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PERSISTENT ROTATION INTERVALS FOR OLD MAPS

MICHAŁ MISIUREWICZ

Institute of Mathematics, Warsaw University
Warszawa, Poland

0. Introduction

This paper is really about continuous maps of the circle onto itself of degree one. However, it is usually much easier to work with their liftings instead. We shall call such liftings *old maps* (see [M2]; *old* stands for *degree one liftings* with the order of letters changed). They are characterized by the property $F(x+1) = F(x)+1$ for all $x \in \mathbf{R}$.

For any old continuous map $F\colon \mathbf{R} \to \mathbf{R}$, the set

$$L(F) = \text{closure}\left\{\limsup_{n\to\infty} \frac{1}{n}(F^n(x)-x)\colon\ x \in \mathbf{R}\right\}$$

is a closed interval (perhaps degenerated to a point) and it is called the *rotation interval* of F (see [NPT], [BGMY]). We shall denote the left and right endpoints of $L(F)$ by $\varrho_l(F)$ and $\varrho_r(F)$ respectively. By $\mathcal{O}$ we denote the space of all old continuous maps with the C^0 topology (uniform); by $\mathcal{O}^r$ the space of all old C^r maps with the C^r topology (uniform) ($r = 1, 2, \ldots, \infty, \omega$, where by C^ω we understand real analytic).

If $\mathcal{A}$ is some space of old maps, we shall say that $F \in \mathcal{A}$ has an $\mathcal{A}$-*persistent rotation interval* if there exists a neighbourhood U of F in $\mathcal{A}$ such that $L(G) = L(F)$ for all $G \in U$. Bamon, Malta and Pacifico proved the following theorem [BMP] for $r = 1, 2, \ldots$:

(i) *The set of all maps with $\mathcal{O}^r$-persistent rotation interval is open and dense in $\mathcal{O}^r$,*
(ii) *If F has $\mathcal{O}^r$-persistent rotation interval then $\varrho_l(F)$ and $\varrho_r(F)$ are rational.*

The aim of this paper is to generalize the above result to other spaces of old continuous maps. Even more important than the generalization itself is showing that the whole phenomenon of persistence of the rotation interval has nothing to do with the smoothness of the maps under consideration. Some related results were obtained independently by L. Jonker [J].

The translation of the results into the language of maps of the circle is very simple and I leave it to the reader.

The natural projection $x \mapsto \exp(2\pi i x)$ of $\boldsymbol{R}$ onto S^1 will be denoted by e. Whenever the fractions like $\frac{p}{q}$ will appear, it will be understood that p and q are relatively prime integers and $q > 0$. Any rational number can be written uniquely in this way.

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1. Old continuous non-decreasing maps

We shall denote the space of all old continuous non-decreasing maps with C^0 topology (uniform) by $\mathcal{N}$. For $\alpha \in \boldsymbol{R}$, by R_α we shall denote the translation by α: $R_\alpha(x) = x+\alpha$. If $F \in \mathcal{O}$ then we set $F_\alpha = R_\alpha \circ F$, i.e. $F_\alpha(x) = F(x)+\alpha$.

We shall use the following simple observations:

(1) If $F, G \in \mathcal{O}$ and either F or G belongs to $\mathcal{N}$ then $F \leqslant G$ implies $F^n \leqslant G^n$ for all $n \geqslant 0$.

(2) If $F \in \mathcal{N}$ then $\varrho_l(F) = \varrho_r(F)$ (to prove it, look at the standard proof of the existence of a rotation number for a homeomorphism).

Because of (2), we shall write $\varrho(F)$ instead of $\varrho_l(F)$ or $\varrho_r(F)$ if $F \in \mathcal{N}$.

(3) If $F \in \mathcal{N}$ then $F_\alpha \in \mathcal{N}$ for all $\alpha \in \boldsymbol{R}$ and $\varrho(F_\alpha)$ is a non-decreasing function of α.

(4) $\varrho(F)$ depends continuously on $F \in \mathcal{N}$.

(5) $\frac{p}{q} \in L(F)$ if and only if there exists $z \in \boldsymbol{R}$ such that $F^q(z)-p = z$.

LEMMA 1. *If $F \in \mathcal{N}$, $\alpha > 0$ and $n \geqslant 1$ then $F_\alpha^n \geqslant (F^n)_\alpha$.*

Proof. We have

$$F_\alpha^n(x) = F_\alpha(F_\alpha^{n-1}(x))+\alpha \geqslant F(F^{n-1}(x))+\alpha = F^n(x)+\alpha = (F^n)_\alpha(x). \quad \blacksquare$$

LEMMA 2 ([I], [M2]). *If $F \in \mathcal{N}$ and $\varrho(F)$ is irrational then $\alpha > 0$ (respectively $\alpha < 0$) implies $\varrho(F_\alpha) > \varrho(F)$ (respectively $\varrho(F_\alpha) < \varrho(F)$).*

We shall say that $F \in \mathcal{N}$ has a *stable rotation number* p/q if there exist $x, y \in \boldsymbol{R}$ such that $F^q(x)-p < x$ and $F^q(y)-p > y$. Clearly, this property is open (i.e. for a given p/q, the set of $F \in \mathcal{N}$ having stable rotation number p/q is open) and if F has a stable rotation number p/q then $\varrho(F) = p/q$. These properties justify the name.

If F has no stable rotation number then at least one of the following three cases occur:

Case 1. $\varrho(F)$ is irrational,

Case 2. $\varrho(F) = \frac{p}{q}$ and $F^q(x) - p \leqslant x$ for all $x \in \mathbf{R}$,

Case 3. $\varrho(F) = \frac{p}{q}$ and $F^q(x) - p \geqslant x$ for all $x \in \mathbf{R}$.

LEMMA 3. (a) *If Case* 2 *occurs but Case* 3 *does not then for every* $\varepsilon > 0$ *there exists* $\delta \in (0, \varepsilon)$ *such that* F_δ *has a stable rotation number* $\frac{p}{q}$.

(b) *If Case* 3 *occurs but Case* 2 *does not then for every* $\varepsilon > 0$ *there exists* $\delta \in (-\varepsilon, 0)$ *such that* F_δ *has a stable rotation number* p/q.

Proof. We shall prove (a); the proof of (b) is analogous. Since Case 3 does not occur, there is $y \in \mathbf{R}$ such that $F^q(y) - p < y$. If δ is small enough, then also $F^q_\delta(y) - p < y$. Since $\varrho(F) = p/q$, there exists $z \in \mathbf{R}$ such that $F^q(z) = z$. For $\delta > 0$ we have by Lemma 1,

$$F^q_\delta(z) - p \geqslant F^q(z) + \delta - p = z + \delta > z$$

and therefore F_δ has a stable rotation number p/q. ∎

Remark 1. If both cases 2 and 3 occur then $F^q = R_p$.

2. Endpoints of the rotation interval

For $F \in \mathcal{O}$ we define the maps $\underline{F}$ and $\bar{F}$ by $\underline{F}(x) = \inf_{y \geqslant x} F(y)$, $\bar{F}(x) = \sup_{y \leqslant x} F(y)$ (see [M1], [CGT], [M2]). Make again simple observations:

(6) $\bar{F}, \underline{F} \in \mathcal{N}$,

(7) $\underline{F}_\delta = \underline{F_\delta}$, $\bar{F}_\delta = \overline{F_\delta}$,

(8) $\underline{F}$ *and* $\bar{F}$ *depend continuously on* $F \in \mathcal{O}$.

The following fact was stated without proof in [M1]; it was proved in a more general situation in [M2]. We are going to give a simple proof below.

LEMMA 4. *If* $F \in \mathcal{O}$ *then* $L(F) = [\varrho(\underline{F}), \varrho(\bar{F})]$.

Proof. Since $\underline{F} \leqslant F \leqslant \bar{F}$, then by (1), $L(F) \subset [\varrho(\underline{F}), \varrho(\bar{F})]$. To prove the equality it is enough to find points $y, z \in \mathbf{R}$ such that $F^n(y) = \underline{F}^n(y)$ and $F^n(z)$

$= \bar{F}^n(z)$ for $n = 1, 2, \ldots$ We shall show the existence of such y; the proof for z is analogous.

Assume that such y does not exist. Then for each $x \in \mathbf{R}$ there exists $n > 0$ such that $F^n(x) \neq \underline{F}^n(x)$. Since $F(t) \neq \underline{F}(t)$ only if t is in the interior of one of the intervals on which $\underline{F}$ is constant, this means that for some $k < n$, $F^k(x) = \underline{F}^k(x)$ and $\underline{F}^k(x)$ is in the interior of one of such intervals. Then there exists a neighbourhood $U(x)$ of x such that $\underline{F}^{k(x)}$ is constant on $U(x)$ $(k(x) = k)$. From the cover $\{U(x)\colon x \in [0, 1]\}$ of $[0, 1]$ one can choose a finite subcover $\{U(x_i)\}_{i=1}^{s}$ and then for $m = \max k(x_i)$, $\underline{F}^m$ is constant on each $U(x_i)$. Hence $\underline{F}$ is constant on the whole $[0, 1]$ – a contradiction since $\underline{F}$ is old. ∎

THEOREM. *Let $\mathscr{A}$ be a topological space whose elements are old maps. Assume that:*

(a) *If $F \in \mathscr{A}$ and U is a neighbourhood of F in $\mathscr{A}$ then there exists $\varepsilon > 0$ such that $F_\delta \in U$ for all $\delta \in (-\varepsilon, \varepsilon)$.*

(b) *If U is an open subset of $\mathscr{O}$ then $U \cap \mathscr{A}$ is an open subset of $\mathscr{A}$ (in the topology of $\mathscr{A}$).*

(c) *If for some p/q and $F \in \mathscr{A} \cap \mathscr{N}$ we have $F^q = R_p$ then for every neighbourhood U of F in $\mathscr{A}$ there exists $G \in U$ and $x, y \in \mathbf{R}$ such that $G^q(x) - p = x$ and $G^q(y) - p \neq y$.*

Then:

(i) *The set of all maps with $\mathscr{A}$-persistent rotation interval is open and dense in $\mathscr{A}$.*

(ii) *$F \in \mathscr{A}$ has $\mathscr{A}$-persistent rotation interval if and only if $\underline{F}$ and $\bar{F}$ have stable rotation numbers.*

Remark 2. Clearly, if $\underline{F}$ and $\bar{F}$ have stable rotation numbers then $\varrho_l(F)$ and $\varrho_r(F)$ are rational.

Remark 3. It follows from (ii) and from the fact that $\mathscr{O}$ satisfies (a)–(c) (see Proposition later) that if $\mathscr{A}$ satisfies (a)–(c) then $F \in \mathscr{A}$ has $\mathscr{A}$-persistent rotation interval if and only if it has $\mathscr{O}$-persistent interval.

Proof of Theorem. We start by proving (ii). If $\bar{F}$ and $\underline{F}$ have stable rotation numbers then by (8), F has $\mathscr{O}$-persistent rotation interval. Hence, by (b), F has also $\mathscr{A}$-persistent rotation interval. On the other hand, if F has $\mathscr{A}$-persistent rotation interval, then by (a), for some $\varepsilon > 0$ we have $\varrho(\underline{F}_\delta) = \varrho(\underline{F})$ and $\varrho(\bar{F}_\delta) = \varrho(\bar{F})$ for all $\delta \in (-\varepsilon, \varepsilon)$. This cannot occur in any of the three cases when $\underline{F}$ (respectively $\bar{F}$) has no stable rotation number. This proves (ii).

Now we prove (i). The set of all maps with $\mathscr{A}$-persistent rotation interval is open by the definition. Suppose that it is not dense in $\mathscr{A}$. Then there exists $F \in \mathscr{A}$ and its neighbourhood U in $\mathscr{A}$ such that no element of U has $\mathscr{A}$-persistent rotation interval.

Assume first that $F \in \mathcal{N}$. By (a), there exists $\varepsilon > 0$ such that $F_\delta \in U$ for all $\delta \in (-\varepsilon, \varepsilon)$. By (ii), these F_δ have no stable rotation number. By Lemma 2, one of them has rational rotation number. By Lemma 3, both cases 2 and 3 occur for it. Hence, by Remark 1 and (c), there exists $G \in U$ for which either $G \notin \mathcal{N}$ or exactly one of the cases 2, 3 occurs. The second possibility is ruled out by Lemma 3. In such a way our situation is reduced to the case of $F \notin \mathcal{N}$.

If $F \notin \mathcal{N}$ then $\underline{F}$ and $\bar{F}$ have intervals on which they are constant and therefore neither $\underline{F}_\delta^q = R_p$ nor $\bar{F}_\delta^q = R_p$ is possible. Then, by (a) and Lemmas 2 and 3, there exists δ such that $F_\delta \in U$ and $\underline{F}_\delta$ has a stable rotation number. By the same argument, there exists η arbitrarily close to δ such that $F_\eta \in U$ and $\bar{F}_\eta$ has a stable rotation number. If η is sufficiently close to δ, then $\underline{F}_\eta$ has the same stable rotation number as $\underline{F}_\delta$ and hence by (ii), F_η has $\mathcal{A}$-persistent rotation interval. Thus the set of all maps with $\mathcal{A}$-persistent rotation intervals is dense in $\mathcal{A}$. ∎

Proposition. *For $\mathcal{A} = C$, $\mathcal{A} = C^1$, $\mathcal{A} = C^2, \ldots, \mathcal{A} = C^\infty$ and $\mathcal{A} = C^\omega$ the assumptions of Theorem are satisfied.*

Proof. The assumptions (a) and (b) are obviously satisfied. Assume that $F \in \mathcal{A} \cap \mathcal{N}$ and $F^q = R_p$. The function $\Phi \colon \mathbf{R} \to \mathbf{R}$ given by

$$\Phi(x) = \prod_{i=0}^{q-1} \Big(1 - \cos\big(2\pi(x - F^i(0))\big)\Big)$$

is non-negative, periodic of period 1, real analytic, and $\Phi(x) = 0$ if and only if $x = F^m(0) + n$ for some $m, n \in \mathbf{Z}$, $m \geqslant 0$. If $\varepsilon > 0$ then $G = F + \varepsilon\Phi$ is an old map, $G \in \mathcal{A}$, $G \geqslant F$ and $G^q(0) = p$. Moreover, if y is not of the form $F^m(0) + n$, then $G(y) > F(y)$, and thus $G^q(y) \geqslant F^{q-1}(G(y)) > F^{q-1}(F(y)) = y + p$ (the strict inequality is due to the fact that F is strictly increasing, a consequence of the assumption $F^q = R_p$). By taking ε small enough we can get G arbitrarily close to F. This shows that (c) is also satisfied. ∎

Remark 4. It would be nice to have (c) replaced by a simpler assumption, for example

(c′) In each non-empty set $U \subset \mathcal{A}$ there is F which is not of the form R_α.

The natural idea to do this seemed to be to prove

(*) If $F \in \mathcal{N}$, $F^q = R_p$ for some p/q and $F \neq R_{p/q}$ then for every $\varepsilon > 0$ there exists $\delta \in (-\varepsilon, \varepsilon)$ and r/s such that $\varrho(F_\delta) = r/s$ and $F^s \neq R_r$.

Unfortunately, as was shown by J. Graczyk, (*) is false even for analytic F. This leaves the question about simplifying (c) open.

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