ANNALES
POLONICI MATHEMATICI
XLIX (1989)

Local generalized solutions of mixed problems for quasilinear hyperbolic systems of functional partial differential equations in two independent variables

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Abstract. A theorem of existence and uniqueness of generalized (in the sense almost everywhere) solutions of mixed problems for quasilinear hyperbolic systems of functional partial differential equations in two independent variables is proved.

1. Introduction. Let $a_0, b, \Omega > 0$ be given constants. We denote by $\|u\| = \max_{1 \leq i \leq n} |u_i|$ the norm of u in $\boldsymbol{R}^n$. Let us denote by I_{a_0} the rectangle $I_{a_0} = \{(x, y): 0 \leq x \leq a_0, 0 \leq y \leq b\}$.

We consider quasilinear hyperbolic systems of functional partial differential equations in diagonal form

$$(1) \quad D_x u(x, y) + \lambda(x, y, u(x, y), (Vu)(x, y)) D_y u(x, y) = f(x, y, u(x, y), (Vu)(x, y)), \quad (x, y) \in I_{a_0},$$

with the initial condition

$$(2) \qquad u(0, y) = \varphi(y), \quad y \in [0, b],$$

and the boundary conditions[1]

$$(3) \quad \begin{aligned} u_i(x, 0) &= h_{0i}(x, u(x, 0)), \quad i \in J_0 = \{i: \operatorname{sgn} \lambda_i(0, 0, 0, 0) = 1\}, \\ u_i(x, b) &= h_{bi}(x, u(x, b)), \quad i \in J_b = \{i: \operatorname{sgn} \lambda_i(0, b, 0, 0) = -1\}, \end{aligned}$$

where

$$u(x, y) = \operatorname{col}(u_1(x, y), \ldots, u_n(x, y)), \quad \varphi(y) = \operatorname{col}(\varphi_1(y), \ldots, \varphi_n(y)),$$

$$\lambda(x, y, u(x, y), (Vu)(x, y)) = \operatorname{diag}[\lambda_1(x, y, u(x, y), (V_1 u)(x, y)), \ldots, \lambda_n(x, y, u(x, y), (V_n u)(x, y))],$$

$$f(x, y, u(x, y), (Vu)(x, y)) = \operatorname{col}(f_1(x, y, u(x, y), (V_1 u)(x, y)), \ldots \ldots, f_n(x, y, u(x, y), (V_n u)(x, y))),$$

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[1] One for each bicharacteristic entering the domain at $y = 0, b$.

$D_x = \partial/\partial x$, $D_y = \partial/\partial y$, and V_i, $i = 1, \ldots, n$, are operators of the Volterra type.

System (1) contains as a particular case ($(V_i u)(x, y) = u(\alpha_i(x, y), \beta_i(x, y))$) the system of differential equations with a retarded argument and hence the unretarded case ($\alpha_i(x, y) = x$, $\beta_i(x, y) = y$) which was widely studied [1]–[4], [6], [7]. A few kinds of integral-differential systems can be derived from system (1) by specializing the operators V_i, $i = 1, \ldots, n$ (for instance, $(V_i u)(x, y) = \int_{\alpha_i(x,y)}^{\beta_i(x,y)} K_i(s, t, x, y) u(s, t)\, ds dt$).

In this paper, we seek local generalized (in the sense almost everywhere) solutions of mixed problem (1)–(3).

Existence and uniqueness of continuous generalized solutions of mixed problems for hyperbolic systems of partial differential equations in two independent variables have been investigated by Filimonov [4], Myshkis, Filimonov [6], [7], Abolinia, Myshkis [1]. By continuous generalized solutions they understand the functions satisfying the integral system obtained from the differential system by integration along characteristic curves.

In view of the groupal property of the characteristic lines and the Chain Rule Differentiation Lemma (4ii) of [3] it is possible to prove that the continuous generalized solutions satisfy the differential systems almost everywhere.

2. Assumptions and lemmas.

ASSUMPTION H_1. Suppose that

(i_1) the functions $\operatorname{sgn}\lambda_1(\cdot, 0, \cdot, \cdot)$, $\operatorname{sgn}\lambda_i(\cdot, b, \cdot, \cdot)$: $E_{a_0} = [0, a_0] \times \bar{\Omega} \times \bar{\Omega} \to \boldsymbol{R}$, $i = 1, \ldots, n$, are constant in E_{a_0}, where $\bar{\Omega} = [-\Omega, \Omega]^n \subset \boldsymbol{R}^n$;

(ii_1) $\lambda_i(\cdot, y, u, v)$: $[0, a_0] \to \boldsymbol{R}$, $i = 1, \ldots, n$, are measurable for every $(y, u, v) \in E = [0, b] \times \bar{\Omega} \times \bar{\Omega}$;

(iii_1) there are a constant $\Lambda > 0$ and integrable functions l_j: $[0, a_0] \to \boldsymbol{R}_+ = [0, +\infty)$, $j = 1, 2, 3$, such that for all $(y, u, v), (\bar{y}, \bar{u}, \bar{v}) \in E$, almost everywhere (a.e.) in $[0, a_0]$, we have

$$|\lambda_i(x, y, u, v)| \leqslant \Lambda,$$

$$|\lambda_i(x, y, u, v) - \lambda_i(x, \bar{y}, \bar{u}, \bar{v})| \leqslant l_1(x)|y - \bar{y}| + l_2(x)\|u - \bar{u}\| + l_3(x)\|v - \bar{v}\|,$$

$i = 1, \ldots, n$;

(iv_1) there are constants $\varepsilon_0 \in (0, b)$ and $\Lambda_0 > 0$ such that

$$\lambda_i(x, y, u, v) \geqslant \Lambda_0 \quad \text{for } i \in J_0,\ y \in [0, \varepsilon_0],\ (x, u, v) \in E_{a_0},$$

and

$$-\lambda_i(x, y, u, v) \geqslant \Lambda_0 \quad \text{for } i \in J_b,\ y \in [b - \varepsilon_0, b],\ (x, u, v) \in E_{a_0}.$$

We denote by $B(a)$ the set of continuous functions $u\colon I_a \to \boldsymbol{R}^n$ Lipschitzian with respect to both variables.

ASSUMPTION H_2. Suppose that

(i_2) $V_i\colon B(a) \to B(a)$, $i = 1, \ldots, n$;

(ii_2) there are integrable functions $c, d\colon [0, a] \to \boldsymbol{R}_+$ such that for every $u \in B(a)$ we have

$$\|(V_i u)(x, \cdot)\|_* \leqslant c(x)\|u(x, \cdot)\|_* + d(x), \qquad i = 1, \ldots, n, \text{ a.e. in } [0, a],$$

where

$$\|u(x, \cdot)\|_* = \sup_{y,\bar{y}\in[0,b]} \frac{\|u(x, y) - u(x, \bar{y})\|}{|y - \bar{y}|};$$

(iii_2) there is an integrable function $m\colon [0, a] \to \boldsymbol{R}_+$ such that for all $u, v \in B(a)$, $x \in [0, a]$, we have

$$\|V_i u - V_i v\|_x \leqslant m(x)\|u - v\|_x, \qquad i = 1, \ldots, n, \tag{4}$$

where

$$\|u\|_x = \sup_{(t,y)\in I_x} \|u(t, y)\|, \qquad I_x = [0, x] \times [0, b].$$

Remark 1. From (4) we conclude that V_i, $i = 1, \ldots, n$, satisfy the following Volterra condition: if $u, \bar{u} \in B(a)$ and $u(t, y) = \bar{u}(t, y)$, for $(t, y) \in I_x$, then $(V_i u)(x, y) = (V_i \bar{u})(x, y)$, $(x, y) \in I_a$, $i = 1, \ldots, n$.

For $u \in B(a)$, we consider in I_a the following problem:

$$D_t g(t; x, y) = \lambda_i\big(t, g(t; x, y), u(t, g(t; x, y)), (V_i u)(t, g(t; x, y))\big), \tag{5}$$

$i = 1, \ldots, n$,

$$g(x; x, y) = y, \tag{6}$$

where equality (5) is satisfied for almost every $t \in [0, a]$.

Because of assumptions (ii_1), (iii_1) of H_1 and (ii_2) of H_2, and $u \in B(a)$, we see that the functions $\lambda_i(x, y, u(x, y), (V_i u)(x, y))$, $i = 1, \ldots, n$, satisfy the Carathéodory conditions. Thus, for every $u \in B(a)$, there is a unique solution $g_i = g_i[u]$ of problem (5), (6).

Remark 2. Note that, since $g_i = g_i[u]$ is the unique solution of problem (5), (6), g_i satisfies the following groupal property

$$g_i(t'; t, g_i(t; x, y)) = g_i(t'; x, y), \qquad t, t' \in [0, x], \ (x, y) \in I_a. \tag{7}$$

We denote by $\tau_i(x, y, u)$ the smallest value of the argument x for which the solution $g_i = g_i[u](t; x, y)$ of problem (5), (6) is defined. Then the point $(\tau_i(x, y, u), g_i(\tau_i(x, y, u); x, y))$ belongs to the boundary of I_a.

We introduce the following notations:

$$I^u_{\varphi i} = \{(x, y)\colon (x, y)\in I_a,\ \tau_i(x, y, u) = 0\},$$

$$I^u_{0i} = \{(x, y)\colon (x, y)\in I_a,\ \tau_i(x, y, u) > 0,\ g_i[u](\tau_i(x, y, u); x, y) = 0\},$$

$$I^u_{hi} = \{(x, y)\colon (x, y)\in I_a,\ \tau_i(x, y, u) > 0,\ g_i[u](\tau_i(x, y, u); x, y) = h\}.$$

Let $B(a, P, Q)$ be the set of all functions u, $u\in B(a)$, satisfying the following conditions:

$$\|u(x, y)\| \leqslant \Omega, \qquad \|u(x, y) - u(\bar{x}, \bar{y})\| \leqslant P|x-\bar{x}| + Q|y-\bar{y}|,$$

for all $(x, y), (\bar{x}, \bar{y})\in I_a$.

Note that $B(a, P, Q)$ is closed (convex) subset of the Banach space $C(I_a)\cap L_\infty(I_a)$ with the norm $\|u\|_a = \sup\limits_{I_a}\|u(x, y)\|$.

Put

$$L_1 = L_1(a) = \int_0^a \{l_1(t) + l_2(t)Q + l_3(t)[c(t)Q + d(t)]\}\, dt,$$

$$L_2 = L_2(a) = \int_0^a [l_2(t) + l_3(t)m(t)]\, dt.$$

LEMMA 1. *If Assumptions* (ii_1), (iii_1) *of* H_1 *and* H_2 *are satisfied, then for all* $(x, y), (\bar{x}, \bar{y})\in I_a$, $u, v\in B(a, P, Q)$, *the following inequality*

(8) $$|g_i[u](t; x, y) - g_i[v](t; \bar{x}, \bar{y})| \leqslant (A|x-\bar{x}| + |y-\bar{y}| + L_2\|u-v\|_a)\exp(L_1),$$

$t\in[0, a]$, $i = 1, \ldots, n$, *holds*.

Proof. Let $\bar{x}\leqslant x$. For $t\in[0, \bar{x}]$, we have

$$\begin{aligned}
&|g_i[u](t; x, y) - g_i[v](t; \bar{x}, \bar{y})| \\
&\leqslant |y-\bar{y}| + \Big|\int_{\bar{x}}^{t} \{[l_1(s) + l_2(s)Q + l_3(s)(c(s)Q + d(s))]\,|g_i[u](s; x, y) - \\
&\qquad - g_i[v](s; \bar{x}, \bar{y})| + [l_2(s) + l_3(s)m(s)]\,\|u-v\|_a\}\, ds\Big| + \\
&\quad + \Big|\int_{x}^{\bar{x}} \lambda_i\big(t, g_i[u](t; x, y), u(t, g_i[u](t; x, y)), (V_i u)(t; g_i[u](t; x, y))\big)\, dt\Big| \\
&\leqslant |y-\bar{y}| + A|x-\bar{x}| + L_2\|u-v\|_a + \Big|\int_{\bar{x}}^{t} [l_1(s) + l_2(s)Q + \\
&\qquad + l_3(s)(c(s)Q + d(s))]\,|g_i[u](s; x, y) - g_i[v](s; \bar{x}, \bar{y})|\, ds\Big|.
\end{aligned}$$

Hence, and by Gronwall's inequality we get (8). This ends the proof.

LEMMA 2. *If Assumptions* H_1 *and* H_2 *are satisfied and* a, $0 < a \leqslant a_0$, *is sufficiently small such that*

$$\Lambda a \leqslant \varepsilon_0, \tag{9}$$

where ε_0 *is given in* (iv_1) *of* H_1, *then for all* $(x, y), (x, \bar{y}) \in \bar{I}^u_{0i}$ *or* $(x, y), (x, \bar{y}) \in \bar{I}^u_{bi}$ (*where the bar means closure of the set*) *and* $u \in B(a, P, Q)$, *we have*

$$|\tau_i(x, y, u) - \tau_i(x, \bar{y}, u)| \leqslant \Lambda_0^{-1} \exp(L_1)|y - \bar{y}|, \tag{10}$$

and for $(x, y) \in \bar{I}^u_{0i} \cap \bar{I}^v_{0i}$ *or* $(x, y) \in \bar{I}^u_{bi} \cap \bar{I}^v_{bi}$, *and* $u, v \in B(a, P, Q)$, *we have*

$$|\tau_i(x, y, u) - \tau_i(x, y, v)| \leqslant \Lambda_0^{-1} L_2 \exp(L_1)\|u - v\|_a, \quad i = 1, \ldots, n. \tag{11}$$

Proof. First we prove inequality (10). Let us suppose that $(x, y), (x, \bar{y}) \in \bar{I}^u_{0i}$ and $y > \bar{y}$. Hence, since the characteristic lines of the same family corresponding to the function u cannot intersect, we have $\tau_i(x, y, u) < \tau_i(x, \bar{y}, u)$.

It follows from (8) and by $g_i(\tau_i(x, \bar{y}, u); x, \bar{y}) = 0$ that

$$|g_i(\tau_i(x, \bar{y}, u); x, y) - 0| \leqslant \exp(L_1)(y - \bar{y}). \tag{12}$$

Now, by using the mean value theorem, we get

$$\begin{aligned} g_i(\tau_i(x, \bar{y}, u); x, y) &= g_i(\tau_i(x, \bar{y}, u); x, y) - g_i(\tau_i(x, y, u); x, y) \\ &= D_\xi g_i(\xi; x, y)[\tau_i(x, \bar{y}, u) - \tau_i(x, y, u)], \end{aligned} \tag{13}$$

where $\tau_i(x, \bar{y}, u) \leqslant \xi \leqslant \tau_i(x, y, u)$.

From the estimate $|dy/dx| \leqslant \Lambda$ for any characteristic, it follows that $y \leqslant \Lambda a$, provided $(x, y) \in \bar{I}^u_{0i}$. It is obvious that if $(x, y) \in \bar{I}^u_{0i}$ then $i \in J_0$. Therefore, by (9), for the points $(x, y, u, v) \in I_{a_0} \times \bar{\Omega} \times \bar{\Omega}$ with $y \leqslant \Lambda a$ we have $\lambda_i(x, y, u, v) \geqslant \Lambda_0$, $0 \leqslant y \leqslant \xi_0$, $i \in J_0$. In particular, this inequality holds true for the points of the set $\bar{I}^u_{0i} \times \bar{\Omega} \times \bar{\Omega}$. Hence by (13) we obtain

$$\tau_i(x, \bar{y}, u) - \tau_i(x, y, u) \leqslant \Lambda_0^{-1} g_i(\tau_i(x, \bar{y}, u); x, y). \tag{14}$$

Combining (12) and (14), we have

$$\tau_i(x, \bar{y}, u) - \tau_i(x, y, u) \leqslant \Lambda_0^{-1} \exp(L_1)(y - \bar{y}).$$

The case $y < \bar{y}$ we consider analogously. In a similar way we can prove (10) if $(x, y), (x, \bar{y}) \in \bar{I}^u_{bi}$.

Now we prove (11). Let assume that $(x, y) \in \bar{I}^u_{0i} \cap \bar{I}^v_{0i}$ and $\tau_i(x, y, u) < \tau_i(x, y, v)$. Since $g_i[v](\tau_i(x, y, v); x, y) = 0$, by Lemma 1, we have

(15) $$|g_i[u](\tau_i(x, y, v); x, y) - 0| \leqslant L_2 \exp(L_1) \|u - v\|_a.$$

Now, in virtue of the mean value theorem, we get

$$\begin{aligned} g_i[u](\tau_i(x, y, v); x, y) &= g_i[u](\tau_i(x, y, v); x, y) - g_i[u](\tau_i(x, y, u); x, y) \\ &= D_\xi g_i[u](\xi; x, y)[\tau_i(x, y, v) - \tau_i(x, y, u)], \end{aligned}$$

where $\tau_i(x, y, u) \leqslant \xi \leqslant \tau_i(x, y, v)$.

Hence, by (15), we have

$$\begin{aligned} \tau_i(x, y, v) - \tau_i(x, y, u) &\leqslant \Lambda_0^{-1} g_i[u](\tau_i(x, y, v); x, y) \\ &\leqslant \Lambda_0^{-1} L_2 \exp(L_1) \|u - v\|_a. \end{aligned}$$

Analogously, we can prove (11) with $\tau_i(x, y, u) > \tau_i(x, y, v)$ and $(x, y) \in \bar{I}^u_{bi} \cap \bar{I}^v_{bi}$. Hence, the assertion is proved.

Assumption H_3. Suppose that

(i_3) $f_i(\cdot, y, u, v)$: $[0, a_0] \to \boldsymbol{R}$, $i = 1, \ldots, n$, are measurable for every $(y, u, v) \in E$;

(ii_3) there are a constant $F > 0$, and integrable functions k_j: $[0, a_0] \to \boldsymbol{R}_+$, $j = 1, 2, 3$, such that for all $(y, u, v), (\bar{y}, \bar{u}, \bar{v}) \in E$, a.e. in $[0, a_0]$, we have

$$|f_i(x, y, u, v)| \leqslant F,$$

$$|f_i(x, y, u, v) - f_i(x, \bar{y}, \bar{u}, \bar{v})| \leqslant k_1(x)|y - \bar{y}| + k_2(x)\|u - \bar{u}\| + k_3(x)\|v - \bar{v}\|,$$

$i = 1, \ldots, n$.

Assumption H_4. Suppose that

(i_4) the functions h_{0i}: $[0, a_0] \times \bar{\Omega} \to \boldsymbol{R}$, $i \in J_0$, are independent of u_j for $j \in J_0$, and the functions h_{bi}: $[0, a_0] \times \bar{\Omega} \to \boldsymbol{R}$, $i \in J_b$, are independent of u_j, for $j \in J_b$;

(ii_4) there are constants $H_j \geqslant 0$, $j = 1, 2$, such that for all $(x, u), (\bar{x}, \bar{u}) \in [0, a_0] \times \bar{\Omega}$ we have

$$|h_{0i}(x, u) - h_{0i}(\bar{x}, \bar{u})| \leqslant H_1|x - \bar{x}| + H_2\|u - \bar{u}\|, \quad i \in J_0,$$

$$|h_{bi}(x, u) - h_{bi}(\bar{x}, \bar{u})| \leqslant H_1|x - \bar{x}| + H_2\|u - \bar{u}\|, \quad i \in J_b;$$

(iii_4) the compatibility conditions

(16) $$h_{0i}(0, \varphi(0)) = \varphi_i(0), \quad i \in J_0,$$

(17) $$h_{bi}(0, \varphi(b)) = \varphi_i(b), \quad i \in J_b,$$

are satisfied;

(iv$_4$) there is a constant $\Phi \geq 0$ such that for all $y, \bar{y} \in [0, b]$ we have

$$\|\varphi(y) - \varphi(\bar{y})\| \leq \Phi |y - \bar{y}|,$$

and $\max_{[0,b]} \|\varphi(y)\| = \Phi_1 < \Omega$.

We write $B_\varphi(a, P, Q)$ to denote the following set

$$B_\varphi(a, P, Q) = \{u\colon u \in B(a, P, Q),\ u(0, y) = \varphi(y),\ y \in [0, b]\},$$

where we assume that $Q \geq \Phi$, which assures that $B_\varphi(a, P, Q)$ is not empty.

Let us consider in $B_\varphi(a, P, Q)$ the following ball

$$B_\varphi(a, P, Q, \omega) = \{u\colon u \in B_\varphi(a, P, Q),\ \max_{I_a} \|u(x, y) - \varphi(y)\| \leq \omega\},$$

where $0 \leq \omega \leq \Omega - \Phi_1$. Obviously, for $u \in B_\varphi(a, P, Q, \omega)$ we have

$$\|u(x, y)\| \leq \omega + \Phi_1 \leq \Omega.$$

Hence, for $u \in B_\varphi(a, P, Q, \omega)$ the points $(x, y, u(x, y), (V_i u)(x, y))$, $i = 1, \ldots, n$, where $(x, y) \in I_a$, belong to $I_{a_0} \times \bar{\Omega} \times \bar{\Omega}$. Thus, for every $u \in B_\varphi(a, P, Q, \omega)$ the corresponding family of characteristic is defined.

3. The operator S **and its properties.** Now we consider in $B(a)$ the operator S defined by

(18) $$(Su)_i(x, y) = (Ru)_i(x, y) + (Zu)_i(x, y), \quad i = 1, \ldots, n,$$

where

(19) $$(Ru)_i(x, y) = \begin{cases} \varphi_i(g_i(0; x, y)), & (x, y) \in I^u_{\varphi i}, \\ h_{0i}(\tau_i(x, y, u), u(\tau_i(x, y, u), 0)), & (x, y) \in I^u_{0i}, \\ h_{bi}(\tau_i(x, y, u), u(\tau_i(x, y, u), b)), & (x, y) \in I^u_{bi}, \end{cases}$$

and

(20) $$(Zu)_i(x, y) = \int_{\tau_i(x,y,u)}^{x} f_i(t, g_i(t; x, y), u(t, g_i(t; x, y)), (V_i u)(t, g_i(t; x, y)))\,dt, \quad i = 1, \ldots, n.$$

From now on we make the assumption: $2\Lambda a < b$ which yields $\bar{I}^u_{0i} \cap \bar{I}^v_{bi} = \emptyset$, and the assumption: $\Lambda a \leq \varepsilon_0$, which guarantees that the inequalities $\lambda_i(x, y, u, v) \geq \Lambda_0$ and $-\lambda_i(x, y, u, v) \geq \Lambda_0$ are satisfied in the sets $\bar{I}^u_{0i}$ and $\bar{I}^u_{bi}$, respectively.

LEMMA 3. *If Assumptions* H_1–H_4 *are satisfied, then for every* $u \in B_\varphi(a, P, Q, \omega)$, *the function* $Su\colon I_a \to \boldsymbol{R}^n$ *is continuous.*

Proof. By the definition of $(Su)_i$ it follows that $(Su)_i$ is continuous in each of the sets $\bar{I}^u_{\varphi i}$, $\bar{I}^u_{0i}$ and $\bar{I}^u_{bi}$. Hence, it is enough to show that $(Su)_i$ is continuous on the lines $y = g_i(x; 0, 0)$ and $y = g_i(x; 0, b)$. Let us consider the first line $y = g_i(x; 0, 0)$. Obviously, for the points (x, y) of this line, we have $\tau_i(x, y, u) = 0$. Hence, by (20), we have

$$[(Zu)_i(x, y)]_{\text{left}} - [(Zu)_i(x, y)]_{\text{right}} = 0.$$

Furthermore, by (19) and by compatibility condition (16), we get

$$[(Ru)_i(x, y)]_{\text{left}} - [(Ru)_i(x, y)]_{\text{right}}$$
$$= h_{0i}(0, u(0, 0)) - \varphi_i(g_i(0; x, y)) = h_{0i}(0, \varphi(0)) - \varphi_i(0) = 0.$$

Therefore

$$[(Su)_i(x, y)]_{\text{left}} - [(Su)_i(x, y)]_{\text{right}} = 0.$$

The case of the points belonging to the line $y = g_i(x; 0, b)$ is analogous. This proves the lemma.

LEMMA 4. *If Assumptions* H_1–H_4 *are satisfied, then for every* $u \in B_\varphi(a, P, Q, \omega)$ *the function Su satisfies in* $\bar{I}^u_{\varphi i}$ *a Lipschitz condition in y with some constant* Q^S_φ.

Proof. Let $(x, y), (x, \bar{y}) \in \bar{I}^u_{\varphi i}$. Then by (8) we have

$$|(Ru)_i(x, y) - (Ru)_i(x, \bar{y})| = |\varphi_i(g_i(0; x, y)) - \varphi_i(g_i(0; x, \bar{y}))|$$
$$\leqslant \Phi\, |g_i(0; x, y) - g_i(0; x, \bar{y})| \leqslant \Phi \exp(L_1)\, |y - \bar{y}|.$$

Furthermore

$$|(Zu)_i(x, y) - (Zu)_i(x, \bar{y})|$$
$$\leqslant \int_0^x \{k_1(t) + k_2(t) Q + k_3(t)[c(t) Q + d(t)]\}\, |g_i(t; x, y) - g_i(t; x, \bar{y})|\, dt$$
$$\leqslant K_1 \exp(L_1)\, |y - \bar{y}|,$$

where

$$K_1 = K_1(a) = \int_0^a \{k_1(t) + k_2(t) Q + k_3(t)[c(t) Q + d(t)]\}\, dt.$$

Hence

$$|(Su)_i(x, y) - (Su)_i(x, \bar{y})| \leqslant (\Phi + K_1) \exp(L_1)\, |y - \bar{y}|,$$

thus we can take $Q^S_\varphi = (\Phi + K_1)\exp(L_1)$. This proves the lemma.

LEMMA 5. *If Assumptions* H_1–H_4 *are satisfied, then for* $(x, y) \in \bar{I}^u_{\varphi i}$, $u \in B_\varphi(a, P, Q, \omega)$, *the function* $(Su)_i$, $i = 1, \ldots, n$, *is Lipschitzian in* x *with some constant* P^S_φ.

Proof. Since $2\Lambda a < b$, it follows that, in virtue of the theorem on prolongation of a solution to the boundary for an ordinary differential equation, for any two points (x, y), $(\bar{x}, y) \in \bar{I}^u_{\varphi i}$ $(x \leqslant \bar{x})$ we can find the point $(x, \bar{y}) \in \bar{I}^u_{\varphi i}$, such that $\bar{y} = g_i(x; \bar{x}, y)$.

Since the points $(x, \bar{y})$ and $(\bar{x}, y)$ belong to the same characteristic $\xi = g_i(t; \bar{x}, y)$, thus we have

$$|(Su)_i(\bar{x}, y) - (Su)_i(x, y)|$$
$$\leqslant |(Su)_i(\bar{x}, y) - (Su)_i(x, \bar{y})| + |(Su)_i(x, \bar{y}) - (Su)_i(x, y)|$$
$$\leqslant \int_x^{\bar{x}} F\,dt + Q^S_\varphi |y - \bar{y}| \leqslant F|x - \bar{x}| + Q^S_\varphi |y - \bar{y}|.$$

On the other hand, we have

$$|y - \bar{y}| = |y - g_i(x; \bar{x}, y)|$$
$$= \Big|\int_x^{\bar{x}} \lambda_i\big(t, g_i(t; \bar{x}, y), u\big(t, g_i(t; \bar{x}, y)\big), (V_i u)\big(t, g_i(t; \bar{x}, y)\big)\big)\,dt\Big| \leqslant \Lambda |x - \bar{x}|.$$

Hence, we see that

$$|(Su)_i(x, y) - (Su)_i(\bar{x}, y)| \leqslant (Q^S_\varphi \Lambda + F)|x - \bar{x}|.$$

It means that we can take $P^S_\varphi = \Lambda Q^S_\varphi + F$. This ends the proof.

LEMMA 6. *If Assumptions* H_1–H_4 *are satisfied, then in the sets* $\bar{I}^u_{0i}$ *and* $\bar{I}^u_{bi}$ *the function* $(Su)_i$, $i = 1, \ldots, n$, *satisfies a Lipschitz condition in* y *with some constant* Q^S_0.

Proof. Observe that the assumption $2\Lambda a < b$ gives $\bar{I}^u_{0i} \cap \bar{I}^u_{bi} = \emptyset$. Let us take (x, y), $(x, \bar{y}) \in \bar{I}^u_{0i}$ and $y \leqslant \bar{y}$ (the proof for $\bar{I}^u_{bi}$ is similar). Then, Lemma 2 gives

$$|(Ru)_i(x, y) - (Ru)_i(x, \bar{y})| \leqslant (H_1 + H_2 P)|\tau_i(x, y, u) - \tau_i(x, \bar{y}, u)|$$
$$\leqslant (H_1 + H_2 P)\Lambda_0^{-1} \exp(L_1)|y - \bar{y}|.$$

Furthermore, because of $\tau_i(x, y, u) \geqslant \tau_i(x, \bar{y}, u)$ for $y \leqslant \bar{y}$, and by Lemma 2 we find

$$|(Zu)_i(x, y) - (Zu)_i(x, \bar{y})|$$
$$\leqslant \int_{\tau_i(x,y,u)}^{x} \big[f_i\big(t, g_i(t; x, y), u\big(t, g_i(t; x, y)\big), (V_i u)\big(t, g_i(t; x, y)\big)\big) -$$

$$-f_i\big(t, g_i(t; x, \bar{y}), u(t, g_i(t; x, \bar{y})), (V_i u)(t, g_i(t; x, \bar{y}))\big)\big] dt + \int_{\tau_i(x,\bar{y},u)}^{\tau_i(x,y,u)} F dt$$

$$\leqslant \exp(L_1) K_1 |y-\bar{y}| + F|\tau_i(x, y, u) - \tau_i(x, \bar{y}, u)|$$

$$\leqslant \exp(L_1)(K_1 + F\Lambda_0^{-1})|y-\bar{y}|.$$

Therefore

$$|(Su)_i(x, y) - (Su)_i(x, \bar{y})| \leqslant \exp(L_1)[\Lambda_0^{-1}(H_1 + H_2 P + F) + K_1]|y-\bar{y}|.$$

Thus, we can put $Q_0^S = \exp(L_1)[\Lambda_0^{-1}(H_1 + H_2 P + F) + K_1]$, and the proof is complete.

Conclusion. From Lemmas 4 and 6 it follows that the function $(Su)_i$, $i = 1, \ldots, n$, satisfies in I_a a Lipschitz condition in y with the constant

$$Q^S = \exp(L_1)[\max\{\Phi, \Lambda_0^{-1}(H_1 + H_2 P + F)\} + K_1].$$

Obviously, $Q^S \geqslant Q_\varphi^S$ and $Q^S \geqslant Q_0^S$. If the points (x, y) and $(x, \bar{y})$ belong to the different sets $\bar{I}_{\varphi i}^u$, $\bar{I}_{0i}^u$, $\bar{I}_{bi}^u$, then this case reduces, in view of Lemma 3, to the one already considered.

LEMMA 7. *If Assumptions* H_1–H_4 *are satisfied, then the function* $(Su)_i$, $i = 1, \ldots, n$, *satisfies in* I_a *a Lipschitz condition in* x *with some constant* P^S.

Proof. It is obvious that for any two points (x, y), $(\bar{x}, y) \in I_a$, $x \leqslant \bar{x}$, one can find:

(a) if $(\bar{x}, y) \in \bar{I}_{\varphi i}^u$, then the point $(x, \bar{y}) \in I_a$, such that $\bar{y} = g_i(x; \bar{x}, y)$;

(b) if $(\bar{x}, y) \in \bar{I}_{0i}^u \cup \bar{I}_{bi}^u$, then the point $(\bar{x}, \bar{y}) \in I_a$, such that $y = g_i(x; \bar{x}, \bar{y})$.

Proceeding analogously as in the proof of Lemma 5, we can put $P^S = Q^S \Lambda + F$. This ends the proof.

Remark 3. In particular, without loss of generality we may assume that $\Lambda \geqslant 1$. Then, by Lemmas 4, 6 and 7, we conclude that the function Su satisfies in I_a a Lipschitz condition with respect to both variables with the constant P^S.

LEMMA 8. *If Assumptions* H_1–H_4 *are satisfied, and* $a \in (0, a_0]$ *is sufficiently small such that*

$$a \leqslant \omega(H_1 + H_2 P + \Phi\Lambda + F)^{-1}, \tag{21}$$

then the operator S *maps* $B_\varphi(a, P, Q, \omega)$ *into* $B_\varphi(a, P^S, Q^S, \omega)$.

Proof. This lemma will be proved by showing that

$$\|(Su)(x, y) - \varphi(y)\| \leqslant \omega, \tag{22}$$

and

$$(Su)(0, y) = \varphi(y), \qquad y \in [0, b], \tag{23}$$

for $u \in B_\varphi(a, P, Q, \omega)$.

First let $(x, y) \in \bar{I}^u_{\varphi i}$, $i = 1, \ldots, n$. Then

$$|(Su)_i(x, y) - \varphi_i(y)| \leqslant \Phi |g_i(0; x, y) - y| + \int_0^x F dt \leqslant (\Phi \Lambda + F) a.$$

Now let $(x, y) \in \bar{I}^u_{0i}$ (the proof for $(x, y) \in \bar{I}^u_{bi}$ is analogous). Then, taking into consideration compatibility condition (16), and initial condition (2), we see that

$$\begin{aligned} &|(Ru)_i(x, y) - \varphi_i(y)| \\ &\leqslant \left|h_{0i}\big(\tau_i(x, y, u), u(\tau_i(x, y, u), 0)\big) - h_{0i}(0, u(0, 0))\right| + \left|h_{0i}(0, \varphi(0)) - \varphi_i(y)\right| \\ &\leqslant (H_1 + H_2 P)\tau_i(x, y, u) + |\varphi_i(0) - \varphi_i(y)| \leqslant (H_1 + H_2 P + \Phi\Lambda) a, \end{aligned}$$

since $y \leqslant \Lambda x \leqslant \Lambda a$ for $(x, y) \in \bar{I}^u_{0i}$.

Furthermore, evidently $|(Zu)_i(x, y)| \leqslant Fa$, so that, combining the previous estimates, we get

$$|(Su)_i(x, y) - \varphi_i(y)| \leqslant (H_1 + H_2 P + \Phi\Lambda + F) a.$$

Hence, by (21), we conclude that (22) holds. It is obvious that (23) is satisfied.

Finally, let us observe that from Lemmas 4, 6 and 7 it follows that Su satisfies in I_a a Lipschitz condition with respect to both variables, and the proof is complete.

From now on we make, additionally, assumption (21).

Note that, generaly speaking, $P^S \geqslant P$, $Q^S \geqslant Q$, therefore $B_\varphi(a, P, Q, \omega) \subset B_\varphi(a, P^S, Q^S, \omega)$. The operator S is defined on all $B(a)$. We shall also use the symbol S to denote the restriction of S to the ball $B_\varphi(a, P, Q, \omega)$.

4. Properties of the operator S^2. Now we are interested in properties of the operator $SS = S^2$.

LEMMA 9. *If Assumptions* H_1–H_4 *are satisfied, then for* $a \in (0, a_0]$ *sufficiently small and* P, Q *sufficiently large, the operator* S^2 *maps* $B_\varphi(a, P, Q, \omega)$ *into itself.*

Proof. Applying Lemma 8 to the function $Su \in B_\varphi(a, P^S, Q^S, \omega)$ we obtain

$$\|(S^2 u)(x, y) - \varphi(y)\| \leqslant \omega, \qquad \|(S^2 u)(x, y)\| \leqslant \Omega, \qquad (S^2 u)(0, y) = \varphi(y),$$

provided $a \leqslant \omega(H_1 + H_2 P^S + \Phi\Lambda + F)^{-1}$.

From Lemmas 4, 5, 6 and 7 it follows that the function $S^2 u$ satisfies in I_a a Lipschitz condition with respect to both variables with the constants P^{SS}, Q^{SS}, respectively. Since now the arguments of the operator S are not arbitrary functions of $B_\varphi(a, P^S, Q^S, \omega)$, but the functions of the form Su, therefore the Lipschitz constants of the function $S^2 u$ can be made more precise.

Indeed, for any two points $(x, y), (x, \bar{y}) \in \bar{I}^u_{\varphi i}$, by Lemma 4, we have

$$|(S^2 u)_i(x, y) - (S^2 u)_i(x, \bar{y})| \leqslant \exp(L^S_1)(\Phi + K^S_1)|y - \bar{y}|,$$

where

$$K^S_1 = \int_0^a \{k_1(t) + k_2(t) Q^S + k_3(t)[c(t) Q^S + d(t)]\}\, dt,$$

$$L^S_1 = \int_0^a \{l_1(t) + l_2(t) Q^S + l_3(t)[c(t) Q^S + d(t)]\}\, dt.$$

Let now $(x, y), (x, \bar{y}) \in \bar{I}^u_{0i}$ (the same proof works in the case $(x, y), (x, \bar{y}) \in \bar{I}^u_{bi}$). Then we have that $i \in J_0$, therefore for $j \notin J_0$ the point $(x, 0)$ belongs to the set $\bar{I}^u_{\varphi i}$. Hence, by Lemma 5, we obtain

$$|(Su)_j(x, 0) - (Su)_j(\bar{x}, 0)| \leqslant P^S_\varphi |x - \bar{x}|,$$

where $j \notin J_0$. According to assumption (i_4) of H_4, the function $h_{0i}(x, Su)$ does not depend on $(Su)_j$ for $j \in J_0$, thus for $(x, y), (x, \bar{y}) \in \bar{I}^{Su}_{0i}$, by Lemmas 2 and 5, we have

$$\begin{aligned}
&|(RSu)_i(x, y) - (RSu)_i(x, \bar{y})| \\
&\quad \leqslant H_1 |\tau_i(x, y, Su) - \tau_i(x, \bar{y}, Su)| + \\
&\qquad\qquad + H_2 \max_{j \notin J_0} |(Su)_j(\tau_i(x, y, Su), 0) - (Su)_j(\tau_i(x, \bar{y}, Su), 0)| \\
&\quad \leqslant H_1 \Lambda_0^{-1} \exp(L^S_1)|y - \bar{y}| + H_2 P^S_\varphi |\tau_i(x, y, Su) - \tau_i(x, \bar{y}, Su)| \\
&\quad \leqslant \Lambda_0^{-1} \exp(L^S_1) \{H_1 + H_2[\Lambda \exp(L_1)(\Phi + K_1) + F]\} |y - \bar{y}|.
\end{aligned}$$

Furthermore, by Lemma 6, we get

$$|(ZSu)_i(x, y) - (ZSu)_i(x, \bar{y})| \leqslant \exp(L^S_1)(K^S_1 + F\Lambda_0^{-1})|y - \bar{y}|.$$

Combining the estimates above we find

$$\begin{aligned}
&|(S^2 u)_i(x, y) - (S^2 u)_i(x, \bar{y})| \\
&\quad \leqslant \big[\Lambda_0^{-1} \exp(L^S_1) \{H_1 + H_2[\Lambda \exp(L_1)(\Phi + K_1) + F]\} + \\
&\qquad\qquad + \exp(L^S_1)(K^S_1 + F\Lambda_0^{-1})\big] |y - \bar{y}|.
\end{aligned}$$

Hence, for the function $S^2 u$ as a Lipschitz constant in y we can take

$$Q^{SS} = \exp(L_1^S)\left(\max\{\Phi, \Lambda_0^{-1}[H_1+H_2(\Lambda\exp(L_1)(\Phi+K_1)+F)+F]\}+K_1^S\right).$$

Consequently, in virtue of Lemma 7, we conclude that the function $S^2 u$ satisfies in I_a a Lipschitz condition in x with the constant $P^{SS} = Q^{SS}\Lambda+F$.

Obviously, without loss of generality we may assume that $\Lambda \geqslant 1$. Hence, in particular, we can take P^{SS} as a Lipschitz constant of the function $S^2 u$ with respect to both variables.

Thus, in order to show that the operator S maps $B_\varphi(a, P, Q, \omega)$ into $B_\varphi(a, P^S, Q^S, \omega)$, and the operator S^2 maps the ball $B_\varphi(a, P, Q, \omega)$ into itself, one needs the following restrictions on the constants $\omega\in(0, \Omega-\Phi_1]$, $P\geqslant 0$, $Q\geqslant\Phi$, $a\in(0, a_0]$:

$$\begin{aligned}&(H_1+H_2P+\Phi\Lambda+F)a\leqslant\omega, \quad \Lambda a\leqslant\varepsilon_0,\ 2\Lambda a<b,\\ &(H_1+H_2P^S+\Phi\Lambda+F)a\leqslant\omega, \quad P^{SS}\leqslant P,\ Q^{SS}\leqslant Q.\end{aligned} \tag{24}$$

Observe that, if ω, P, Q are fixed, and $a\to 0^+$, then

$$P^S\to\Lambda\max\{\Phi, \Lambda_0^{-1}(H_1+H_2P+F)\}+F,$$

$$Q^S\to\max\{\Phi, \Lambda_0^{-1}(H_1+H_2P+F)\},$$

$$P^{SS}\to\Lambda\max\{\Phi, \Lambda_0^{-1}[H_1+H_2(\Lambda\Phi+F)+F]\}+F,$$

$$Q^{SS}\to\max\{\Phi, \Lambda_0^{-1}[H_1+H_2(\Lambda\Phi+F)+F]\}.$$

Therefore, for arbitrary $\omega\in(0, \Omega-\Phi_1]$, if

$$P>\Lambda\max\{\Phi, \Lambda_0^{-1}[H_1+H_2(\Lambda\Phi+F)+F]\}+F,$$

$$Q>\max\{\Phi, \Lambda_0^{-1}[H_1+H_2(\Lambda\Phi+F)+F]\},$$

then, for sufficiently small $a\in(0, a_0]$, all inequalities of (24) are satisfied.

Thus, we have

$$S^2\colon B_\varphi(a, P, Q, \omega)\to B_\varphi(a, P, Q, \omega), \tag{25}$$

which ends the proof.

In the sequel we shall assume that the constants P, Q and a are chosen in such a way that (25) is satisfied.

Now we need to show that the operator S^2 is a contraction. Therefore, first we shall investigate the operator S from this point of view.

LEMMA 10. *If Assumptions* H_1–H_4 *are satisfied, then for all* $(x, y)\in\bar I_{\varphi i}^u\cap\bar I_{\varphi i}^v$, $u, v\in B_\varphi(a, P, Q, \omega)$, *we have*

$$\|(Su)(x, y)-(Sv)(x, y)\|\leqslant q_1\|u-v\|_a,$$

where q_1 *is some constant, such that* $q_1\to 0^+$ *as* $a\to 0^+$.

Proof. Since $(x, y) \in \bar{I}^u_{\varphi i} \cap \bar{I}^v_{\varphi i}$, by Lemma 1, we have

$$|(Ru)_i(x, y) - (Rv)_i(x, y)| \leqslant \Phi L_2 \exp(L_1) \|u - v\|_a.$$

Moreover,

$$|(Zu)_i(x, y) - (Zv)_i(x, y)| \leqslant \int_0^x \big[k_1(t) L_2 \exp(L_1) \|u - v\|_a +$$
$$+ k_2(t) \big\|u(t, g_i[u](t; x, y)) - v(t, g_i[v](t; x, y))\big\| +$$
$$+ k_3(t) \big\|(V_i u)(t, g_i[u](t; x, y)) - (V_i v)(t, g_i[v](t; x, y))\big\|\big]\, dt$$
$$\leqslant \int_0^x \{k_1(t) L_2 \exp(L_1) \|u - v\|_a + k_2(t) \big[Q |g_i[u](t; x, y) - g_i[v](t; x, y)| +$$
$$+ \|u - v\|_a\big] + k_3(t) \big[(c(t) Q + d(t)) |g_i[u](t; x, y) - g_i[v](t; x, y)| +$$
$$+ m(t) \|u - v\|_a\big]\}\, dt$$
$$\leqslant [L_2 \exp(L_1) K_1 + K_2] \|u - v\|_a,$$

where $K_2 = K_2(a) = \int_0^a [k_2(t) + k_3(t) m(t)]\, dt$.

Combining the above estimates, we get

$$|(Su)_i(x, y) - (Sv)_i(x, y)| \leqslant q_1 \|u - v\|_a, \qquad i = 1, \ldots, n,$$

where $q_1 = L_2 \exp(L_1)(\Phi + K_1) + K_2$. This completes the proof.

LEMMA 11. *If Assumptions* H_1–H_4 *are satisfied, then for every* $(x, y) \in \bar{I}^u_{0i} \cap \bar{I}^v_{0i}$ *or* $(x, y) \in \bar{I}^u_{bi} \cap \bar{I}^v_{bi}$, *and* $u, v \in B_\varphi(a, P, Q, \omega)$, *we have*

$$\|(Su)(x, y) - (Sv)(x, y)\| \leqslant q_2 \|u - v\|_a,$$

where q_2 *is a constant.*

Proof. Assume that $(x, y) \in \bar{I}^u_{0i} \cap \bar{I}^v_{0i}$; then by Lemma 2 we find

$$|(Ru)_i(x, y) - (Rv)_i(x, y)|$$
$$\leqslant H_1 |\tau_i(x, y, u) - \tau_i(x, y, v)| + H_2 \big\|u(\tau_i(x, y, u), 0) - v(\tau_i(x, y, v), 0)\big\|$$
$$\leqslant [\Lambda_0^{-1} L_2 \exp(L_1)(H_1 + H_2 P) + H_2] \|u - v\|_a.$$

Furthermore, let us assume that $\tau_i(x, y, u) \leqslant \tau_i(x, y, v)$. Then by using Lemmas 1 and 2 we see that

$$|(Zu)_i(x, y) - (Zv)_i(x, y)|$$
$$\leqslant \int_{\tau_i(x,y,v)}^x \{[k_1(t) + k_2(t) Q + k_3(t)(c(t) Q + d(t))] |g_i[u](t; x, y) - g_i[v](t; x, y)| +$$

$$+[k_2(t)+k_3(t)m(t)]\,\|u-v\|_a\}\,dt+F\,|\tau_i(x,y,u)-\tau_i(x,y,v)|$$
$$\leqslant [L_2\exp(L_1)(K_1+F\Lambda_0^{-1})+K_2]\,\|u-v\|_a.$$

Hence, we have

$$|(Su)_i(x,y)-(Sv)_i(x,y)| \leqslant q_2\,\|u-v\|_a, \quad i = 1, \ldots, n,$$

where $q_2 = L_2\exp(L_1)[\Lambda_0^{-1}(H_1+H_2P+F)+K_1]+H_2+K_2$.
Thus, the Lemma 11 is proved.

LEMMA 12. *If Assumptions* H_1–H_4 *are satisfied, then for every* $(x, y)\in \bar{I}^u_{\varphi i}\cap \bar{I}^v_{0i}$ *(or* $(x, y)\in \bar{I}^u_{\varphi i}\cap \bar{I}^v_{bi}$, *or* $(x, y)\in \bar{I}^u_{0i}\cap \bar{I}^v_{\varphi i}$, *or* $(x, y)\in \bar{I}^u_{bi}\cap \bar{I}^v_{\varphi i}$), *and* $u, v\in B_\varphi(a, P, Q, \omega)$ *we have*

$$\|(Su)(x,y)-(Sv)(x,y)\| \leqslant q_3\,\|u-v\|_a,$$

where q_3 *is a constant.*

Proof. We consider only the case where $(x, y)\in \bar{I}^u_{\varphi i}\cap \bar{I}^v_{0i}$ (the remaining cases may be handled in the same way). Since $\tau_i(x, y, u) = 0$ for $(x, y)\in \bar{I}^u_{\varphi i}$, and $y\leqslant \Lambda a$ for $(x, y)\in \bar{I}^v_{0i}$, by Lemma 2, we have

$$|\tau_i(x,y,v)| \leqslant \Lambda_0^{-1}\,L_2\exp(L_1)\,\|u-v\|_a.$$

Since $i\in J_0$ for $(x, y)\in \bar{I}^v_{0i}$ in virtue of assumption (iv$_1$) of H_1, we have $\lambda_i(x, y, u, v)\geqslant \Lambda_0$, because of $i\in J_0$, and $y\leqslant \varepsilon_0$. Thus the function $g_i[u](t; x, y)$ is increasing in t, for $t\in[0, x]$, whence

$$g_i[u](0;x,y) \leqslant g_i[u](\tau_i(x,y,v);x,y).$$

In view of Lemma 1, we get

$$\big|g_i[u](\tau_i(x,y,v);x,y)-g_i[v](\tau_i(x,y,v);x,y)\big| \leqslant L_2\exp(L_1)\,\|u-v\|_a.$$

Hence, we obtain

$$g_i[u](0;x,y) \leqslant L_2\exp(L_1)\,\|u-v\|_a.$$

In view of compatibility conditions (iii$_4$) of H_4 and initial condition (2), we find

$$|(Ru)_i(x,y)-(Rv)_i(x,y)|$$
$$\leqslant \big|\varphi_i(g_i(0;x,y))-\varphi_i(0)\big|+\big|h_{0i}\big(\tau_i(x,y,v),\,v(\tau_i(x,y,v),0)\big)-h_{0i}(0,v(0,0))\big|$$
$$\leqslant \Phi\,|g_i(0;x,y)|+(H_1+H_2P)\,|\tau_i(x,y,v)|$$
$$\leqslant L_2\exp(L_1)[\Phi+(H_1+H_2P)\Lambda_0^{-1}]\,\|u-v\|_a.$$

Moreover,

$$|(Zu)_i(x, y)-(Zv)_i(x, y)|$$
$$\leqslant \int_{\tau_i(x,y,v)}^{x} \{[k_1(t)+k_2(t)Q+k_3(t)(c(t)Q+d(t))]\,|g_i[u](t; x, y)-$$
$$-g_i[v](t; x, y)|+[k_2(t)+k_3(t)m(t)]\,\|u-v\|_a\}\,dt+\int_0^{\tau_i(x,y,v)} F\,dt$$
$$\leqslant [L_2\exp(L_1)(K_1+F\Lambda_0^{-1})+K_2]\,\|u-v\|_a.$$

Combining the estimates above, we conclude that

$$|(Su)_i(x, y)-(Sv)_i(x, y)| \leqslant q_3\,\|u-v\|_a,$$

where $q_3 = L_2\exp(L_1)[\Phi+\Lambda_0^{-1}(H_1+H_2P+F)+K_1]+K_2$. Thus, the proof is finished.

Conclusion. From the assumption $2\Lambda a < b$ it follows that $\bar{I}^u_{0i} \cap \bar{I}^v_{bi} = \emptyset$, $\bar{I}^u_{bi} \cap \bar{I}^v_{0i} = \emptyset$. Thus, the cases considered in Lemmas 10, 11 and 12 cover all rectangle I_a. These lemmas show that in I_a we have

$$\|Su-Sv\|_a \leqslant q_4\,\|u-v\|_a, \tag{26}$$

where $q_4 = q_3+H_2$ (since $q_1 \leqslant q_3$).

Note that, generally, it has not to be $H_2 < 1$. Thus, in general case the operator S is not a contraction.

LEMMA 13. *If Assumptions* H_1–H_4 *are satisfied, then for all* $u, v \in B_\varphi(a, P, Q, \omega)$, *we have*

$$\|(S^2u)(x, y)-(S^2v)(x, y)\| \leqslant q^S\,\|u-v\|_a,$$

where the constant $q^S \to 0^+$ *as* $a \to 0^+$.

Proof. Let $(x, y) \in \bar{I}^{Su}_{\varphi i} \cap \bar{I}^{Sv}_{\varphi i}$. Using Lemma 10, we get

$$\|(S^2u)(x, y)-(S^2v)(x, y)\| \leqslant q_1^S\,\|Su-Sv\|_a,$$

with $q_1^S = L_2\exp(L_1^S)(\Phi+K_1^S)+K_2$.

Let $(x, y) \in \bar{I}^{Su}_{\varphi i} \cap \bar{I}^{Sv}_{0i}$ (analogously we can consider the cases $(x, y) \in \bar{I}^{Su}_{\varphi i} \cap \bar{I}^{Sv}_{bi}$, $(x, y) \in \bar{I}^{Su}_{0i} \cap \bar{I}^{Sv}_{\varphi i}$, $(x, y) \in \bar{I}^{Sv}_{\varphi i} \cap \bar{I}^{Su}_{bi}$). Then the assumptions of Lemma 12 are satisfied, and we have

$$\|(S^2u)(x, y)-(S^2v)(x, y)\| \leqslant q_3^S\,\|Su-Sv\|_a,$$

where $q_3^S = L_2\exp(L_1^S)[\Phi+\Lambda_0^{-1}(H_1+H_2P^S+F)+K_1^S]+K_2$.

Let now $(x, y) \in \bar{I}^{Su}_{0i} \cap \bar{I}^{Sv}_{0i}$ (the case $(x, y) \in \bar{I}^{Su}_{bi} \cap \bar{I}^{Sv}_{bi}$ is similar). This implies that $i \in J_0$. Thus, the point $(x, 0)$ belongs to $\bar{I}^u_{\varphi j} \cap \bar{I}^v_{\varphi j}$ for $j \notin J_0$. Consequently, by Lemma 10, we obtain

$$|(Su)_j(x, 0) - (Sv)_j(x, 0)| \leqslant q_1 \|u - v\|_a.$$

In virtue of assumption (i_4) of H_4, the function $h_{0i}(x, Su)$, $i \in J_0$, does not depend on $(Su)_j$ for $j \in J_0$. Therefore

$$\begin{aligned} |(RSu)_i(x, y) - (RSv)_i(x, y)| &\leqslant H_1 |\tau_i(x, y, Su) - \tau_i(x, y, Sv)| + \\ &+ H_2 \max_{j \notin J_0} \big[|(Su)_j(\tau_i(x, y, Su), 0) - (Sv)_j(\tau_i(x, y, Su), 0)| + \\ &+ |(Sv)_j(\tau_i(x, y, Su), 0) - (Sv)_j(\tau_i(x, y, Sv), 0)| \big] \\ &\leqslant H_1 \Lambda_0^{-1} L_2 \exp(L_1^S) \|Su - Sv\|_a + H_2 q_1 \|u - v\|_a + \\ &+ H_2 [\Lambda \exp(L_1)(\Phi + K_1) + F] |\tau_i(x, y, Su) - \tau_i(x, y, Sv)| \\ &\leqslant \Lambda_0^{-1} L_2 \exp(L_1^S) \{H_1 + H_2 [\Lambda \exp(L_1)(\Phi + K_1) + F]\} \|Su - Sv\|_a + \\ &+ H_2 q_1 \|u - v\|_a. \end{aligned}$$

We may assume that $\tau_i(x, y, Su) \leqslant \tau_i(x, y, Sv)$; then we have

$$\begin{aligned} |(ZSu)_i(x, y) - (ZSv)_i(x, y)| &\leqslant \int\limits_{\tau_i(x,y,Sv)}^{x} \{[k_1(t) + k_2(t) Q^S + \\ &+ k_3(t)(c(t) Q^S + d(t))] |g_i[Su](t; x, y) - g_i[Sv](t; x, y)| + \\ &+ (k_2(t) + k_3(t) m(t)) \|Su - Sv\|_a\} dt + \int\limits_{\tau_i(x,y,Su)}^{\tau_i(x,y,Sv)} F\, dt \\ &\leqslant [K_1^S L_2 \exp(L_1^S) + K_2 + F \Lambda_0^{-1} L_2 \exp(L_1^S)] \|Su - Sv\|_a. \end{aligned}$$

Hence, in $\bar{I}^{Su}_{0i} \cap \bar{I}^{Sv}_{0i}$ (also in $\bar{I}^{Su}_{bi} \cap \bar{I}^{Sv}_{bi}$), we get

$$\|(S^2 u)(x, y) - (S^2 v)(x, y)\| \leqslant q_2^S \|Su - Sv\|_a + H_2 q_1 \|u - v\|_a,$$

where

$$q_2^S = L_2 \exp(L_1^S) \{\Lambda_0^{-1} [H_1 + H_2(\Lambda \exp(L_1)(\Phi + K_1) + F) + F] + K_1^S\} + K_2.$$

Thus, combining the estimates above, we find (we remind that $\bar{I}^{Su}_{0i} \cap \bar{I}^{Sv}_{bi} = \bar{I}^{Su}_{bi} \cap \bar{I}^{Sv}_{0i} = \emptyset$, because of $2\Lambda a < b$) in I_a

$$\begin{aligned} &\|(S^2 u)(x, y) - (S^2 v)(x, y)\| \\ &\qquad \leqslant [q_2^S + L_2 \exp(L_1^S)(\Phi + \Lambda_0^{-1} H_2 P^S)] \|Su - Sv\|_a + H_2 q_1 \|u - v\|_a. \end{aligned}$$

Finally, by (26), we obtain

$$\begin{aligned} &\|(S^2 u)(x, y) - (S^2 v)(x, y)\| \\ &\qquad \leqslant \{[q_2^S + L_2 \exp(L_1^S)(\Phi + \Lambda_0^{-1} H_2 P^S)] q_4 + H_2 q_1\} \|u - v\|_a. \end{aligned}$$

It should be remarked that $q_2^S \to 0^+$, $L_2 \to 0^+$, $L_1^S \to 0^+$, $q_1 \to 0^+$, $q_4 \to H_2$, $P^S \to \text{const}$ as $a \to 0^+$.

Taking

$$q^S = [q_2^S + L_2 \exp(L_1^S)(\Phi + A_0^{-1} H_2 P^S)] q_4 + H_2 q_1,$$

where certainly $q^S \to 0^+$ as $a \to 0^+$, we get the assertion of Lemma 13.

5. The main result.

THEOREM. *If Assumptions* H_1–H_4 *are satisfied, then for any* $\omega \in (0, \Omega - \Phi_1]$, *and any sufficiently large constants* P, Q, *there are a number* a, $a \in (0, a_0]$, *and a function* u: $I_a \to \boldsymbol{R}^n$, $u \in B_\varphi(a, P, Q, \omega)$, *which satisfies* (1) *a.e. in* I_a *and* (2), (3) *everywhere in* $[0, b]$, $[0, a]$, *respectively. Furthermore,* u *is unique in* $B_\varphi(a, P, Q, \omega)$.

Proof. Let us choose P, Q and a in such a way that inequalities (24) are satisfied. Then, by Lemma 9, we see that

$$S\colon B_\varphi(a, P, Q, \omega) \to B_\varphi(a, P^S, Q^S, \omega),$$

$$S^2\colon B_\varphi(a, P, Q, \omega) \to B_\varphi(a, P, Q, \omega).$$

Next, let us take $a \in (0, a_0]$, so that $q^S < 1$. Then, by Lemma 13, we conclude that the operator S^2 is a contraction. Hence, in view of completeness of $B_\varphi(a, P, Q, \omega)$, it follows that there exists a function $u \in B_\varphi(a, P, Q, \omega)$ such that $S^2 u = u$. It is known [5] that the fixed point of any power of an operator is the fixed point of this operator. Thus $Su = u$. For this fixed element, we derive from (18) the integral equations

$$u_i(x, y) = (Ru)_i(x, y) + (Zu)_i(x, y), \quad i = 1, \ldots, n, \tag{27}$$

where R and Z are defined by (19), (20), respectively.

It remains to prove that the fixed point u of the operator S satisfies system (1) a.e. in I_a and conditions (2), (3) everywhere in $[0, b]$ and $[0, a]$, respectively.

First we consider the case where $(x, y) \in I_{\varphi i}^u$. By taking $y = g_i(x; 0, \eta)$ in (27), we have

$$\begin{aligned} &u_i(x, g_i(x; 0, \eta)) \\ &\quad = \varphi_i(\eta) + \int_0^x f_i\big(t, g_i(t; 0, \eta), u(t, g_i(t; 0, \eta)), (V_i u)(t, g_i(t; 0, \eta))\big)\, dt, \\ &u_i(0, \eta) = \varphi_i(\eta), \end{aligned} \tag{28}$$

since (7) yields $g_i(t; x, g_i(x; 0, \eta)) = g_i(t; 0, \eta)$.

Note that, for every $\eta \in [0, b]$, the first member of (28) is absolutely continuous in x as the superposition of two Lipschitzian functions. In view of the Chain Rule Differentiation Lemma (4ii) [3], we derive that the relations

$$\frac{d}{dx} u_i(x, g_i(x; 0, \eta)) = D_x u_i + D_x g_i D_y u_i, \quad i = 1, \ldots, n,$$

hold a.e. in I_a. By differentiation of (28), and using relation (5), we get

$$D_x u_i(x, g_i(x; 0, \eta)) + \lambda_i(x, g_i(x; 0, \eta), u(x, g_i(x; 0, \eta)), (V_i u)(x, g_i(x; 0, \eta))) D_y u_i(x, g_i(x; 0, \eta))$$
$$= f_i(x, g_i(x; 0, \eta), u(x, g_i(x; 0, \eta)), (V_i u)(x, g_i(x; 0, \eta))), \quad i = 1, \ldots, n.$$

In other words, these relations hold a.e. in the rectangle I_a of (x, η) space. By taking $y = g_i(x; 0, \eta)$, and using the property that this transformation preserves sets of measure zero (being Lipschitzian), we obtain

$$D_x u_i(x, y) + \lambda_i(x, y, u(x, y), (V_i u)(x, y)) D_y u_i(x, y) = f_i(x, y, u(x, y), (V_i u)(x, y)), \quad i = 1, \ldots, n,$$

and these relations hold a.e. in $I^u_{\varphi i}$ of the (x, y) space.

Now we consider the case where $(x, y) \in I^u_{0i}$. By taking $y = g_i(x; \eta, 0)$ in (27) and using the equality $\tau_i(x, g_i(x; \eta, 0)) = \eta$, we have

$$(29) \quad \begin{aligned} u_i(x, g_i(x; \eta, 0)) &= h_{0i}(\eta, u(\eta, 0)) + \int_\eta^x f_i(t, g_i(t; \eta, 0), u(t, g_i(t; \eta, 0)), (V_i u)(t, g_i(t; \eta, 0))) dt, \\ u_i(\eta, 0) &= h_{0i}(\eta, u(\eta, 0)), \quad i = 1, \ldots, n, \end{aligned}$$

since (7) yields $g_i(t; x, g_i(x; \eta, 0)) = g_i(t; \eta, 0)$.

Note that for every $\eta \in [0, a]$ the first member of (29) is absolutely continuous in x. By differentiating (29) with respect to x, in view of the Chain Rule Differentiation Lemma (4ii), and making use of (5), we derive

$$D_x u_i(x, g_i(x; \eta, 0)) + \lambda_i(x, g_i(x; \eta, 0), u(x, g_i(x; \eta, 0)), (V_i u)(x, g_i(x; \eta, 0))) D_y u_i(x, g_i(x; \eta, 0))$$
$$= f_i(x, g_i(x; \eta, 0), u(x, g_i(x; \eta, 0)), (V_i u)(x, g_i(x; \eta, 0))),$$

and this relations hold a.e. in $[0, a] \times [0, a]$ of (x, η) space. Finally, by taking $y = g_i(x; \eta, 0)$, that is, returning to the variables xy, we obtain

$$(30) \quad D_x u_i(x, y) + \lambda_i(x, y, u(x, y), (V_i u)(x, y)) D_y u_i(x, y) = f_i(x, y, u(x, y), (V_i u)(x, y)), \quad i = 1, \ldots, n, \ (x, y) \in I^u_{0i} \text{ (a.e.)}.$$

Since the transformation $y = g_i(x; \eta, 0)$ preserves sets of measure zero, we conclude that (30) holds a.e. in I_{0i}^u as stated. Analogously we can consider the case where $(x, y) \in I_{bi}^u$. The theorem is thereby proved.

Remark 4. It is easy to see that we can seek a solution of problem (1)–(3) in a slightly larger class than $B(a)$. Namely, we can require that u is Lipschitzian with respect to y for every x, and Lipschitzian with respect to x only for $y = 0$ and $y = b$. In this case the function Su is Lipschitzian with respect to both variables for every x and y, but the above assumption does not extend the class of generalized solutions of problem (1)–(3).

Remark 5. Note that the case where the initial condition (2) is replaced by the following $u(x, y) = \varphi(x, y)$, $(x, y) \in [-\delta, 0] \times [0, b]$, $\delta > 0$, can also be studied analogously.

Remark 6. All above results can be extend to the general diagonal case ($r+1$ independent variables). However, it leads to certain technical complications and does not require the application of new methods, and does not provide us with important results.

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Reçu par la Rédaction le 20.03.1987