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Oscillatoriness of solutions of a second order differential equation

by PAVEL ŠOLTÉS and EVA ŠVIDROŇOVÁ (Košice, Czechoslovakia)

Abstract. The present paper contains theorems on the oscillatory behaviour of solutions of the equation

(2) $$(r(t)x')' + p(t)f(x)h(x') = 0,$$

which are a generalization of certain oscillation theorems of [1] [4]. The paper consists of two sections; the first section deals with conditions under which a solution $x(t)$ of (2) with $h(x') = 1$ either is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. The second section is concerned with the same question under the additional condition $p(t) \geqslant 0$.

Bobisud [1] and H. Onose [2] have discussed the oscillatory behaviour of solutions of

$$x'' + a(t)f(x) = 0.$$

In this paper we shall investigate the properties of solutions of the equation

(1) $$x'' + a(t)x' + b(t)f(x)h(x') = 0,$$

where $a(t) \in C_0(J)$, $b(t) \in C_0(J)$, $f(x) \in C_0(\boldsymbol{R})$, $h(y) \in C_0(\boldsymbol{R})$ and $J = \langle t_0, \infty)$, $\boldsymbol{R} = (-\infty, \infty)$, $t_0 \in \boldsymbol{R}$.

Throughout this paper we assume that $h(x) > 0$ for all $x \in \boldsymbol{R}$ and $f(x) \operatorname{sgn} x > 0$ for $x \neq 0$.

If we denote

$$r(t) = \exp\Big[\int_{t_0}^{t} a(s)\,ds\Big],$$

then equation (1) can be written in the form

(2) $$(r(t)x')' + p(t)f(x)h(x') = 0,$$

where $p(t) = r(t)b(t)$. In the sequel we assume that the solutions of (1) are defined on J. A solution of (1) is said to be oscillatory if it has arbitrarily great zeros.

I. In the first section we assume that $h(y) = 1$; thus we investigate the oscillatoriness of solutions of the equation

$$(r(t)x')' + p(t)f(x) = 0, \tag{3}$$

where $r(t) \in C_0(J)$, $r(t) > 0$, $p(t) \in C_0(J)$ and $f(x) \in C_1(J)$ is a function satisfying assumptions as above.

THEOREM 1. *Suppose that the following conditions hold*:

1° $f'(x) \geqslant 0$ *on* $(-\infty, 0) \cup (0, \infty)$ *and for every* $\varepsilon > 0$

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty,$$

2° $0 < \lim_{t \to \infty} \int_{z_0}^{t} p(s)\,ds$ *for sufficiently lagre* z_0;

3° *there exist a function* $\varphi(t) \in C_1(J)$, $\varphi(t) > 0$ *and a constant* k_0 *such that for every* $t \in J$

$$|\varphi'(t)r(t)| \leqslant k_0;$$

4° $$\int^{\infty} \varphi(s)p(s)\,ds = \int^{\infty} \frac{ds}{\varphi(s)r(s)} = +\infty.$$

Then for every nonoscillatory solution $x(t)$ *of equation* (3) *we have* $\lim_{t \to \infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (3). Then there exists $t_1 \geqslant t_0$ such that $x(t) \neq 0$ for every $t \geqslant t_1$. Suppose e.g. $x(t) > 0$. Integrating (3) from t_1 to $t \geqslant t_1$ we get

$$\frac{r(t)x'(t)}{f(x(t))} - \frac{r(t_1)x'(t_1)}{f(x(t_1))} + \int_{t_1}^{t} \frac{f'(x(s))r(s)x'^2(s)}{f^2(x(s))}\,ds + \int_{t_1}^{t} p(s)\,ds = 0.$$

Taking into consideration Hypotheses 1 and 2 we obtain for sufficiently large t

$$\frac{r(t)x'(t)}{f(x(t))} < \frac{r(t_1)x'(t_1)}{f(x(t_1))}. \tag{4}$$

Thus there exists $t_2 \geqslant t_1$ that one of the following two inequalities holds:

1° $x'(t) \geqslant 0$ for every $t \geqslant t_2$,

2° $x'(t) < 0$ for every $t \geqslant t_2$.

Dividing (3) by $f(x(t))$, multiplying by $\varphi(t)$ and integrating from t_2 to $t \geqslant t_2$, we get

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}-\int_{t_2}^{t}\frac{\varphi'(s)r(s)x'(s)}{f(x(s))}ds+$$

$$+\int_{t_2}^{t}\frac{\varphi(s)r(s)x'^2(s)f'(x(s))}{f^2(x(s))}ds+\int_{t_2}^{t}\varphi(s)p(s)ds=0;$$

hence we have for $t \geqslant t_2$

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}- \tag{5}$$

$$-\int_{t_2}^{t}\frac{\varphi'(s)r(s)x'(s)}{f(x(s))}ds+\int_{t_2}^{t}\varphi(s)p(s)ds \leqslant 0.$$

Let 1° hold. Then according to (5) and the hypotheses of the theorem we get

$$\int_{t_2}^{t}\varphi(s)p(s)ds \leqslant \frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}+k_0\int_{x(t_2)}^{x(t)}\frac{du}{f(u)}<\infty$$

for every $t \geqslant t_2$, which leads to a contradiction with Hypothesis 4.

Now suppose that 2° holds and $\lim_{t\to\infty} x(t)=C>0$. From (5) we obtain for $t \geqslant t_2$

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}+\int_{t_2}^{t}\varphi(s)p(s)ds \leqslant \frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}-k_0\int_{x(t_2)}^{x(t)}\frac{du}{f(u)}$$

$$\leqslant \frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}+k_0\int_{C}^{x(t_2)}\frac{du}{f(u)}<\infty;$$

in view of Hypothesis 4 it follows that

$$\lim_{t\to\infty}\frac{\varphi(t)r(t)x'(t)}{f(x(t))}=-\infty.$$

Therefore there exists $t_3 \geqslant t_2$ such that for every $t \geqslant t_3$ we have

$$\frac{x'(t)}{f(x(t))} \leqslant -\frac{1}{\varphi(t)r(t)}, \tag{6}$$

and this implies

$$\int_{x(t_3)}^{x(t)} \frac{du}{f(u)} \leqslant -\int_{t_3}^{t} \frac{ds}{\varphi(s)r(s)}.$$

In limit as $t \to \infty$, the last inequality contradicts Hypothesis 1, and so $C = 0$.

A similar argument is used in the case where $x(t) < 0$ for every $t \geqslant t_1$.

Remark 1. If $r(t) = 1$ and $\varphi(t) = t$, then from our Theorem 1 we get Theorem 1 of [1].

The rest theorem is a generalization of Theorem 2 of [1] and Theorem 2 of [2].

THEOREM 2. *Suppose, in addition to assumptions* 1°, 3° *and* 4° *of Theorem* 1, *that*

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{z_0} \frac{du}{f(u)} < \infty, \quad \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{-z_0} \frac{du}{f(u)} < \infty \tag{7}$$

for every $z_0 > 0$.

Let $\varphi'(t)r(t)$ *be differentiable on* J *and*

$$(\varphi'(t)r(t))' \geqslant 0.$$

Then every solution $x(t)$ *of* (3) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (3) and let e.g. $x(t) > 0$ for every $t \geqslant t_1 \geqslant t_0$.

One of the following three cases necessarily occurs:

1° $x'(t) \geqslant 0$ for every $t \geqslant t_2 \geqslant t_1$,

2° $x'(t) < 0$ for every $t \geqslant t_2 \geqslant t_1$,

3° $x'(t)$ is oscillatory on $\langle t_2, \infty)$.

Proceeding as in the proof of Theorem 1 we show that case 1° is impossible.

Let 2° hold and let $\lim_{t \to \infty} x(t) = C \geqslant 0$. From relation (5) we get for $t \geqslant t_2$

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))} + \int_{t_2}^{t} \varphi(s)p(s)\,ds \leqslant \frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))} - k_0 \int_{x(t_2)}^{x(t)} \frac{du}{f(u)}$$

$$\leqslant \frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))} + k_0 \lim_{t \to \infty} \int_{x(t)}^{x(t_2)} \frac{du}{f(u)} < \infty,$$

and hence, in view of Hypothesis 4, we have

$$\lim_{t\to\infty}\frac{\varphi(t)r(t)x'(t)}{f(x(t))}=-\infty .$$

Taking inequality (6) and Hypothesis 4 into account we obtain

$$\lim_{t\to\infty}\int_{x(t)}^{x(t_3)}\frac{du}{f(u)}\geqslant\int_{t_3}^{\infty}\frac{ds}{\varphi(s)r(s)}=\infty .$$

But in view of (7) this is impossible.

Now consider case 3°. Inequality (5) implies

$$\text{(8)}\quad \frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}-\varphi'(t)r(t)\int_{x(t_2)}^{x(t)}\frac{du}{f(u)}+$$

$$+\int_{t_2}^{t}\Big([\varphi'(s)r(s)]'\int_{x(t_2)}^{x(s)}\frac{du}{f(u)}\Big)ds+\int_{t_2}^{t}\varphi(s)p(s)\,ds\leqslant 0.$$

We first prove that $\lim_{t\to\infty}x(t)=0$.

Suppose that $\liminf_{t\to\infty}x(t)=C>0$. If, according to (8), we choose t_2 large enough in order that

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}+\int_{t_2}^{t}\varphi(s)p(s)\,ds$$

$$\leqslant k_0\int_{C/2}^{\infty}\frac{du}{f(u)}+[\varphi'(t)r(t)-\varphi'(t_2)r(t_2)]\int_{C/2}^{\infty}\frac{du}{f(u)}<\infty ,$$

for $t\geqslant t_2$, then

$$\lim_{t\to\infty}\frac{\varphi(t)r(t)x'(t)}{f(x(t))}=-\infty .$$

This is a contradiction, because of the same reason as before. Therefore $\liminf_{t\to\infty}x(t)=0$.

Assume that $\limsup_{t\to\infty}x(t)=C_1>0$. Then from (8) we obtain for sufficiently large t_2

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}+\int_{t_2}^{t}\varphi(s)p(s)\,ds$$

$$\leqslant k_0\int_0^{2c_1}\frac{du}{f(u)}-\int_0^{2c_1}\frac{du}{f(u)}[\varphi'(t)r(t)-\varphi'(t_2)r(t_2)]<\infty,$$

and this again leads to a contradiction.

Consequently $\lim_{t\to\infty} x(t)=0$.

Applying once more relation (8), we get

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}-\frac{\varphi(t_2)r(t_2)x'(t_2)}{f(x(t_2))}+\int_{t_2}^{t}\varphi(s)p(s)\,ds$$

$$\leqslant k_0\int_0^{\infty}\frac{du}{f(u)}+[\varphi'(t)r(t)-\varphi'(t_2)r(t_2)]\int_0^{\infty}\frac{du}{f(u)}<\infty,$$

which yields a contradiction. This completes the proof.

THEOREM 3. *Suppose that* $f'(x)\geqslant\varepsilon>0$ *on* $(-\infty,0)\cup(0,\infty)$ *and there exists a function* $\varphi(t)\in C_1(J)$ *such that* $\varphi(t)>0$ *and*

$$\int_{t_0}^{\infty}\frac{\varphi'^2(s)r(s)}{\varphi(s)}ds=K<\infty.$$

Suppose, moreover, that Hypothesis 4 of Theorem 1 is true. Then every nonoscillatory solution $x(t)$ *of* (3) *satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (3), i.e., let $x(t)\neq 0$ for each $t\geqslant t_1\geqslant t_0$. We now define the function

$$v(t)=\frac{\varphi(t)r(t)x'(t)}{f(x(t))}.$$

It is easily verified that

$$v'(t)+\frac{f'(x(t))}{\varphi(t)r(t)}v^2(t)-\frac{\varphi'(t)}{\varphi(t)}v(t)=-\varphi(t)p(t). \tag{9}$$

Since for each real number t we have

$$at^2+bt\geqslant-\frac{b^2}{4a},$$

with suitable constants a and b, $a > 0$, from (9) we see that

$$v'(t) \leqslant \frac{1}{4\varepsilon} \frac{\varphi'^2(t) r(t)}{\varphi(t)} - \varphi(t) p(t).$$

Integrating the last inequality from t_1 to $t \geqslant t_1$ we get

$$\frac{\varphi(t) r(t) x'(t)}{f(x(t))} \to -\infty \quad \text{as } t \to \infty.$$

Thus there exists $t_2 \geqslant t_1$ such that $x(t)x'(t) < 0$ for all $t \geqslant t_2$. We now proceed exactly as in the proof of Theorem 2.

Obviously, we have

THEOREM 4. *Assume that the hypotheses of Theorem* 3 *are satisfied. If* (7) *holds, then every solution* $x(t)$ *of equation* (3) *is oscillatory.*

From the proof of Theorem 3 we get as a consequence:

THEOREM 3'. *Suppose that* $f'(x) \geqslant \varepsilon > 0$ *on* $(-\infty, 0) \cup (0, \infty)$ *and there exists a function* $\varphi(t) \in C_1(J)$ *such that* $\varphi(t) > 0$ *and for arbitrary constant* $k_1 > 0$

$$\int^{\infty} \left[k_1 \frac{\varphi'^2(s) r(s)}{\varphi(s)} - \varphi(s) p(s) \right] ds = -\infty,$$

and

$$\int^{\infty} \frac{ds}{\varphi(s) r(s)} = +\infty.$$

Then every nonoscillatory solution $x(t)$ *of* (3) *satisfies* $\lim_{t \to \infty} x(t) = 0$.

THEOREM 4'. *Assume that the hypotheses of Theorem* 3' *are satisfied. If* (7) *hold, then every solution* $x(t)$ *of equation* (3) *is oscillatory.*

Remark 2. It is evident from the proofs of Theorems 2 through 4' that instead of the assumptions $\varphi(t) > 0$, $r(t) > 0$ it is sufficient to assume $\varphi(t) r(t) > 0$ for $t \in J$.

II. The purpose of this section is to study the oscillatory behaviour of solutions of equation (2) under the assumptions $p(t) \geqslant 0$, $r(t) > 0$ for $t \in J$ and

$$\int^{\infty} \frac{ds}{r(s)} = +\infty.$$

THEOREM 5. *Let* $f(x)$ *be a nondecreasing function on* $\boldsymbol{R}$ *and let* $r(t)$

$\geqslant r_0 > 0$ *for each* $t \in J$. *Suppose that there exists a function* $\varphi(t) \in C_1(J)$, $\varphi(t) > 0$ *such that*

$$\int^{\infty} \{\varphi'(s)\}_+ r(s)\, ds = K < \infty. \tag{10}$$

If

$$\int^{\infty} \varphi(s) p(s)\, ds = +\infty, \tag{11}$$

then every solution $x(t)$ *of equation* (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (2). We may assume that $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Now it follows from (2) that

$$r(t)x'(t) \leqslant r(t_1)x'(t_1).$$

We distinguish the two cases:

1° $x'(t) \geqslant 0$ for each $t \geqslant t_1$;

2° there exists $t_2 \geqslant t_1$ such that $x'(t) < 0$ for every $t \geqslant t_2$.

In case 2° we clearly obtain

$$x(t) \leqslant x(t_2) + r(t_2)x'(t_2) \int_{t_2}^{t} \frac{ds}{r(s)} \to -\infty \qquad \text{as } t \to \infty.$$

Taking this and the condition $x(t) > 0$ into account we arrive at a contradiction.

We now show that also case 1° is impossible. Multiplying (2) by $\varphi(t)$, integrating from t_1 to t and using the hypotheses we get

$$f(x(t_1))h(\beta) \int_{t_1}^{t} \varphi(s)p(s)\, ds \leqslant \varphi(t_1)r(t_1)x'(t_1) + \frac{r(t_1)x'(t_1)}{r_0} \int_{t_1}^{t} \{\varphi'(s)\}_+ r(s)\, ds,$$

where $\beta \in \left\langle 0, \frac{r(t_1)x'(t_1)}{r_0} \right\rangle$ is a number such that $h(\beta) \leqslant h(x'(t))$ for all $t \geqslant t_1$. Letting $t \to +\infty$ in the last inequality we get a contradiction.

The theorem is proved.

THEOREM 6. *Assume that the hypotheses of Theorem 5 are satisfied but instead of assumption* (10) *let us now suppose that*

$$\int^{\infty} \frac{\{\varphi'(s)\}_+}{\varphi(s)}\, ds = K < \infty. \tag{10'}$$

Then every solution $x(t)$ *of* (2) *is oscillatory.*

Proof. Let $x(t) > 0$ and $x'(t) \geqslant 0$ for $t \geqslant t_1$. From equation (2) we obtain

$$\varphi(t) r(t) x'(t) \leqslant \varphi(t_1) r(t_1) x'(t_1) + \int_{t_1}^{t} \frac{\{\varphi'(s)\}_+}{\varphi(s)} \varphi(s) r(s) x'(s) \, ds$$

and according the Bellman's lemma we see that

$$\varphi(t) r(t) x'(t) \leqslant C_1 \exp \int_{t_1}^{t} \frac{\{\varphi'(s)\}_+}{\varphi(s)} ds \leqslant C_1 K,$$

where $C_1 = \varphi(t_1) r(t_1) x'(t_1)$. In view of this, equation (2) implies

$$\varphi(t) r(t) x'(t) + f(x(t_1)) h(\beta) \int_{t_1}^{t} \varphi(s) p(s) \, ds$$

$$\leqslant C_1 + \int_{t_1}^{t} \{\varphi'(s)\}_+ r(s) x'(s) \, ds \leqslant C_1 + C_1 K \int_{t_1}^{t} \frac{\{\varphi'(s)\}_+}{\varphi(s)} ds$$

$$= C_1 + C_1 K^2 < \infty .$$

This together with the fact that $x'(t) \geqslant 0$ and condition (11), gives a contradiction and the proof is complete.

Remark 3. Theorems 5 and 6 generalize Theorems 4 and 5 of [3] and also Theorems 2 and 3 of paper [4].

The following theorem is a generalization of Theorem 4 of [4].

THEOREM 7. *Let $f(x) \in C_1(\boldsymbol{R})$, $f'(x) \geqslant 0$ on $(-\infty, 0) \cup (0, \infty)$ and $r(t) \geqslant r_0 > 0$ for each $t \in J$. Suppose that there exists a function $\varphi(t) \in C_1(J)$, $\varphi(t) \geqslant 0$ such that*

$$|\varphi'(t) r(t)| \leqslant k_0 < \infty .$$

If (11) holds and for any $\varepsilon > 0$ we have

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \qquad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty,$$

then every solution $x(t)$ of equation (2) is oscillatory.

Proof. It is evident that if $x(t) > 0$ then $x'(t) \geqslant 0$ for $t \geqslant t_1$. From equation (2) we get

$$(12)\qquad \frac{\varphi(t)r(t)x'(t)}{f(x(t))}+\int_{t_1}^{t}\frac{\varphi(s)r(s)x'^2(s)f'(x(s))}{f^2(x(s))}ds+\int_{t_1}^{t}\varphi(s)p(s)h(x'(s))ds$$
$$=\frac{\varphi(t_1)r(t_1)x'(t_1)}{f(x(t_1))}+\int_{t_1}^{t}\frac{\varphi'(s)r(s)x'(s)}{f(x(s))}ds,$$

from which we see that

$$h(\beta)\int_{t_1}^{t}\varphi(s)p(s)ds\leqslant\frac{\varphi(t_1)r(t_1)x'(t_1)}{f(x(t_1))}+k_0\int_{x(t_1)}^{x(t)}\frac{du}{f(u)},$$

where $\beta\in\left\langle 0,\frac{r(t_1)x'(t_1)}{r_0}\right\rangle$ is a number such that $h(\beta)\leqslant h(x'(t))$ for all $t\geqslant t_1$. Relation (12) implies

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}\to-\infty\quad\text{ as } t\to\infty,$$

and this contradicts the fact that $x'(t)\geqslant 0$.

Thereby the theorem is proved.

Also the following theorem is a generalization of Theorem 4 of [4].

THEOREM 8. *Let* $f(x)\in C_1(\boldsymbol{R})$ *and let* $f'(x)\geqslant 0$ *on the set* $(-\infty,0)\cup(0,\infty)$. *Let, further,* $r(t)\geqslant r_0>0$ *for every* $t\in J$.

If there exists a function $\varphi(t)\in C_1(J)$, $\varphi(t)>0$, *such that*

$$\int^{\infty}\frac{\varphi'^2(s)r(s)}{\varphi(s)}ds=K<\infty,$$

then every solution $x(t)$ *of equation* (2) *is oscillatory.*

Proof. Let $x(t)>0$, $x'(t)\geqslant 0$ for $t\geqslant t_1$. From the relation (12) we get

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}+\varepsilon\int_{t_1}^{t}\varphi(s)r(s)\left(\frac{x'(s)}{f(x(s))}\right)^2ds-$$
$$-\int_{t_1}^{t}\varphi'(s)r(s)\frac{x'(s)}{f(x(s))}ds+h(\beta)\int_{t_1}^{t}\varphi(s)p(s)ds\leqslant\frac{\varphi(t_1)r(t_1)x'(t_1)}{f(x(t_1))}.$$

Hence for $t\geqslant t_1$ we obtain

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))}+h(\beta)\int_{t_1}^{t}\varphi(s)p(s)ds\leqslant\frac{\varphi(t_1)r(t_1)x'(t_1)}{f(x(t_1))}+\frac{1}{4\varepsilon}\int_{t_1}^{t}\frac{\varphi'^2(s)r(s)}{\varphi(s)}ds.$$

The last inequality implies that $x'(t) < 0$ for sufficiently large t and this contradicts the hypothesis. Thus the theorem is proved.

In the following theorems we assume that

$$\int^{\infty} \frac{ds}{r(s)} < \infty. \tag{13}$$

THEOREM 9. *Let* $\inf_{y\in \mathbf{R}} h(y) = h_0 > 0$ *and suppose that there exists a function* $\varphi(t) \in C_1(J)$, $\varphi(t) > 0$ *such that* $|\varphi'(t) r(t)| \leqslant k_0 < \infty$ *and*

$$\int^{\infty} \varphi(s) p(s)\, ds = \int^{\infty} \frac{ds}{\varphi(s) r(s)} = +\infty.$$

If (13) *holds, then any nonoscillatory solution* $x(t)$ *of equation* (2) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (2) and let e.g. $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then from equation (2) we have

$$r(t) x'(t) \leqslant r(t_1) x'(t_1).$$

Thus the two cases are possible:

1° $x'(t) \geqslant 0$ for every $t \geqslant t_1$;

2° there exists $t_2 \geqslant t_1$ such that $x'(t) < 0$ for every $t \geqslant t_2$.

Suppose that 1° holds. According to (13) there exists a number $k_1 > 0$ such that for every $t \geqslant t_1$

$$x(t_1) \leqslant x(t) \leqslant k_1.$$

Since $f(x) \in C_0(\mathbf{R})$ there exists a number $\alpha \in \langle x(t_1), k_1 \rangle$ such that for every $t \geqslant t_1$

$$0 < f(\alpha) \leqslant f\big(x(t)\big).$$

From equation (2) we obtain

$$\varphi(t) r(t) x'(t) - \int_{t_1}^{t} \varphi'(s) r(s) x'(s)\, ds + f(\alpha) h_0 \int_{t_1}^{t} \varphi(s) p(s)\, ds \leqslant \varphi(t_1) r(t_1) x'(t_1), \tag{14}$$

and so

$$f(\alpha) h_0 \int_{t_1}^{t} \varphi(s) p(s)\, ds \leqslant \varphi(t_1) r(t_1) x'(t_1) + k_0 \big(k_1 - x(t_1)\big).$$

The last inequality contradicts the assumption

$$\int^{\infty} \varphi(s)\,p(s)\,ds = \infty$$

for sufficiently large t.

Now suppose that 2° holds and let $\lim_{t\to\infty} x(t) = C > 0$. From relation (14) we get

$$\varphi(t)r(t)x'(t) + f(\alpha_1)h_0 \int_{t_2}^{t} \varphi(s)p(s)\,ds \leqslant \varphi(t_2)r(t_2)x'(t_2) + k_0\bigl(x(t_2) - C\bigr) < \infty,$$

where $\alpha_1 \in \langle C, x(t_2)\rangle$. It is easy to see that $\lim_{t\to\infty} \varphi(t)r(t)x'(t) = -\infty$, since $x(t) > 0$, we have a contradiction for every $t \geqslant t_1$. This completes the proof.

THEOREM 10. *Let* $f(x) \in C_1(\boldsymbol{R})$ *and* $f'(x) \geqslant \varepsilon > 0$ *on* $(-\infty, 0) \cup (0, \infty)$. *Let, further,* $r(t) \geqslant r_0 > 0$ *for every* $t \in J$. *If there exists a function* $\varphi(t) \in C_1(J)$, $\varphi(t) > 0$ *such that*

$$\int_{t_0}^{\infty} \frac{ds}{\varphi(s)r(s)} = K < \infty$$

and for arbitrary constant $k_1 > 0$ *we have*

$$\int^{\infty} \left[k_1\,\varphi(s)p(s) - \frac{1}{4\varepsilon}\,\frac{\varphi'^2(s)r(s)}{\varphi(s)}\right] ds = +\infty,$$

then every nonoscillatory $x(t)$ *of equation* (2) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 8 we get

$$\frac{\varphi(t)r(t)x'(t)}{f(x(t))} + \int_{t_1}^{t} \left[h(\beta)\,\varphi(s)p(s) - \frac{1}{4\varepsilon}\,\frac{\varphi'^2(s)r(s)}{\varphi(s)}\right] ds \leqslant \frac{\varphi(t_1)r(t_1)x'(t_1)}{f(x(t_1))};$$

hence it follows that

$$\lim_{t\to\infty} \frac{\varphi(t)r(t)x'(t)}{f(x(t))} = -\infty.$$

This shows that there exists $t_2 \geqslant t_1$ such that for every $t \geqslant t_2$

$$\int_{x(t_2)}^{x(t)} \frac{du}{f(u)} < -\int_{t_2}^{t} \frac{ds}{\varphi(s)r(s)}.$$

If $\lim_{t \to \infty} x(t) = C > 0$, then from the last inequality we obtain

$$\int_{t_2}^{t} \frac{ds}{\varphi(s)r(s)} < \int_{C}^{x(t_2)} \frac{du}{f(u)},$$

and we have a contradiction because of $f(x) \in C_0(\mathbf{R})$, $f(x) > 0$ for $x \in \langle C, x(t_2) \rangle$.

The last theorem is quite evident:

THEOREM 11. *Assume that the hypotheses of Theorem* 10 *are satisfied and that, moreover,*

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{z_0} \frac{du}{f(u)} < \infty, \quad \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{-z_0} \frac{du}{f(u)} < \infty.$$

Then every solution $x(t)$ of equation (2) *is oscillatory.*

Remark 4. From the proof of Theorems 5 through 8 and Theorem 10 is easy to see that the assumption $r(t) \geqslant r_0 > 0$ can be replaced by the condition $\inf_{y \in \mathbf{R}} h(y) = h_0 > 0$.

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