A POISSON KERNEL ON HEISENBERG TYPE NILPOTENT GROUPS

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The classical nilpotent Lie groups which appear in the Iwasawa decomposition of semisimple Lie groups are embedded in a larger class of so-called groups of type H (see [8] and [12]). This class has been introduced by Kaplan in [7] and has recently attracted considerable attention (cf., e.g., [1], [2], [12], and [13]). If such a group $N$ is extended by the one-parameter group of dilations $A$, the group $S = NA$ may be regarded as an analogy of a rank one symmetric space, $N$ being the boundary; cf. [2], where the Riemannian structure of $S$ which generalizes that of the symmetric space has been investigated.

J. Cygan has proposed to study the harmonic functions on $S$ with respect to the Laplace–Beltrami operator on $S$ and has written a formula for what should be the Poisson kernel $P_a$, $a \in A$. In this paper we verify that in fact this formula is correct. We prove that the function $P(n, a) = P_a(n)$ is harmonic on $S$, and consequently so is the function $\varphi \ast P_a(n)$ for every $\varphi \in L^p(N)$, $1 \leq p \leq \infty$. We also show that

$$\lim_{a \to 0} \varphi \ast P_a(n) = \varphi(n) \text{ a.e.}$$

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1. Introduction. Let $n$ be a nilpotent 2-step algebra with an inner product. Denote by $V$ the orthogonal complement to its center $Z$. Then, for every $v \in V$, $\text{ad}_v$ maps $V$ into $Z$ and we have the orthogonal decomposition of $V$ given by

$$V = \text{Ker} \text{ad}_v \oplus D(v).$$
n is said to be of *Heisenberg type* (shortly, of *type H*) if for every unit vector \( v \in V \) the mapping \( \text{ad}_v : D(v) \to Z \) is a surjective isometry.

Every Lie algebra of type H arises as follows [7]. Let \( U \) and \( V \) be vector spaces with positive definite quadratic forms \(|\cdot|^2\). By definition, a composition of these quadratic forms is a bilinear map \( \mu : U \times V \to V \) which satisfies

\[
|\mu(u, v)| = |u||v|, \quad u \in U, \ v \in V,
\]

and for a \( u_0 \)

\[
\mu(u_0, v) = v, \quad v \in V.
\]

Let \( Z \) be the orthogonal complement to \( R_{u_0} \) and let \( \pi : U \to Z \) be the orthogonal projection. Define a bilinear map \( \Phi : V \times V \to U \) by

\[
\langle u, \Phi(v, v') \rangle = \langle \mu(u, v), v' \rangle.
\]

Then \( \pi\Phi \) is skew-symmetric [7], and \( n = \langle u, \Phi(v, v') \rangle \) with the bracket

\[
[(v, z), (v', z')] = (0, \pi\Phi(v, v'))
\]

and the inner product

\[
\langle (v, z), (v', z') \rangle = \langle v, v' \rangle + \langle z, z' \rangle
\]

is an algebra of type H.

Let \( N \) be a connected and simple connected Lie group whose Lie algebra is \( n \). If we identify \( N \) with \( n \) by the exponential map, the multiplication in \( N \) is given by

\[
(v, z)(v', z') = (v + v', z + z' + \frac{1}{2} \pi\Phi(v, v')).
\]

We denote by \( A \) the multiplicative group of \( R^+ \). Let

(1) \[ S = NA \]

be a semi-direct product of \( N \) and \( A \), \( A \) acting on \( N \) as dilations \( \delta_a(v, z) = (av, a^2 z) \). Thus we identify \( S \) with \( V \times Z \times R^+ \) and

\[
(v, z, a)(v', z', a') = (v + av', z + a^2 z' + \frac{1}{2} a \pi\Phi(v, v'), aa').
\]

\( S \) has a Lie algebra \( s = n \oplus R \) with the bracket

\[
[(v, z, \log a), (v', z', \log a')] = ((\log a)v' - (\log a') v, 2(\log a)z' - 2(\log a') z + \pi\Phi(v, v'), 0).
\]

In the Lie algebra \( s \) we select an inner product

\[
\langle (v, z, \log a), (v', z', \log a') \rangle_s = \langle v, v' \rangle + \langle z, z' \rangle + 4(\log a)(\log a')
\]

and the left-invariant metric it defines we denote also by \( \langle \cdot, \cdot \rangle_s \).

The above construction includes the noncompact rank one symmetric spaces as particular cases, which of course has been the reason to study
groups of type H (cf. [9]). Somewhat more about this inclusion is in [2].

Let $M$ be a rank one noncompact symmetric space. The group of isometries of $M$ is semisimple. Denote by $G$ its connected component. Let $\mathfrak{g}$ be the Lie algebra of $G$, $B$ the Killing form of $\mathfrak{g}$, $\theta$ the Cartan involution, and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ the Cartan decomposition [15]. Denote by $\mathfrak{a}$ a maximal Abelian subalgebra of $\mathfrak{p}$, which by assumption is one-dimensional. The positive part of the corresponding system of restricted roots is equal to $\{x, 2x\}$ or $\{x\}$ for a root $x$. Then

$$n = \mathfrak{g}^x \oplus \mathfrak{g}^{2x}$$

is a nilpotent type H subalgebra of $\mathfrak{g}$ with the center $\mathfrak{g}^{2x}$. For the Iwasawa decomposition

$$\mathfrak{g} = n \oplus \mathfrak{a} \oplus \mathfrak{t}$$

we write $N, A$, and $K$ for the corresponding subgroups of $G$. The map

$$N \times A \times K \ni (n, a, k) \rightarrow n a k \in G$$

is a diffeomorphism and $S = NA$ is a closed solvable subgroup of $G$ diffeomorphic to $M$.

The Poisson kernel for $M$ in terms of $NA$ is a family $P_a$ of convolution kernels on $N$ (see [5]):

$$P_a(n) = \frac{cc^d}{\left[ (c^2 + (4b)^{-1} |X_a|^2)^2 + b^{-1} |X_{2a}|^2 \right]^{d/2}},$$

where

$$n = \exp(X_a + X_{2a}), \quad X_a \in \mathfrak{g}^x, \quad X_{2a} \in \mathfrak{g}^{2x},$$

$$c = e^{2(\log a)}, \quad |X|^2 = -B(X, \theta X),$$

$$d = \dim \mathfrak{g}^x + 2 \dim \mathfrak{g}^{2x}, \quad b = \dim \mathfrak{g}^x + 4 \dim \mathfrak{g}^{2x},$$

and $c$ is such that

$$\int_N P_a(n) = 1.$$ 

Now, if $N$ is a group of type H and $S$ is given by (1), then the inner product

$$\langle (v, z, \log a), (v', z', \log a') \rangle_{\theta} = b \left( \langle v, v' \rangle + \langle z, z' \rangle + 2(\log a)(\log a') \right)$$

on $S$ corresponds to the product $-B(\cdot, \theta \cdot)$ (cf., e.g., [2]). So following a suggestion of J. Cygan we write

$$P_a(n) = \frac{ca^d}{\left[ (a^2 + \frac{1}{4} |v|^2)^2 + |z|^2 \right]^{d/2}},$$
where \( n = \exp(v, z) \), \( d = \dim V + 2 \dim Z \), and \( c \) is such that \( \int_n P_a(n) = 1 \). The conjecture that \( P_a(n) \) is the Poisson kernel for \( S \) can be also somewhat metamathematically supported by the fact that the fundamental solution for the sublaplacian on the nilpotent groups, which appear in the Iwasawa decomposition of the rank one symmetric spaces, has been rewritten for the groups of type \( H \) in a similar way and in this shape is the fundamental solution for the corresponding sublaplacian (see [7] and [9]).

2. The Laplace–Beltrami operator. Let \( \{e_i\} \), \( i = 1, \ldots, \dim V \), and \( \{e_r\}, \ r = \dim V + 1, \ldots, \dim V + \dim Z \), be orthonormal bases in \( V \) and in \( Z \), respectively, and \( e_0 \) a vector in \( R \) such that \( \langle e_0, e_0 \rangle = \frac{1}{2} \cdot \{e_i, e_r, e_0\} \) is a \( \langle \cdot, \cdot \rangle_S \)-orthonormal basis of \( S \). Denote by \( E_\beta \) the left-invariant vector field on \( S \) corresponding to \( e_\beta \) for \( \beta = 0, 1, \ldots, \dim V + \dim Z \).

**Theorem 2.1.** The Laplace–Beltrami operator \( \Delta \) is of the form

\[
\Delta = \sum_\beta E_\beta^2 - \frac{d}{2} E_0.
\]

**Proof.** The proof is based on the following facts:

1. If \( V \) is a Riemannian connection on \( M \), then \( (\text{div} X)_p \) is equal to the trace of \( T_p M \ni w \rightarrow \nabla_w X \in T_p M \) (cf. [4] and [10]).

2. We have \( \Delta g = \text{div}(\text{grad} g) \) for \( g \in C^\infty(M) \).

3. The Riemannian connection on \( S \) expressed in terms of \( E_\beta \) (cf. [2]) is given by

\[
\nabla_{E_i} E_i = \frac{1}{2} E_0, \quad \nabla_{E_r} E_r = E_0, \quad \nabla_{E_0} E_0 = 0.
\]

Now

\[
\text{div}(\text{grad} g) = \sum_\beta \langle E_\beta, \nabla_{E_\beta} (\text{grad} g) \rangle_S
\]

\[
= \sum_\beta E_\beta \langle E_\beta, (\text{grad} g) \rangle_S - \sum_\beta \langle \nabla_{E_\beta} E_\beta, (\text{grad} g) \rangle_S
\]

\[
= \sum_\beta E_\beta^2 g - \frac{d}{2} \langle E_0, (\text{grad} g) \rangle_S = \sum_\beta E_\beta^2 g - \frac{d}{2} E_0 g.
\]

**Theorem 2.2.** We have \( \Delta P = 0 \).

**Proof.** We write \( h = a^2 + \frac{1}{2} |u|^2 \) and \( f = h^2 + |z|^2 \). Then \( P = a^d f^{-d/2} \). To calculate \( E_\beta g \) for \( g \in C^\infty(S) \) we use the formula

\[
E_\beta g(v, z, a) = \frac{d}{dt} g((v, z, a)(\exp t E_\beta))|_{t=0}.
\]
Hence by a simple calculation we obtain

\[ E_0 h = a^2, \quad E_r h = 0, \quad E_i h = \frac{1}{2} a v_i, \]
\[ E_0 f = 2 h a^2, \quad E_r f = 2 a^2 z_r, \quad E_i f = a (h v_i + \langle \mu(z, v), e_i \rangle), \]

where

\[ v = \sum_i v_i e_i, \quad z = \sum_r z_r e_r. \]

Moreover,

\[
\sum_\beta (E_\beta f)^2 = 4 h^2 a^4 + 4 a^4 |z|^2 + a^2 h^2 |v|^2 + \\
+ 2 a h \sum_i v_i \langle \mu(z, v), e_i \rangle + a^2 \sum_i \langle \mu(z, v), e_i \rangle^2 .
\]

But

\[
\sum_i v_i \langle \mu(z, v), e_i \rangle = \langle \mu(z, v), v \rangle = \langle z, \pi \Phi(v, v) \rangle = 0
\]

and

\[
\sum_i \langle \mu(z, v), e_i \rangle^2 = |\mu(z, v)|^2 = |z|^2 |v|^2.
\]

Hence, finally,

\[
\sum_\beta (E_\beta f)^2 = 4 h a^2 f .
\]

On the other hand, it is easy to see that

\[
E_i^2 |z|^2 = \frac{1}{2} a^2 |\pi \Phi(v, e_i)|^2
\]

and

\[
\Delta |z|^2 = 2 (\dim Z) a^4 + \frac{1}{2} a^2 \sum_i |\pi \Phi(v, e_i)|^2
\]

\[
= 2 (\dim Z) a^4 + \frac{1}{2} a^2 \sum_i \langle \mu(e_r, v), e_i \rangle^2
\]

\[
= 2 (\dim Z) a^4 + \frac{1}{2} a^2 \sum_r |\mu(e_r, v)|^2 = 2 (\dim Z) a^2 h ,
\]

\[
\Delta |v|^2 = 2 (\dim V) a^2 , \quad \Delta a = \frac{1 - d}{4} a , \quad \Delta a^d = 0 ,
\]

\[
\Delta h = a^2 (1 + \frac{1}{2} \dim V - \frac{1}{2} d) , \quad \Delta f = 4 h a^2 .
\]

Finally we have

\[
\Delta P = a^d \Delta f^{-d/2} + 2 \sum_\beta (E_\beta a^d) (E_\beta f^{-d/2})
\]

\[
= a^d \Delta f^{-d/2} + 2 (E_0 a^d) (E_0 f^{-d/2})
\]
\[
= a^d \left( \left( -\frac{d}{2} \right) \left( -\frac{d}{2} - 1 \right) f^{-d/2 - 2} \sum_\beta (E_\beta f)^2 + \right. \\
+ \left( -\frac{d}{2} \right) f^{-d/2 - 1} \Delta f + d \left( -\frac{d}{2} \right) f^{-d/2 - 1} E_0 f \right) \\
= a^d \left( -\frac{d}{2} \right) f^{-d/2 - 1} \left( \left( -\frac{d}{2} - 1 \right) 4ha^2 + 4ha^2 + 2dha^2 \right) = 0.
\]

3. Fatou's theorem. We define a gauge by

\[
\|(v, z)\| = \left( \frac{1}{16} |v|^4 + |z|^2 \right)^{1/4},
\]

which is homogeneous of degree 1, i.e., for dilations \( \delta_a \)

\[
\|\delta_a (v, z)\| = a \| (v, z) \|
\]

(see [3]). \( N \) with \( \| \cdot \| \) is a space of homogeneous type. Moreover, \( \| \cdot \| \) is subadditive, i.e.,

\[
\|n n_1\| \leq \|n\| + \|n_1\|, \quad n, n_1 \in N
\]

(see [1]). We have the following estimate for \( P_a(n) \):

(2)

\[
P_a(n) = \frac{cd^d}{(a^d + \|n\|^d)^{d/2}}.
\]

Consequently, \( P_a(n) \in L^p(N), \ 1 \leq p \leq \infty \). Moreover, for every left-invariant differential operator \( \partial \) on \( N \) we have

(3)

\[
\partial P_a \in L^p(N), \quad 1 \leq p \leq \infty .
\]

Indeed, it can be easily seen that \((E_{\beta_1} \ldots E_{\beta_k} f)^{-1}\) is bounded, where as above \( f = (a^2 + \frac{1}{4} |v|^2 + |z|^2) \). It follows from (3) that for every \( \varphi \in L^p(N) \) and every left-invariant differential operator \( \partial \) we have

\[
\partial (\varphi \ast P_a) = \varphi \ast \partial P_a.
\]

Now we are ready to prove an analogy of the classical Fatou's theorem (cf. [11] and [16]).

**Theorem 3.1.** If \( \varphi \in L^p(N) \), then

\[
\lim_{a \to 0} \varphi \ast P_a(n) = \varphi(n) \ a.e.
\]

**Proof.** The proof goes by a routine way. First we show

(a) If \( \varphi \in C_c(N) \), then

\[
\lim_{a \to 0} \varphi \ast P_a(n) = \varphi(n).
\]
Then we prove

(b) The operator $T$ defined by

$$T\varphi(n) = \sup_{a > 0} |\varphi * P_a(n)|$$

is of weak type $(1, 1)$ (see [14]).

From (a) and (b) we infer that Theorem 3.1 holds for $\varphi \in L^1(N)$ (cf., e.g., [14], p. 60).

Since

$$P_a(n) = a^{-d} P_1(\delta_{a^{-1}}(n)) \quad \text{and} \quad \int P_1(n) dn = 1,$$

$P_a(n)$ is an approximate identity and (a) follows.

To prove (b), let $B_r(n) = \{n_1: \|nn_1^{-1}\| < r\}$. Denote by $m$ the standard Lebesgue measure on $N$. Since $N$ with $\|\cdot\|$ is a space of homogeneous type, the operator which assigns to each function $\varphi \in L^1(N)$ its maximal function

$$\varphi^*(n) = \sup_{r > 0} \frac{1}{m(B_r(n))} \int_{B_r(n)} |\varphi(n_1)| dn_1 = \sup_{r > 0} \frac{1}{m(B_r(0))} \int_{B_r(0)} |\varphi(nn_1^{-1})| dn_1$$

is of weak type $(1, 1)$.

Now it is sufficient to show that there is a constant $D$ such that for every $\varphi \in L^1(N)$

$$T\varphi(n) \leq D \varphi^*(n).$$

Notice that

$$P_a(n) \leq \frac{ca^d}{[a^4 + \|n\|^4]^{d/2}} \leq ca^{-d},$$

and if $\|n\| \geq a$, then

$$\frac{ca^d}{[a^4 + \|n\|^4]^{d/2}} \leq \frac{ca \|n\|^{d-1}}{\|n\|^{2d}} = ca \|n\|^{d-1}.$$

By a standard method [6], since $m(B_r(0)) = r^d m(B(0))$, we have

$$\left| \int_N \varphi(nn_1^{-1}) P_a(n_1) dn_1 \right|$$

$$\leq \int_{\|n_1\| < a} |\varphi(nn_1^{-1})| P_a(n_1) dn_1 + \sum_{k=1}^{\infty} \int_{2^{k-1}a \leq \|n_1\| < 2^k a} |\varphi(nn_1^{-1})| P_a(n_1) dn_1$$

$$\leq ca^{-d} \int_{\|n_1\| < a} |\varphi(nn_1^{-1})| dn_1 + \sum_{k=1}^{\infty} ca(2^{k-1}a)^{-d-1} \int_{B_{2^k a}(0)} |\varphi(nn_1^{-1})| dn_1$$

$$\leq c\varphi^*(n) + c \cdot 2^{d+1} \left( \sum_{k=1}^{\infty} 2^{-k} \right) \varphi^*(n)$$

$$= (c + c \cdot 2^{d+1}) \varphi^*(n).$$
To prove Theorem 3.1 for $\varphi$ in $L^p(N)$, $p > 1$, we write for every $r > 0$

$$\varphi = \psi_r + \xi_r,$$

where $\psi_r \in L^1$ and $\xi_r$ is bounded and vanishes in the ball $B_r(0)$. Thus for $n \in B_{r/2}(0)$ we have

$$\varphi \ast P_a(n) = \psi_r \ast P_a(n) + \xi_r \ast P_a(n)$$

$$= \psi_r \ast P_a(n) + \int_{\|n_1\| > r/2} \xi_r(n n_1^{-1}) P_a(n_1) \, dn_1.$$ 

Since by (2) we get

$$\lim_{a \to 0} \int_{\|n_1\| > r/2} \xi_r(n n_1^{-1}) P_a(n_1) \, dn_1 \leq \lim_{a \to 0} \|\xi_r\|_{L^\infty} \int_{\|n_1\| > r/2} P_a(n_1) \, dn_1 = 0,$$

we have

$$\lim_{a \to 0} \varphi \ast P_a(n) = \varphi(n) \quad \text{a.e. for } n \in B_{r/2}(0),$$

and since $r$ is arbitrary, the equality holds for all $n$ in $N$.

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