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On generalized recurrent Kaehlerian manifolds of second order I*

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Abstract. Lichnérowicz [3] called a Riemannian space satisfying $R_{hijk,lm} = a_{lm} R_{hijk}$, where a_{lm} is a non-zero tensor, a recurrent space of second order. Recently, Ray [4] considered a generalization of such spaces. Our aim in the present paper is to define generalized recurrent Kaehler manifolds of second order and study some relations existing between them. We have thereby generalized certain results of Hasegava [2] and myself [5].

Introduction. A non-flat Riemanian space V_n whose curvature tensor R_{hijk} satisfies the relation

$$R_{hijk,lm} = R_{hijk,l}\beta_m + \gamma_{lm} R_{hijk},$$

where β_m and γ_{lm} are not both zero, has been called a *generalized recurrent space of second order* or a *generalized 2-recurrent space* [4]. If $\beta_m = 0$, the space reduces to what is known to be a 2-recurrent space as named by Lichnérowicz [3]. β_m and γ_{lm} are respectively called the *vector* and *tensor* of recurence of the space.

The main purpose of the present paper is to define generalized second order recurrent, Ricci-recurrent, H-projective recurrent and Bochner-recurrent Kaehlerian manifolds and study some relations existing between them.

1. Preliminaries. Let M be a Kaehlerian manifold of real dimension n $(= 2m)$ and (g, J) its Kaehlerian structure. That is, g is a Riemannian metric and J a complex structure in M such that(¹)

(1.1) $$J^i{}_a J^a{}_j = -\delta^i_j, \quad J_{ij} = -J_{ji}, \quad J^i{}_{j,h} = 0, \quad (J_{ij} = g_{ia} J^a{}_j),$$

where g_{ij} and $J^i{}_j$ are local components of g and J respectively and the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij}.

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(¹) The indices $a, b, c, \ldots, i, j, k, \ldots$ run over the same range from 1 to n.

It is well known that the holomorphically projective curvature tensor is given by

$$(1.2)\qquad P_{hijk} = H_{hijk} - \frac{1}{n+2}(R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk}),$$

where $H_{ij} = R_{ia}J^a{}_j$. This tensor P_{hijk}, which is invariant under any holomorphically projective correspondence, corresponds to the Weyl projective curvature tensor of a Riemannian space, and is called *H-projective curvature tensor*.

As a complex analogue to the Weyl's conformal curvature tensor Bochner [1] (see also Yano and Bochner [8]) introduces the following curvature tensor in a Kaehlerian manifold:

$$(1.3)\qquad B_{hijk} = R_{hijk} - \frac{1}{n+4}(R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk} +$$
$$+ R_{hk}g_{ij} - R_{ik}g_{hj} + H_{hk}J_{ij} - H_{ik}J_{hj} - 2H_{jk}J_{hi}) +$$
$$+ \frac{R}{(n+2)(n+4)}(g_{ij}g_{hk} - g_{hj}g_{ik} + J_{ij}J_{hk} - J_{hj}J_{ik} - 2J_{hi}J_{jk}).$$

Bochner introduced this curvature tensor using a complex local coordinate system. We call this tensor the *Bochner curvature tensor*. The form (1.3) of the Bochner curvature tensor with respect to a real coordinate system has been given by Tachibana [6].

We shall consider in M a tensor U_{hijk} given by

$$(1.4)\qquad U_{hijk} = R_{hijk} - \frac{R}{n(n+2)}(g_{ij}g_{hk} - g_{hj}g_{ik} + J_{ij}J_{hk} - J_{hj}J_{ik} - 2J_{hi}J_{jk}),$$

which has been called the *H-concircular curvature tensor*. The H-projective curvature tensor and the Bochner curvature tensor coincide with H-concircular curvature tensor if and only if M is an Einstein manifold.

2. Generalized recurrent, Ricci-recurrent, H-projective recurrent and H-concircular recurrent Kaehlerian manifolds of second order.

DEFINITION 2.1. A Kaehlerian manifold M satisfying

$$(2.1)\qquad R_{hijk,lm} = R_{hijk,l}\beta_m + \gamma_{lm}R_{hijk},$$

and

$$(2.2)\qquad R_{hk,lm} = R_{hk,l}\beta_m + \gamma_{lm}R_{hk},$$

where β_m and γ_{lm} are not both zero, shall be called *generalized recurrent Kaehlerian manifold of second order* (or briefly G2 *recurrent Kaehlerian manifold*) and *generalized Ricci-recurrent Kaehlerian manifold of second order* (briefly G2 *Ricci-recurrent Kaehlerian manifold*) respectively.

DEFINITION 2.2. A Kaehlerian manifold M satisfying

$$P_{hijk,lm} = P_{hijk,l}\beta_m + \gamma_{lm} P_{hijk}, \tag{2.3}$$

and

$$U_{hijk,lm} = U_{hijk,l}\beta_m + \gamma_{lm} U_{hijk}, \tag{2.4}$$

where β_m and γ_{lm} are not both zero, shall be called *generalized H-projective recurrent Kaehlerian manifold of second order* (or briefly G2 *H-projective recurrent Kaehlerian manifold*) and *generalized H-concircular recurrent Kaehlerian manifold of second order* (or briefly G2 *H-concircular recurrent Kaehlerian manifold*) respectively.

Now, we have the following

Remark 2.1. A G2 recurrent Kaehlerian manifold is G2 Ricci-recurrent, but the converse is not necessarily true.

From (2.2) it follows that

$$H_{hk,lm} = H_{hk,l}\beta_m + \gamma_{lm} H_{hk}. \tag{2.5}$$

Hence, in view of (1.2), (2.2), (2.5) and Remark 2.1, it is easy to verify the following

THEOREM 2.1. *A* G2 *recurrent Kaehlerian manifold is* G2 *H-projective recurrent.*

Remark 2.2. The converse of Theorem 2.1 is not necessarily true.

THEOREM 2.2. *In a* G2 *H-projective recurrent Kaehlerian manifold the relations*

$$R_{ij,lm} - R_{ij,l}\beta_m - \gamma_{lm} R_{ij} = \frac{1}{n}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R) g_{ij}, \tag{2.6a}$$

or equivalently,

$$H_{ij,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{ij} = \frac{1}{n}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R) J_{ij} \tag{2.6b}$$

hold.

Proof. A G2 H-projective recurrent Kaehlerian manifold satisfies

$$P_{hijk,lm} - P_{hijk,l}\beta_m - P_{hijk}\gamma_{lm} = 0. \tag{2.7}$$

Substituting the value of P_{hijk} from (1.2) in (2.7) we have

$$\begin{aligned} &R_{hijk,lm} - R_{hijk,l}\beta_m - \gamma_{lm} R_{hijk} \\ &= \frac{1}{n+2} g_{hk}(R_{ij,lm} - R_{ij,l}\beta_m - \gamma_{lm} R_{ij}) - g_{ik}(R_{hj,lm} - R_{hj,l}\beta_m - \gamma_{lm} R_{hj}) + \\ &+ J_{hk}(H_{ij,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{ij}) - J_{ik}(H_{hj,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{hj}) - \\ &- 2J_{jk}(H_{hi,lm} - H_{hi,l}\beta_m - \gamma_{lm} H_{hi}). \end{aligned} \tag{2.8}$$

Transvecting (2.8) with g^{ij} and after some simplifications we so obtain relation (2.6a).

THEOREM 2.3. *A* G2 *H-projective recurrent Kaehlerian manifold satisfying the relation*

$$R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R=0, \tag{2.9}$$

is G2 *recurrent.*

Proof. In view of Theorem 2.2 and condition (2.9) we have

$$R_{ij,lm}-R_{ij,l}\beta_m-\gamma_{lm}R_{ij}=0. \tag{2.10}$$

Transvecting (2.10) with $J^j{}_k$ we have

$$H_{ik,lm}-H_{ik,l}\beta_m-\gamma_{lm}H_{ik}=0. \tag{2.11}$$

Using (2.10) and (2.11) in (2.8) we obtain

$$R_{hljk,lm}-R_{hijk,l}\beta_m-\gamma_{lm}R_{hijk}=0, \tag{2.12}$$

i.e., the manifold is G2 recurrent.

In view of Theorem 2.1 and Theorem 2.3 we have the following

THEOREM 2.4. *A necessary and sufficient condition for a* G2 *H-projective recurrent Kaehlerian manifold to be* G2 *recurrent is that*

$$R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R=0 \tag{2.13}$$

holds.

From Theorems 2.2 and 2.4 follows

THEOREM 2.5. *A necessary and sufficient conditon for a* G2 *H-projective recurrent Kaehlerian manifold to be* G2 *recurrent is that the manifold be* G2 *Ricci-recurrent.*

This result can also be stated as

THEOREM 2.6. *If a Kaehlerian manifold satisfies any two of the properties*

1° *the manifold is* G2 *H-projective recurrent;*

2° *the manifold is* G2 *recurrent;*

3° *the manifold is* G2 *Ricci-recurrent;*

then it must also satisfy the third one.

THEOREM 2.7. *The necessary and sufficient condition for a Kaehlerian manifold to be* G2 *H-projective recurrent is that the manifold be* G2 *H-concircular recurrent.*

Proof. We assume that the Kaehlerian manifold M is $G2$ H-concircular recurrent. We then have

$$U_{hijk,lm}-U_{hijk,l}\beta_m-\gamma_{lm}U_{hijk}=0. \tag{2.14}$$

Substituting from (1.4) into (2.14), we get

$$(2.15)\quad R_{hijk,lm}-R_{hijk,l}\beta_m-\gamma_{lm}R_{hijk}=\frac{1}{n(n+2)}(R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R)S_{hijk},$$

where $S_{hijk}=g_{ij}g_{hk}-g_{hj}g_{ik}-J_{ij}J_{hk}-J_{hj}J_{ik}-2J_{hi}J_{jk}$. Contracting (2.15) with g^{hk}, we obtain

$$(2.16)\quad R_{ij,lm}-R_{ij,l}\beta_m-\gamma_{lm}R_{ij}=\frac{1}{n}(R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R)g_{ij}.$$

Substituting from the right-hand side of (2.15), we find that

$$P_{hijk,lm}-P_{hijk,l}\beta_m-\gamma_{lm}P_{hijk}=0.$$

Hence the manifold is G2 H-projective recurrent.

Conversely, let the manifold be G2 H-projective recurrent. Then we have (2.6a) and (2.6b).

Further a G2 H-projective recurrent manifold satisfies relation (2.8). Substituting from (2.6a) and (2.6b) into (2.8), we obtain

$$(2.17)\quad R_{hijk,lm}-R_{hijk,l}\beta_m-\gamma_{lm}R_{hijk}=\frac{S_{hijk}}{n(n+2)}(R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R),$$

i.e.,

$$(2.18)\quad U_{hijk,lm}-U_{hijk,l}\beta_m-\beta_{lm}U_{hijk}=0.$$

Hence the manifold is G2 H-concircular recurrent.

3. Generalized Bochner recurrent Kaehlerian manifold of second order.

DEFINITION 3.1. A Kaehlerian manifold M satisfying

$$(3.1)\quad B_{hijk,lm}=B_{hijk,l}\beta_m+B_{hijk}\gamma_{lm},$$

where β_m and γ_{lm} are not both zero, will be called a *generalized Bochner recurrent Kaehlerian manifold of second order* (or briefly a G2 *Bochner recurrent Kaehlerian manifold*).

THEOREM 3.1. *A* G2 *H-projective recurrent Kaehlerian manifold is* G2 *Bochner recurrent.*

Proof. A G2 H-projective recurrent Kaehlerian manifold satisfies relations (2.6a) and (2.6b). Now from (1.3) we have

$$\begin{aligned}(3.2)\quad B_{hijk,lm}-B_{hijk,l}\beta_m-B_{hijk}\gamma_{lm}&=R_{hijk,lm}-R_{hijk,l}\beta_m-R_{hijk}\gamma_{lm}-\\
&-\frac{1}{n+4}\{g_{hk}(R_{ij,lm}-R_{ij,l}\beta_m-\gamma_{lm}R_{ij})-g_{ik}(R_{hj,lm}-R_{hj,l}\beta_m-\gamma_{lm}R_{hj})+\\
&+J_{hk}(H_{ij,lm}-H_{ij,l}\beta_m-\gamma_{lm}H_{ij})-J_{ik}(H_{hj,lm}-H_{hj,l}\beta_m-\gamma_{lm}H_{hj})-\\
&-2J_{jk}(H_{hi,lm}-H_{hi,l}\beta_m-\gamma_{lm}H_{hi})+g_{ij}(R_{hk,lm}-R_{hk,l}\beta_m-\gamma_{lm}R_{hk})-\\
&-g_{hj}(R_{ik,lm}-R_{ik,l}\beta_m-\gamma_{lm}R_{ik})+J_{ij}(H_{hk,lm}-H_{hk,l}\beta_m-\gamma_{lm}H_{hk})-\\
&-J_{hj}(H_{ik,lm}-H_{ik,l}\beta_m-\gamma_{lm}H_{ik})-2J_{hi}(H_{jk,lm}-H_{jk,l}\beta_m-H_{jk}\gamma_{lm})\}+\\
&+\frac{S_{hijk}}{(n+2)(n+4)}(R_{,lm}-R_{,l}\beta_m-\gamma_{lm}R),\end{aligned}$$

where $S_{hijk} = g_{ij} g_{hk} - g_{hj} g_{ik} + J_{ij} J_{hk} - J_{hj} J_{ik} - 2J_{hi} J_{jk}$. Using (2.3), (2.6a) and (2.6b) in (3.2), after some simplifications we obtain

$$B_{hijk,lm} - B_{hijk,l}\beta_m - \gamma_{lm} B_{hijk} = 0. \tag{3.3}$$

Hence the manifold is G2 Bochner recurrent.

Remark 3.1. The converse of Theorem 3.1 is not necessarily true.

THEOREM 3.2. *The necessary and sufficient condition for a* G2 *Bochner recurrent Kaehlerian manifold M to be* G2 *H-projective recurrent is that equation*

$$R_{ij,lm} - R_{ij,l}\beta_m - \gamma_{lm} R_{ij} = \frac{1}{n}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R) g_{ij}, \tag{3.4a}$$

or equivalently

$$H_{ij,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{ij} = \frac{1}{n}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R) J_{ij}, \tag{3.4b}$$

holds.

Proof. That the conditon is necessary follows from Theorem 2.2.

Conversely, we assume that a G2 Bochner recurrent Kaehlerian manifold satisfies condition (3.4). In view of (3.1) and (1.3) we have

$$\begin{aligned}
(3.5)\quad & R_{hijk,lm} - R_{hijk,l}\beta_m - \gamma_{lm} R_{hijk} \\
&= \frac{1}{n+4}\{g_{hk}(R_{ij,lm} - R_{ij,l}\beta_m - \gamma_{lm} R_{ij}) - g_{ik}(R_{hj,lm} - R_{hj,l}\beta_m - \gamma_{lm} R_{hj}) + \\
&\quad + J_{hk}(H_{ij,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{ij}) - J_{ik}(H_{hj,lm} - H_{hj,l}\beta_m - \gamma_{lm} H_{hj}) - \\
&\quad - 2J_{jk}(H_{hi,lm} - H_{hi,l}\beta_m - \gamma_{lm} H_{hi}) + g_{ij}(R_{hk,lm} - R_{hk,l}\beta_m - \gamma_{lm} R_{hk}) - \\
&\quad - g_{hj}(R_{ik,lm} - R_{ik,l}\beta_m - \gamma_{lm} R_{ik}) + J_{ij}(H_{hk,lm} - H_{hk,l}\beta_m - \gamma_{lm} H_{hk}) - \\
&\quad - J_{hj}(H_{ik,lm} - H_{ik,l}\beta_m - \gamma_{lm} H_{ik}) - 2J_{hi}(H_{jk,lm} - H_{jk,l}\beta_m - \gamma_{lm} H_{jk})\} - \\
&\quad - \frac{S_{hijk}}{(n+2)(n+4)}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R).
\end{aligned}$$

Using (3.4a), (3.4b) in (3.5), after some simplifications we get

$$R_{hijk,lm} - R_{hijk,l}\beta_m - R_{hijk}\gamma_{lm} = \frac{S_{hijk}}{n(n+2)}(R_{,lm} - R_{,l}\beta_m - \gamma_{lm} R). \tag{3.6}$$

which implies

$$U_{hijk,lm} = U_{hijk,l}\beta_m - \gamma_{lm} U_{hijk}. \tag{3.7}$$

By Theorem 2.7, the manifold is G2 H-projective recurrent.

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