - x(t_i^m)| = 0$ *uniformly with respect to i, where x is the unique solution to the boundary differential problem* (0.1), (0.3).

Proof. From our assumptions it follows by Theorem 2.2 that there exists exactly one solution x to problem (0.1), (0.3).

To prove the first assertion of the theorem, we shall use Theorem 3.2. We define a function g^m by the formula

$$g^m(i, u, v) = h_m^2 f\left(t_i^m, u, \frac{v}{h_m}\right), \quad i = 0, \ldots, m,$$

$u, v \in R^d$, and we put in the boundary conditions (0.6)

$$\bar{A} = \frac{A}{h_m}, \quad \bar{B} = \frac{B}{h_m}.$$

A direct result of assumption (0.10) is that the function g^m fulfils condition (0.13) of Theorem 3.2 with the constant $c_m = h_m^2 c$. Since the constant c satisfies condition (0.7), then

$$(4.3) \qquad c_m = h_m^2 c < (1+\tfrac{1}{2}|A|)^{-1} m^{-2} = (m+\tfrac{1}{2}|\bar{A}|)^{-1} m^{-1}.$$

Thus the assumptions of Theorem 3.2 are satisfied and therefore for every m there exists exactly one solution $z^m = (z_0^m, \ldots, z_m^m) \in (R^d)^{m+1}$, of the boundary problem (4.1), (4.2).

To prove the second assertion we first show that all solutions z^m are commonly bounded, that is, there exists a constant $\bar{N}$ such that

$$\|z^m\| \leqslant \bar{N} \quad \text{for any natural } m. \tag{4.4}$$

Define a sequence of vectors $v^m = (v_0^m, \ldots, v_m^m) \in R^{m+1}$ by the relation $v_i^m = \frac{1}{2}|z_i^m|^2$, $i = 0, \ldots, m$.

Equation (4.1) and assumption (0.10) give the inequality

$$\langle z_i^m, \nabla\Delta z_i^m\rangle = \left\langle z_i^m, h_m^2 f\left(t_i^m, z_i^m, \frac{\Delta z_i^m}{h_m}\right)\right\rangle \geqslant -h_m^2\bar{K}|z_i^m| - h_m^2\frac{c}{2}|z_i^m|^2 - |\Delta z_i^m|^2,$$

where $\bar{K} = \max\{|f(t, 0, 0)|:\ t \in [0, 1]\}$. Hence and from a formula analogous to (3.7), the vector v^m satisfies the inequality

$$\nabla\Delta v_i^m \geqslant -h_m^2\bar{K}\sqrt{2v_i^m} - h_m^2 c|v_i^m| - \tfrac{1}{2}(|\Delta z_i^m|^2 - |\nabla z_i^m|^2), \tag{4.5}$$

$i = 1, \ldots, m-1$. The boundary conditions (4.2) and identities similar to (3.8), (3.9) imply the relations

$$\begin{aligned} \Delta v_0^m &\geqslant \langle \Delta z_0^m, z_0^m\rangle = \frac{1}{h_m}\langle A\Delta z_0^m, \Delta z_0^m\rangle \geqslant 0, \\ \nabla v_m^m + \tfrac{1}{2}|\nabla z_m^m|^2 &= \langle \nabla z_m^m, z_m^m\rangle = -\frac{1}{h_m}\langle B\nabla z_m^m, \nabla z_m^m\rangle \leqslant 0. \end{aligned} \tag{4.6}$$

From the first condition of (4.2) and Schwarz's inequality we have

$$\begin{aligned} (v_0^m)^2 &= \frac{1}{4h_m^2}\langle A\Delta z_0^m, z_0^m\rangle^2 \leqslant \frac{1}{4h_m^2}\langle A\Delta z_0^m, \Delta z_0^m\rangle\langle A z_0^m, z_0^m\rangle \\ &\leqslant \frac{1}{2h_m}|A|\langle z_0^m, \Delta z_0^m\rangle v_0^m, \end{aligned}$$

and hence, and from an identity similar to (3.8), we obtain the following inequalities:

$$v_0^m = \tfrac{1}{2}|z_0^m|^2 \leqslant \frac{1}{2h_m}|A|\langle z_0^m, \Delta z_0^m\rangle \leqslant \frac{1}{2h_m}|A|\Delta v_0^m. \tag{4.7}$$

Relations (4.5), (4.6), and (4.7) and inequality (4.3) show that the vector v^m satisfies the assumptions of Lemma 3.2 with constants $K = h_m^2\bar{K}$, $L = h_m^2 c$, $\alpha = 2h_m^{-1}|A|$, $\beta = \gamma = 0$. Applying this lemma, we obtain the estimation

$$\|v^m\| \leqslant 2[L_0^{-1}(1 + \tfrac{1}{2}|A|)\bar{K}]^2 + 2L_0^{-1}(1 + \tfrac{1}{2}|A|)\bar{K},$$

where $L_0 = 1 - (1 + \frac{1}{2}|A|)c$, which completes the proof of property (4.4).

For brevity, we shall write $x_i^m = x(t_i^m)$, where x is the solution of the differential problem (0.1), (0.3).

From Taylor's integral formula we have

$$\Delta x_i^m = \int_{t_i^m}^{t_{i+1}^m} x'(t)\,dt, \qquad \nabla\Delta x_i^m = \int_{t_{i-1}^m}^{t_{i+1}^m} x''(t)\,(h_m - |t-t_i^m|)\,dt,$$

$i = 1, \ldots, m-1$. The vector $x^m = (x_0^m, \ldots, x_m^m)$ satisfies the equation

$$\nabla\Delta x_i^m = h_m^2 f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right) + r_i^m, \qquad i = 1, \ldots, m-1, \tag{4.8}$$

where

$$r_i^m = \int_{t_{i-1}^m}^{t_{i+1}^m} x''(t)\,(h_m - |t-t_i^m|)\,dt - h_m^2 f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right).$$

Using the fact that x is the solution to equation (0.1), we shall write the above formula in the following form:

$$r_i^m = \int_{t_{i-1}^m}^{t_{i+1}^m} \left[f\big(t, x(t), x'(t)\big) - f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right)\right](h_m - |t-t_i^m|)\,dt.$$

The continuity of the functions f, x, x' implies the existence, for any $\eta > 0$, of an index m_0 such that for $m \geqslant m_0$ the following inequality holds:

$$\left|f\big(t, x(t), x'(t)\big) - f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right)\right| < \eta, \qquad i = 1, \ldots, m-1,$$

$t \in [t_{i-1}^m, t_{i+1}^m]$. Hence

$$|r_i^m| \leqslant \eta \int_{t_{i-1}^m}^{t_{i+1}^m} (h_m - |t-t_i^m|)\,dt = \eta h_m^2, \qquad m \geqslant m_0.$$

The last relation implies

$$\lim_{m\to\infty} m^2 r^m = 0, \tag{4.9}$$

where

$$r^m = \max\{|r_i^m|:\ i = 1, \ldots, m-1\}. \tag{4.10}$$

By definition, let $y^m = z^m - x^m$, or $y_i^m = z_i^m - x_i^m$, $i = 0, \ldots, m$. From property (4.4) and the definition of vector x^m, there exists a constant $\bar{N}_1$ such that

$$\|y^m\| \leqslant \bar{N}_1 \quad \text{for any } m. \tag{4.11}$$

In view of equations (4.1), (4.8), the vector y^m must satisfy the equation

$$(4.12)\qquad \nabla\Delta y_i^m = h_m^2\left[f\left(t_i^m, z_i^m, \frac{\Delta z_i^m}{h_m}\right) - f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right)\right] - r_i^m,$$

$i = 1, \ldots, m-1$. As in the previous proofs, we shall define the vectors $u^m = (u_0^m, \ldots, u_m^m) \in R^{m+1}$ by the formula

$$u_i^m = \tfrac{1}{2}|y_i^m|^2, \qquad i = 0, \ldots, m.$$

To complete the proof of the theorem, we shall check that the vectors u^m behave according to the assumptions of Lemma 3.2.

Assumption (0.9) and definition (4.10) imply

$$\left\langle h_m^2\left[f\left(t_i^m, z_i^m, \frac{\Delta z_i^m}{h_m}\right) - f\left(t_i^m, x_i^m, \frac{\Delta x_i^m}{h_m}\right)\right] - r_i^m, z_i^m - x_i^m\right\rangle$$
$$\geqslant -h_m^2\tfrac{1}{2}c|z_i^m - x_i^m|^2 - |\Delta z_i^m - \Delta x_i^m|^2 - r^m|z_i^m - x_i^m|,$$

$i = 1, \ldots, m-1$, and hence, from (4.12) and a formula analogous to (3.7) the vector u^m obeys the inequality

$$(4.13)\qquad \nabla\Delta u_i^m \geqslant -r^m\sqrt{2u_i^m} - h_m^2 c u_i^m - \tfrac{1}{2}(|\Delta y_i^m|^2 - |\nabla y_i^m|^2),$$

$i = 1, \ldots, m-1$.

Put

$$(4.14)\qquad a_m = y_0^m - A\frac{\Delta y_0^m}{h_m}, \qquad b_m = y_m^m + B\frac{\nabla y_m^m}{h_m}.$$

From the definitions of y^m and x^m we have the following relations:

$$a_m = z_0^m - A\frac{\Delta z_0^m}{h_m} - [x(0) - Ax'(0)] + A\left[\frac{\Delta x_0^m}{h_m} - x'(0)\right],$$
$$b_m = z_m^m + B\frac{\nabla z_m^m}{h_m} - [x(1) + Bx'(1)] + B\left[x'(1) - \frac{\nabla x_m^m}{h_m}\right],$$

and since the function x and vector z^m fulfil, respectively, the boundary conditions (0.3) and (4.2), then

$$(4.15)\qquad a_m = A\left[\frac{\Delta x_0^m}{h_m} - x'(0)\right] = A\left[\frac{x(h_m) - x(0)}{h_m} - x'(0)\right],$$

$$(4.16)\qquad b_m = B\left[x'(1) - \frac{\nabla x_m^m}{h_m}\right] = B\left[x'(1) - \frac{x(1) - x(1-h_m)}{h_m}\right].$$

Now, let $\eta > 0$ be any real number. If $A > 0$, then there exists $\omega > 0$ such that $\langle Av, v\rangle \geqslant \omega^2|v|^2$ for $v \in R^d$.

From equation (4.15) we obtain for sufficiently large m the inequality

$$|a_m| \leqslant 2\omega\sqrt{\eta}. \tag{4.17}$$

From (4.14) we have

$$\langle y_0^m, \Delta y_0^m \rangle = \frac{1}{h_m}\langle A\Delta y_0^m, \Delta y_0^m\rangle + \langle a_m, \Delta y_0^m\rangle,$$

and hence, on account of (4.17), we obtain the estimate

$$\langle y_0^m, \Delta y_0^m\rangle \geqslant \frac{\omega^2}{h_m}|\Delta y_0^m|^2 - 2\omega\sqrt{\eta}|\Delta y_0^m| \geqslant -h_m\eta.$$

The identity

$$\Delta u_0^m = \tfrac{1}{2}|\Delta y_0^m|^2 + \langle y_0^m, \Delta y_0^m\rangle \tag{4.18}$$

gives the inequality

$$\Delta u_0^m \geqslant -h_m\eta \tag{4.19}$$

for sufficiently large m.

If $A = 0$, then from the boundary conditions (0.3) and (4.2), we have $x_0^m = 0$ and $z_0^m = 0$, and thus $y_0^m = 0$. Thus also in this case, identity (4.18) implies inequality (4.19). For matrix B we proceed similarly. If $B > 0$, $\langle Bv, v\rangle \geqslant \varpi^2|v|^2$ for any $v \in R^d$, then from (4.16), for sufficiently large m

$$|b_m| \leqslant 2\varpi\sqrt{\eta}. \tag{4.20}$$

From (4.14) and (4.20), we obtain

$$\langle y_m^m, \nabla y_m^m\rangle \leqslant -\frac{\varpi^2}{h_m}|\nabla y_m^m|^2 + 2\varpi\sqrt{\eta}|\nabla y_m^m| \leqslant h_m\eta$$

or sufficiently large m. Applying identity

$$\nabla u_m^m + \tfrac{1}{2}|\nabla y_m^m|^2 = \langle y_m^m, \nabla y_m^m\rangle, \tag{4.21}$$

we derive the inequality

$$\nabla u_m^m + \tfrac{1}{2}|\nabla y_m^m|^2 \leqslant h_m\eta \tag{4.22}$$

which is true for sufficiently large m.

If $B = 0$, then from the boundary conditions (0.3) and (4.2), we have $y_m^m = 0$, which together with identity (4.21) gives (4.22).

We now verify that the vector u^m fulfils condition (3.13) of Lemma 3.2. Consider two cases: $A > 0$ and $A = 0$. When $A > 0$, from definition (4.14) of the vector a_m we have the equation

$$\langle y_0^m - a_m, y_0^m\rangle^2 = \frac{1}{h_m^2}\langle A\Delta y_0^m, y_0^m\rangle^2. \tag{4.23}$$

Estimating the right-hand side of (4.23) on the basis of (4.14) and Schwarz's inequality, we obtain

$$\frac{1}{h_m^2}\langle A\Delta y_0^m, y_0^m\rangle^2 \leqslant \frac{1}{h_m^2}\langle A\Delta y_0^m, \Delta y_0^m\rangle\langle A y_0^m, y_0^m\rangle$$
$$\leqslant \frac{1}{h_m}\left\langle A\frac{\Delta y_0^m}{h_m}, \Delta y_0^m\right\rangle |A|\, |y_0^m|^2 = \frac{1}{h_m}|A|\, |y_0^m|^2\langle y_0^m - a_m, \Delta y_0^m\rangle.$$

Hence, by (4.11) and (4.15), we have

$$(4.24)\qquad \frac{1}{h_m^2}\langle A\Delta y_0^m, y_0^m\rangle^2 \leqslant \frac{1}{h_m}|A|\, |y_0^m|^2\langle y_0^m, \Delta y_0^m\rangle + 2\eta^2$$

for sufficiently large m. The same relations (4.11) and (4.15) allow us to estimate the left-hand side of equation (4.23):

$$(4.25)\qquad \langle y_0^m - a_m, y_0^m\rangle^2 = |y_0^m|^4 - 2|y_0^m|^2\langle a_m, y_0^m\rangle + \langle a_m, y_0^m\rangle^2$$
$$\geqslant |y_0^m|^4 - 2\eta^2$$

for sufficiently large m. Collecting together the deduced relations (4.23), (4.24), (4.25), we obtain the inequality

$$|y_0^m|^4 \leqslant \frac{1}{h_m}|A|\langle y_0^m, \Delta y_0^m\rangle |y_0^m|^2 + 4\eta^2$$

for sufficiently large m. Hence after calculation we obtain

$$|y_0^m|^2 \leqslant \frac{1}{h_m}|A|\langle y_0^m, \Delta y_0^m\rangle + 2\eta.$$

The definition of the vector u^m and identity (4.18) yield the inequality

$$(4.26)\qquad u_0^m = \tfrac{1}{2}|y_0^m|^2 \leqslant \frac{1}{2h_m}|A|\langle y_0^m, \Delta y_0^m\rangle + \eta \leqslant a\Delta u_0^m + \eta,$$

where $a = \dfrac{1}{2h_m}|A|$.

In the case when $A = 0$, we have $u_0^m = 0$ and $|A| = 0$, and (4.26) becomes trivial.

Relations (4.13), (4.19), (4.22), (4.26) and inequality (4.3) indicate that, for a sufficiently large m, the vector u^m fulfils the assumptions of Lemma 3.2 with $K = r^m$, $L = h_m^2 c$, $\alpha = \dfrac{1}{2h_m}|A|$, $\beta = \eta$, $\gamma = h_m\eta$. From Lemma 3.2 we obtain the estimation

$$\|u^m\| \leqslant 2[L_0^{-1}(1+\tfrac{1}{2}|A|)m^2 r^m]^2 + 2L_0^{-1}(1+\tfrac{1}{2}|A|)(m^2 r^m + \eta) + 2L_0^{-1}(\eta + h_m\eta),$$

where $L_0 = 1-(1+\frac{1}{2}|A|)c$. Now, in view of (4.9), η being arbitrary, we have

$$\lim_{m\to\infty} \|u^m\| = 0 .$$

The definition of the vectors u^m and y^m completes the proof of the second assertion of the theorem.

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