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Existence and non-existence of maximal solutions for $y'' = f(x, y, y')$ *

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1. Introduction. In this paper we shall consider the question of existence of a maximal solution for the initial value problem

(1) $$y'' = f(t, y, y'),$$

(2) $$y(t_0) = a_0, \quad y'(t_0) = \beta_0,$$

where $t_0 \in I$, $(a_0, \beta_0) \in R \times R$.

For the first-order scalar initial value problem

(3) $$y' = f(t, y),$$

(4) $$y(t_0) = a_0,$$

where $f(t, y)$ is continuous on $I \times R$ and $t_0 \in I$, $a_0 \in R$, Peano [9], Montel [8], and Perron [10] have given proofs of the existence of a maximal solution $y_0(t)$ of IVP(3) - (4). By a maximal solution for IVP(3) - (4), we mean a solution $y_0(t)$ of IVP(3) - (4) on its maximal interval of existence such that if $y(t)$ is any solution, then $y(t) \leqslant y_0(t)$ holds on the common interval of existence.

Kamke [4] asked the same question for the initial value problem for the system

(5) $$y' = F(t, y),$$

(6) $$y(t_0) = a_0,$$

where $F(t, y)$ is continuous on $I \times R^n$, $t_0 \in I$, and $a_0 \in R^n$. By a maximal solution in this case, we mean a solution $\varphi(t)$ of IVP(5) - (6) on its maximal interval of existence such that if $y(t)$ is any solution, then $\varphi_i(t) \geqslant y_i(t)$, $i = 1, \ldots, n$, holds on the common interval of existence. For the case

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$n = 2$ Kamke gave a counterexample showing the non-existence of a maximal solution for IVP(5) - (6).

He then showed that if, as an additional assumption imposed on F, each F_i is non-decreasing in $y_j, j \neq i$, then IVP(5) - (6) has a maximal solution in the sense defined above.

There are obviously other ways of generalizing the definition of a maximal solution for systems from that for the scalar case, IVP(3) - (4), Burton and Whyburn [1] and Lakshmikantham and Leela [7] have done so in introducing the concept of a minimax solution of IVP(5) - (6), and then imposing sufficient conditions on the right-hand side of (5) to ensure the existence of such solutions.

Our task for IVP(1) - (2) is different from the previous generalizations. By a *maximal solution* $y_0(t)$ of IVP(1) - (2), we mean a solution $y_0(t)$ on its right maximal interval of existence such that if $y(t)$ is any solution, then $y(t) \leqslant y_0(t)$ holds on the common interval of existence. Walter [12] has shown by an example that if $f(t, y, y')$ is continuous on $I \times R^2$, then IVP(1) - (2) need not have a maximal solution in the sense defined. However, in his example a local maximal solution does exist. We will say that $y_0(t)$ is a *local maximal solution* of IVP(1) - (2) in case there exists an $\varepsilon > 0$ such that $y_0(t)$ is a solution on $[t_0, t_0+\varepsilon]$ and if $y(t)$ is any other solution on $[t_0, w^+)$, then $y(t) \leqslant y_0(t)$ on $[t_0, \min(w^+, t_0+\varepsilon))$. We shall give an example of an IVP for (1) - (2) in which no maximal and no local maximal solution exists and then give conditions under which maximal solutions do exist.

We also shall use the concept of an upper solution. A function $\psi(t)$ is called a *$C^{(1)}$-upper solution* of (1) on I in case $\psi \in C(I) \cap C^{(1)}(\operatorname{int} I)$ and

$$\overline{D}\psi'(t) = \limsup_{\delta \to 0} \frac{\psi'(t+\delta) - \psi'(t-\delta)}{2\delta} \leqslant f\big(t, \psi(t), \psi'(t)\big),$$

on $\operatorname{int} I$. Lower solutions are defined analogously.

2. Assume throughout that $f(t, y, y')$ is continuous on $I \times R \times R$, I an interval and R the reals. Let $t_0 \in \operatorname{int} I$ and $(\alpha_0, \beta_0) \in R \times R$. Define the funnel of solutions of IVP(1) - (2), C_{t_0}, to be the set of all points (t, α, β) such that $t \geqslant t_0$ and there exists a solution $y(t)$ of (1) - (2) such that $y(t) = \alpha$, $y'(t) = \beta$. Also define $z_0(t) = \sup \alpha$, where $(t, \alpha, \beta) \in C_{t_0}$ for some β.

From these definitions, we can observe several results which we incorporate as the first theorem.

THEOREM 1. A. *If a maximal solution $y_0(t)$ for* (1) - (2) *exists, then it is unique.*

B. *If the maximal solution $y_0(t)$ exists on $[t_0, w_0^+)$, its maximal interval of existence, then $y_0(t) = z_0(t)$ on $[t_0, w_0^+)$.*

C. *If the maximal solution $y_0(t)$ of* (1) - (2) *exists on* $[t_0, w_0^+)$, *then the solution trajectory*

$$c_0 = \{(t, \alpha, \beta) | t \in [t_0, w_0^+), y_0(t) = \alpha, y_0'(t) = \beta\}$$

lies on the boundary of C_{t_0}.

D. *If $z_0(t)$ is a solution of* (1) - (2), *then $z_0(t)$ is the maximal solution.*

E. *Local maximal solutions are unique.*

The proofs of each of these statements are straightforward and are omitted.

We now give sufficient conditions on the right-hand side of (1) to ensure that $z_0(t)$ is a solution on some interval and hence a maximal solution. The next theorem is known, but the proof we give appears to be new.

THEOREM 2. *Suppose g is non-decreasing in y for fixed t and y'. Then there exists an $w_0^+ > t_0$ such that $z_0(t)$ is a solution on $[t_0, w_0^+)$ and hence $z_0(t)$ is the maximal solution of* IVP (1) - (2).

Proof. Let $u_n(t)$ be a solution of

$$y'' = f(t, y, y') + \frac{1}{n}, \quad y(t_0) = \alpha_0, \quad y'(t_0) = \beta_0$$

on $[t_0, w_n^+)$. Then $u_n''(t) > f(t, u_n(t), u_n'(t))$ which implies that $u_n(t)$ is a strict $C^{(2)}$-lower solution on $[t_0, w_n^+)$. By the Kamke convergence theorem ([2], p. 14), there exists a solution $y_0(x)$ of IVP (1) - (2) on $[t_0, w_n^+)$ such that, for every $t_1 < w_0^+$, there exists a subsequence $\{u_{n_j}(t)\}_{j=1}^{\infty}$ of $\{u_n(t)\}_{n=1}^{\infty}$ such that $u_{n_j}(t) \to y_0(t)$ and $u_{n_j}'(t) \to y_0'(t)$ uniformly on $[t_0, t_1)$.

We claim that $y_0(t) = z_0(t)$ on $[t_0, w_0^+)$. By definition of $z_0(t)$, $z_0(t) \geq y_0(t)$ on $[t_0, w_0^+)$. Suppose there exists $t_1 \in (t_0, w_0^+)$ such that $y_0(t_1) < z_0(t_1)$. Then since there exists a sequence $\{y_n(t)\}$ of solutions to IVP (1) - (2) such that $y_n(t_1) < z_0(t_1)$ for all n and $y_n(t_1) \to z_0(t_1)$ as $n \to \infty$, there exists an $N > 0$ such that $y_0(t_1) < y_N(t_1) < z_0(t_1)$. There also exists a subsequence $\{u_j(t)\}$ of $\{u_n(t)\}$ such that $u_j(t) \to y_0(t)$ in $C^{(1)}$-norm on $[t_0, t_1]$. For J sufficiently large, $u_J(t_1) < y_N(t_1)$.

Since $u_J(t_0) = y_N(t_0)$, $u_J'(t_0) = y_N'(t_0)$, and $y_N''(t_0) = f(t_0, y_N(t_0), y_N'(t_0)) = f(t_0, u_J(t_0), u_J'(t_0)) < u_J''(t_0)$, there exists a $\delta > 0$ such that $y_N''(t) < u_J''(t)$ for all $t \in [t_0, t_0 + \delta]$. Hence, for $t \in (t_0, t_0 + \delta]$,

$$u_J(t) - y_N(t) = u_J(t_0) - y_N(t_0) + [u_J'(t_0) - y_N'(t_0)](t - t_0) + \\ + [u_J''(t_0) - y_N''(t_0)] \frac{(t - t_0)^2}{2} > 0.$$

Let

$$t_2 = \inf\{t | t \in (t_0, t_1), y_N(t) = u_J(t)\}.$$

Since $y_N(t_0) = u_J(t_0)$, $y_N(t_2) = u_J(t_2)$, and $y_N(t) < u_J(t)$ on (t_0, t_2), there exists a $\bar{t} \in (t_0, t_2)$ such that $u_J - y_N$ attains a relative positive maximum at $\bar{t}$. This means that $u_J(\bar{t}) > y_N(\bar{t})$, $u'_J(\bar{t}) = y'_N(\bar{t})$, and $u''_J(\bar{t}) < y''_N(\bar{t})$. But

$$\begin{aligned} u''_J(\bar{t}) - y''_N(\bar{t}) &= f(\bar{t}, u_J(\bar{t}), u'_J(\bar{t})) + \frac{1}{J} - f(\bar{t}, y_N(\bar{t}), y'_N(\bar{t})) \\ &> f(\bar{t}, u_J(\bar{t}), u'_J(\bar{t})) - f(\bar{t}, y_N(\bar{t}), y'_N(\bar{t})) \\ &\geqslant 0 \end{aligned}$$

since f is non-decreasing in y. From this contradiction we conclude that $y_0(t) = z_0(t)$ on $[t_0, w_0^+)$. By Theorem 1, $z_0(t)$ is the maximal solution to IVP(1) - (2).

The next theorem is new. We shall say that solutions to boundary value problems (BVP) associated with (1) are unique in case $v(t)$ and $w(t)$, solutions of (1) with $v(t_1) = w(t_1)$, $v(t_2) = w(t_2)$, $t_1, t_2 \in I$, imply $v(t) = w(t)$ on (t_1, t_2).

THEOREM 3. *Assume solutions to two-point boundary value problems are unique when they exist. Then there exists an $w_0^+ > t_0$ such that $z_0(t)$ is a solution to (1) on $[t_0, w_0^+)$ and hence is the maximal solution.*

Proof. As a consequence of the Peano existence theorem ([2], p. 10), there exists a $\delta > 0$, $M > 0$, $M' > 0$ such that all solutions $y(t)$ of (1) - (2) exist on $[t_0, t_0 + \delta]$ and satisfy $|y(t)| \leqslant M$, $|y'(t)| \leqslant M'$ on $[t_0, t_0 + \delta]$. Let $\eta = z_0(t_0 + \delta)$.

There exists a sequence $\{y_n(t)\}$ of solutions of (1) - (2) such that $y_n(t_0 + \delta) \leqslant y_{n+1}(t_0 + \delta) \leqslant z_0(t_0 + \delta)$ and $y_n(t_0 + \delta) \to \eta$ as $n \to \infty$. By uniqueness of boundary value problems, we have $y_n(t) \leqslant y_{n+1}(t)$ on $[t_0, t_0 + \delta]$. Since we have a uniform bound on $\{y_n(t)\}$ and $\{y'_n(t)\}$ for all n, the sequences are uniformly bounded and equicontinuous on $[t_0, t_0 + \delta]$. By Ascoli's theorem, there exists a subsequence $\{y_{n_j}(t)\} = \{y_j(t)\}$ which converges in the $C^{(1)}$-norm to some solution $y_0(t)$ of (1) - (2) on $[t_0, t_0 + \delta]$. We claim $y_0(t) \equiv z_0(t)$ on $[t_0, t_0 + \delta]$. By definition, $z_0(t) \geqslant y_0(t)$. By construction $y_0(t_0) = z_0(t_0)$ and $y_0(t_0 + \delta) = \eta = z_0(t_0 + \delta)$. Suppose there exists $t_1 \in (t_0, t_0 + \delta)$ such that $y_0(t_1) < z_0(t_1)$. But then by definition of z_0, there exists a solution $w(t)$ of (1) - (2) such that

$$y_0(t_1) < w(t_1) < z_0(t_1).$$

We then have $y_0(t_0) = w(t_0)$, $y_0(t_1) < w(t_1)$, and $y_0(t_0 + \delta) \geqslant w(t_0 + \delta)$. This contradicts uniqueness. Hence, $z_0(t)$ is a solution to (1) - (2) on $[t_0, t_0 + \delta]$.

Consider next the initial value problem (1) with initial conditions:

$$y(t_0 + \delta) = z_0(t_0 + \delta), \quad y'(t_0 + \delta) = z'_0(t_0 + \delta). \tag{7}$$

Again by the Peano existence theorem there exists $t_2 > t_0 + \delta$ such that all solutions of (1) - (7) exist on $[t_0 + \delta, t_2]$ and are together with their first derivatives uniformly bounded on $[t_0 + \delta, t_2]$. Let $z_1(t) = \sup\{y(t) | y(t)$ a solution of (1) - (7)$\}$. As before, $z_1(t)$ is a solution on $[t_0 + \delta, t_2]$. We now show that $z_1(t) = z_0(t)$ on $[t_0 + \delta, t_2]$. If not, there exists $\varepsilon \in (t_0 + \delta, t_2]$ such that $z_0(\varepsilon) > z_1(\varepsilon)$. Then there exists a solution $y(t)$ to (1) - (2) on $[t_0, \varepsilon]$ such that $z_0(\varepsilon) > y(\varepsilon) > z_1(\varepsilon)$. By definition of z_0, $z_0(t) \geqslant y(t)$ on $[t_0, \varepsilon]$. If $y(t) = z_0(t)$ on $[t_0, t_0 + \delta]$, we have a contradiction to the definition of $z_1(t)$. If $y(t) < z_0(t)$ for some $t \in [t_0, t_0 + \delta]$, then $y(t_0) = z_0(t_0)$, $y(t) < z_0(t)$, and $y(\varepsilon) > z_1(\varepsilon)$ contradict the uniqueness of boundary value problems. Hence, $z_0(t) \leqslant z_1(t)$ on $[t_0 + \delta, t_2]$.

Suppose now that $z_0(\varepsilon) < z_1(\varepsilon)$ for some $\varepsilon \in (t_0 + \delta, t_2]$. Then the function

$$z(t) = \begin{cases} z_0(t), & t \in [t_0, t_0 + \delta], \\ z_1(t), & t \in [t_0 + \delta, t_2] \end{cases}$$

is a solution to (1) - (2) such that $z(\varepsilon) > z_0(\varepsilon)$. This contradicts the definition of $z_0(t)$. We conclude that $z_0(t) = z_1(t)$ on $[t_0 + \delta, t_2]$.

In this way, $z_0(t)$ can be extended as a solution to (1) - (2) to its maximal interval of existence $[t_0, w_0^+)$, where $w_0^+ \leqslant b$, $b = \sup I$. By Theorem 1, $z_0(t)$ is the maximal solution of (1) - (2).

Beckenbach, Jackson ([3], p. 314) and others have introduced the concept of a superfunction. A function $\psi(t)$ is said to be a *superfunction* with respect to solutions of (1) on I in case for any $[t_1, t_2] \subset I$ and any solution $y \in C^{(2)}[t_1, t_2]$, $y(t_i) \leqslant \psi(t_i)$, $i = 1, 2$, implies that $\psi(t) \geqslant y(t)$ on $[t_1, t_2]$. Subfunctions are defined dually. Jackson and Schrader [4] and Schrader [11] have investigated the interplay of uniqueness of boundary value problems and upper solutions being superfunctions.

If in place of uniqueness, we assume that $C^{(1)}$-upper solutions are superfunctions, we have the following theorem:

THEOREM 4. *If $C^{(1)}$-upper solutions are superfunctions, then $z_0(t)$ is a solution on $[t_0, w_0^+)$ and hence the maximal solution of* (1) - (2).

The proof is the same as Theorem 3 except that one evokes the condition that upper solutions are superfunctions rather than uniqueness whenever the uniqueness condition is used in the proof of Theorem 3.

It should be noted that Theorem 3 does not contain Theorem 2 or vice versa. They are just different theorems.

Walter [12] has given an example in which a local maximal solution to an initial value problem for (1) exists, but no maximal solution exists. We now give an example of an initial value problem with continuous right-hand side f in which no maximal and no local maximal solution exists.

Define

$$
(8)\quad F(t,y,y') = \begin{cases}
20t^3+8t^2-t, & t>0,\, y \geqslant t^5,\, y' \geqslant 5t^4+t^3,\\[4pt]
40t^3+8t^2-\dfrac{20y}{t^2}-\dfrac{y}{t^4}, & t>0,\, |y| \leqslant t^5,\, y' \geqslant 5t^4+t^3,\\[4pt]
\dfrac{8y'}{t}-20t^3-t, & t>0,\, y \geqslant t^5,\, |y'| \leqslant 5t^4+t^3,\\[4pt]
\dfrac{8y'}{t}-\dfrac{20y}{t^2}-\dfrac{y}{t^4}, & t>0,\, |y| \leqslant t^5,\, |y'| \leqslant 5t^4+t^3,\\[4pt]
\dfrac{8y'}{t}+20t^3+t, & t>0,\, y \leqslant -t^5,\, |y'| \leqslant 5t^4+t^3,\\[4pt]
-40t^3-8t^2-\dfrac{20y}{t^2}-\dfrac{y}{t^4}, & t>0,\, |y| \leqslant t^5,\, y' \leqslant -5t^4+t^3,\\[4pt]
-20t^3-8t^2+t, & t>0,\, y \leqslant -t^5,\, y' \leqslant -t^4-t^3,\\[4pt]
0, & t=0
\end{cases}
$$

and consider

$$(9)\qquad y'' = F(t,y,y').$$

It is straightforward to verify that $F(t,y,y')$ is continuous on $[0,\infty)\times R\times R$. $y_1(t) = t^5\sin t^{-1}$ and $y_2(t) = 0$ are solutions of (9) with initial conditions:

$$(10)\qquad y(0) = 0, \quad y'(0) = 0.$$

Clearly, neither $y_1(t)$ nor $y_2(t)$ is a maximal or a local maximal solution of (9) - (10).

Observe that no solution $y(t)$ of (9) - (10) exists with $y(t) \geqslant t^5$ on any interval $[0,\varepsilon]$, $\varepsilon > 0$. For if such a solution $y(t)$ did exist on some interval $[0,\varepsilon]$, then, by definition of F,

$$y''(t) = 20t^3+8t^2-t \quad \text{or} \quad y''(t) = \frac{8y'}{t}-20t^3-t.$$

In either case, $y''(t) \leqslant 20t^3+8t^2-t$ on $[0,\varepsilon]$ which implies $y(t) \leqslant t^5 + \frac{2}{3}t^4-\frac{1}{6}t^3$ on $[0,\varepsilon]$. But $t^5+\frac{2}{3}t^4-\frac{1}{6}t^3 < t^5$ for $t\in(0,1/4)$ since $\frac{2}{3}t^4-\frac{1}{6}t^3 = t^3[\frac{2}{3}t-\frac{1}{6}] < 0$ for $0<t<1/4$. Hence, there can exist no solution $y(t)$ of (9) - (10) such that $y(t) \geqslant t^5$ on any interval $[0,\varepsilon]$, $\varepsilon > 0$.

If there exists a local maximal solution $y_0(t)$ on $[0,\varepsilon]$, then we must have $y_0(t) \geqslant y_1(t)$, $y_0(t) \geqslant y_2(t)$, and $y_0(\varepsilon_n) < \varepsilon_n^5$ for a sequence with $\varepsilon_n > \varepsilon_{n+1} > 0$ and $\varepsilon_n \to 0$ as $n\to\infty$. Note that if $0 < y_0(t) < t^5$, then $y_0'(t) \geqslant 0$. Since if $0<y_0(t)<t^5$ and $y_0'(t)<0$, then

$$y_0''(t) = \frac{8y'}{t}-\frac{20y}{t^2}-\frac{y}{t^4} \quad \text{or} \quad y_0''(t) = -40t^3-8t^2-\frac{20y}{t^2}-\frac{y}{t^4}.$$

In either case, $y_0''(t) < 0$. This in turn implies that y_0 would cross y_2, which contradicts the maximality of y_0. Hence, whenever $y_0(t) < t^5$, $y_0'(t) \geqslant 0$.

We must then have that $y_0\left(\frac{2}{(4n+1)\pi}\right) \geqslant \left[\frac{2}{(4n+1)\pi}\right]^5$ for $n \geqslant N$, where $\frac{2}{(4N+1)\pi} < \varepsilon$, and $y_0(t) < t^5$ for some $t \in \left(\frac{2}{(4n+5)\pi}, \frac{2}{(4n+1)\pi}\right)$ for infinitely many $n \geqslant N$.

Choose the sequence $\{\varepsilon_n\}$ so that the function $z_0(t) = t^5 - y_0(t)$ has a positive maximum at $\varepsilon_n \in \left(\frac{2}{(4n+5)\pi}, \frac{2}{(4n+1)\pi}\right)$, for infinitely many $n \geqslant N$. Then $y_0(\varepsilon_n) < \varepsilon_n^5$, $\varepsilon_n > \varepsilon_{n+1} > 0$, and $\varepsilon_n \to 0$ as $n \to \infty$. Also $y_0'(\varepsilon_n) = 5\varepsilon_n^4$ and $0 < y_0(\varepsilon_n) < \varepsilon_n^5$ imply

$$y_0''(\varepsilon_n) = \frac{8}{\varepsilon_n} y_0'(\varepsilon_n) - \frac{20}{\varepsilon_n^2} y_0'(\varepsilon_n) - \frac{1}{\varepsilon_n^4} y_0(\varepsilon_n).$$

Hence,

$$z_0''(\varepsilon_n) = -20\varepsilon_n^3 + \frac{y_0(\varepsilon_n)}{\varepsilon_n^4}[20\varepsilon_n^2 + 1] \geqslant -20\varepsilon_n^3 + \varepsilon_n^{-4} y_0(\varepsilon_n). \tag{11}$$

Since $\frac{2}{(4n+5)\pi} < \varepsilon_n < \frac{2}{(4n+1)\pi}$ and $y_0(\varepsilon_n) \geqslant \left[\frac{2}{(4n+5)\pi}\right]^5$, from (11) we have

$$\begin{aligned} z_0''(\varepsilon_n) &\geqslant -20\left[\frac{2}{(4n+1)\pi}\right]^3 + \frac{[(4n+1)\pi]^4}{2^4} \cdot \frac{2^5}{[(4n+5)\pi]^5} \\ &= -\frac{160}{n^3\pi^3[4+1/n]^3} + \frac{2}{\pi}\left[\frac{(4+1/n)^4}{(4+5/n)^4}\right] \cdot \frac{1}{4n+5}. \end{aligned} \tag{12}$$

From (12), for n sufficiently large, $z_0''(\varepsilon_n) > 0$. But the sequence $\{\varepsilon_n\}$ was chosen so that z_0 has a positive maximum at each ε_n and hence $z_0''(\varepsilon_n) \leqslant 0$. This is a contradiction. We conclude that there cannot exist a local maximal or a maximal solution to (9) - (10).

3. Differential inequalities are closely related to maximal solutions. In fact, the usual proof of the existence of the maximal solution for (3) - (4) depends on the basic properties of a differential inequality associated with (3) - (4). In this section we shall find sufficient conditions to answer the following question affirmatively.

Question. If ψ is a function of class $C^{(1)}$ on $[a, b] \subset I$ such that $\bar{D}\psi'(t) \leqslant f(t, \psi(t), \psi'(t))$, does there exist a solution $y(t)$ of the initial value problem

$$y'' = f(t, y, y'), \quad y(t_0) = \psi(t_0), \quad y'(t_0) = \psi'(t_0) \tag{13}$$

with $t_0 \in [a, b)$ such that $y(t) \geqslant \psi(t)$ on $[t_0, t_0 + \varepsilon]$ for some $\varepsilon > 0$?

In addition to giving sufficient conditions under which the affirmative answer to the above question holds, we shall give an example showing that f continuous does not ensure the existence of a solution to (13) which dominates ψ to the right.

We first establish a lemma which will be needed later:

LEMMA 5. *Let $\psi(t)$ be a function of class $C^{(1)}$ on $[a, b] \subset \operatorname{int} I$. Then for any $\varepsilon > 0$, there exist $\delta > 0$, $C_0 > 0$, $\bar{C}_0 > 0$ (depending only on ψ and ε) such that, for any $t_0 \in [a, b]$ and any μ, σ with $|\psi(t_0) - \mu| \leqslant \varepsilon$, $|\psi'(t_0) - \sigma| \leqslant \varepsilon$, every solution of*

$$y'' = f(t, y, y'), \quad y(t_0) = \mu, \quad y'(t_0) = \sigma \tag{14}$$

exists on $[t_0 - \sigma, t_0 + \sigma]$ and $|y'(t)| \leqslant C_0$, $|y(t)| \leqslant \bar{C}_0$ on $[t_0 - \delta, t_0 + \delta]$.

Proof. Let $K_1 = \max|\psi(t)| + \varepsilon$, $K_2 = \max|\psi(t)| + \varepsilon$ on $[a, b]$, and let $M = \max|f(t, y, y')|$ for $t \in [a, b]$, $|y| \leqslant K_1 + 1$, and $|y'| \leqslant K_2 + 1$. As a consequence of the Peano theorem ([2], p. 11), there exists a $\delta > 0$ such that any solution $y(t)$ of (1) with $y(t_0) = \mu$, $y'(t_0) = \sigma$ exists and satisfies $|y(t)| \leqslant K_1 + 1$, $|y'(t)| \leqslant K_2 + 1$ on $[t_0 - \delta,\ t_0 + \delta]$.

Moreover, we have that

$$|y'(t) - y'(t_0)| = |y''(\xi)||t - t_0| \leqslant M\delta$$

which implies

$$|y'(t)| \leqslant M\delta + |y'(t_0)| \leqslant M\delta + \max_{[a,b]}|\psi'(t)| + \varepsilon = C_0.$$

Similarly,

$$|y(t)| \leqslant C_0\delta + \max_{[a,b]}|\psi(t)| + \varepsilon = \bar{C}_0.$$

Using the idea of a modified form of (1) as introduced by Jackson and Schrader [5], we can obtain information about the behavior of a maximal solution relative to an upper solution.

DEFINITION 6. Let $\alpha(t)$, $\beta(t) \in C^{(1)}[a, b]$ with $\alpha(t) \leqslant \beta(t)$ on $[a, b]$ and let $c > 0$ be such that $|\alpha'(t)| < c$, $|\beta'(t)| < c$ on $[a, b]$ and $C_0 \leqslant c$. Then define

$$F^*(t, y, y') = \begin{cases} f(t, y, c), & y' \geqslant c, \\ f(t, y, y'), & |y'| \leqslant c, \\ f(t, y, -c), & y' \leqslant -c, \end{cases}$$

and

$$F(t, y, y') = \begin{cases} F^*\big(t, \beta(t), y'\big) + [y - \beta(t)]^{\frac{1}{2}}, & y \geqslant \beta(t), \\ F^*(t, y, y'), & \alpha(t) \leqslant y \leqslant \beta(t), \\ F^*\big(t, \alpha(t), y'\big) - [\alpha(t) - y]^{\frac{1}{2}}, & y \leqslant \alpha(t). \end{cases}$$

$F(t, y, y')$ is called the *modification* of $f(t, y, y')$ relative to the triple $\alpha(t)$, $\beta(t)$, and c.

LEMMA 7. *For any $c > 0$, let F^* be as in Definition 6. Let $\psi(t)$ be as in Lemma 5. Then given $\varepsilon > 0$ there exists $\delta'' > 0$ such that every solution of*

$$y'' = F^*(t, y, y'), \quad y(t_0) = \mu, \quad y'(t_0) = \sigma \tag{15}$$

with $|\psi(t_0) - \mu| \leqslant \varepsilon$, $|\psi_0'(t_0) - \sigma| \leqslant \varepsilon$, $t_0 \in [a, b]$, exists on $[t_0 - \delta'', t_0 + \delta'']$, and satisfies $|y'(t)| \leqslant C_0$ on $[t_0 - \delta'', t_0 + \delta'']$, where C_0 is as in Lemma 5.

Proof. As in Lemma 5, there exist $\delta' > 0$, $K' > 0$, $C' > 0$ such that every solution of (15) exists on $[t_0 - \delta', t_0 + \delta']$ and satisfies $|y(t)| \leqslant K'$, $|y'(t)| \leqslant C'$ on $[t_0 - \delta', t_0 + \delta']$.

If $y(t)$ is a solution of (15), then

$$\begin{aligned} |y'(t) - y'(t_0)| &= |y''(\xi)||t - t_0| \\ &= \left|F^*\big(\xi, y(\xi), y'(\xi)\big)\right||t - t_0| \\ &\leqslant M'|t - t_0|, \end{aligned}$$

where $M' = \max|F^*(t, y, y')|$ for $t \in [a, b]$, $|y| \leqslant K'$, $|y'| \leqslant C'$. Hence,

$$\begin{aligned} |y'(t)| &\leqslant |y'(t_0)| + M'|t - t_0| \\ &\leqslant \max_{[a,b]}|\psi'(t)| + \varepsilon + M'|t - t_0| \end{aligned}$$

for all $t \in [t_0 - \delta', t_0 + \delta']$. Since $C_0 = \max\limits_{[a,a]}|\psi'(t)| + \varepsilon + M\delta$, there exists $0 < \delta'' \leqslant \delta'$ such that $M'\delta'' \leqslant M\delta$ and such that

$$|y'(t)| \leqslant C_0.$$

THEOREM 8. *Let ψ be a $C^{(1)}$-upper solution on $[a, b] \subset I$ and assume there exists a local maximal solution $y_0(t)$ to (13). Then either:*

(i) *there exists an $\varepsilon > 0$ such that $y_0(t) \geqslant \psi(t)$ on $[t_0, t_0 + \varepsilon]$, or*

(ii) *$y_0(t) - \psi(t)$ oscillates positive and negative infinitely often as $t \to t_0^+$.*

Proof. The only behaviour which must be ruled out is $y_0(t) < \psi(t)$ on $(t_0, t_0 + \eta)$ for some $\eta > 0$. Suppose this were the case. Let $\varepsilon > 0$ be given, then by Lemma 5 there exists $\delta > 0$ and $C_0 > 0$ such that any solution $y(t)$ of

$$y'' = f(t, y, y'), \quad y(t_0) = \psi(t_0), \quad y'(t_0) = \psi'(t_0)$$

exists and satisfies $|y'(t)| \leqslant C_0$ on $[t_0, t_0 + \delta]$.

Let $F(t, y, y')$ be the modification of $f(t, y, y')$ relative to the lower solution $y_0(t)$, the upper solution $\psi(t)$, and C_0. Then by Lemma 7, there exists $\delta' > 0$ such that all solutions $z(t)$ of

$$y'' = F^*(t, y, y'), \quad y(t_0) = \psi(t_0), \quad y'(t_0) = \psi'(t_0) \tag{16}$$

exist and satisfy $|z'(t)| \leqslant C_0$ on $[t_0, t_0+\delta']$. By Theorem 2.5 (cf. [3]), there exists a solution $w(t)$ to the boundary value problem

$$y'' = F(t, y, y'), \quad y(t_0) = \psi(t_0), \quad y(t_0+\delta'') = \psi(t_0+\delta''),$$

where $0 < \delta'' < \min[\delta, \delta', \eta]$ with $y_0(t) \leqslant w(t) \leqslant \psi(t)$ on $[t_0, t_0+\delta'']$. Hence, $y_0'(t_0) = w'(t_0) = \psi'(t_0)$ and $w(t)$ is then a solution of (16). Therefore, $|w'(t)| \leqslant C_0$ on $[t_0, t_0+\delta'']$ which implies by the definition of the modification of f that $w(t)$ is in fact a solution of the initial value problem $y'' = f(t, y, y')$, $y(t_0) = \psi(t_0)$, $y'(t_0) = \psi'(t_0)$ on $[t_0, t_0+\delta'']$. But $w(t_0+\delta'') = \psi(t_0+\delta'') > y_0(t_0+\delta)$ contradicts the fact that y_0 is the local maximal solution. We conclude that either case (i) or (ii) holds if there exists a local maximal solution.

We now proceed to show that if solutions to boundary value problems are unique, if $f(t, y, y')$ is non-decreasing in y, or if upper solutions are superfunctions, then we have behavoir as in (i). We shall then give an example which behaves as in (ii).

THEOREM 9. *Assume solutions to boundary value problems for* (1) *are unique if they exist. Let $\psi(t)$ be a $C^{(1)}$-upper solution on $[a, b] \subset I$ and let $y_0(t)$ be the maximal solution of* (13):

$$y'' = f(t, y, y'), \quad y(t_0) = \psi(t_0), \quad y'(t_0) = \psi'(t_0),$$

where $t_0 \in [a, b)$. Then there exists an $\varepsilon > 0$ such that $\psi(t) \leqslant y_0(t)$ on $[t_0, t_0+\varepsilon]$.

Proof. Assume that there exists no $\varepsilon > 0$ such that $y_0(t) \geqslant \psi(t)$ on $[t_0, t_0+\varepsilon]$. By Theorem 8, there then must exist sequences $\{\sigma_n\}$, $\{\mu_n\}$ such that $t_0 < \sigma_n < b$, $t_0 < \mu_n < b$, $\sigma_n \to t_0$, $\mu_n \to t_0$ as $n \to \infty$, and $\psi(\sigma_n) < y_0(\sigma_n)$, $\psi(\mu_n) > y_0(\mu_n)$ for all n. By Theorem 3.12 (cf. [3]) there exists a $\delta_0 > 0$ such that for any $t_1 > t_0$ with $t_1 - t_0 < \delta_0$ the boundary value problem $y'' = f(t, y, y')$, $y(t_0) = \psi(t_0)$, $y(t_1) = \psi(t_1)$ has a solution $z(t; t_1)$ such that $z(t; t_1) \leqslant \psi(t)$ on $[t_0, t_1]$. Choose t_1 such that $y_0(t_1) = \psi(t_1)$ and $t_1 - t_0 < \delta_0$ which is possible because $y_0(t) - \psi(t)$ oscillates positive and negative infinitely often as $t \to t_0^+$. Then for n sufficiently large, $t_0 < \sigma_n < t_1$ and $y_0(\sigma_n) > \psi(\sigma_n) \geqslant z(\sigma_n; t_1)$. But this contradicts uniqueness of boundary value problems.

Hence, there exists an $\varepsilon > 0$ such that $y_0(t) \geqslant \psi(t)$ on $[t_0, t_0+\varepsilon]$.

If we assume that $C^{(1)}$-upper solutions are super functions, then it follows that the maximal solution dominates the upper solution ψ on their common interval of existence. Schrader [11] has shown that if solutions to boundary value problems are unique and if solutions to initial value problems for (1) are extendable to I, then $C^{(1)}$-upper solutions are superfunctions.

THEOREM 10. *Assume that $C^{(1)}$-upper solutions are superfunctions. Then, if $\psi(t)$ is a $C^{(1)}$-upper solution on $[a, b] \subset I$ and if $y_0(t)$ is the maximal* *solution to the initial value problem on $[t_0, w_0^+)$, $y_0(t) \geqslant \psi(t)$ on $\big[t_0, \min(b, w_0^+)\big)$.*

Proof. By Theorem 4, the maximal solution $y_0(t)$ of (13) exists on $[t_0, w_0^+]$. Since $\psi(t)$ is a superfunction, case (ii) of Theorem 8 is impossible. Hence, there exists an $\varepsilon > 0$ such that $y_0(t) \geqslant \psi(t)$ on $[t_0, t_0+\varepsilon]$. If there exists $t_1 \in (t_0, t_0+\varepsilon]$ with $y_0(t_1) > \psi(t_1)$, then $y_0(t) > \psi(t)$ on $\big[t_1, \min(b, w_0^+)\big)$.

The only possibility remaining is that $y_0(t) = \psi(t)$ on $[t_0, t_0+\varepsilon]$ and $y_0(t) < \psi(t)$ on $[t_0+\varepsilon, t_0+\varepsilon_1]$ for some $\varepsilon_1 > \varepsilon$. By Theorem 4, the initial value problem $y'' = f(t, y, y')$, $y(t_0+\varepsilon) = \psi(t_0+\varepsilon)$, $y'(t_0+\varepsilon) = \psi'(t_0+\varepsilon)$ has a maximal solution $y(t)$ and since $\psi(t)$ is a superfunction $y(t) \geqslant \psi(t)$ on $[t_0+\varepsilon, t_0+\varepsilon_2]$ for some $\varepsilon_2 > \varepsilon$. But then

$$z(t) = \begin{cases} y_0(t), & t \in [t_0, t_0+\varepsilon], \\ y(t), & t \in [t_0+\varepsilon, t_0+\varepsilon_2] \end{cases}$$

is a solution to (13) such that $z(t) > y_0(t)$ for some t. This contradicts the maximality of y_0. Hence, $y_0(t) \geqslant \psi(t)$ on $\big[t_0, \min(w_0^+, b)\big)$.

THEOREM 11. *Let $\psi(t)$ be a $C^{(1)}$-upper solution of* (1) *on $[a, b] \subset I$. Suppose the maximal solution to* (13) *exists for $t_0 \in [a, b]$. Then either*:

(i) *$y_0'(t) \geqslant \psi'(t)$ on some right neighborhood of t_0, or*

(ii) *$y_0'(t) - \psi'(t)$ changes sign at an infinite number of points t_n, where $t_n > t_0$ and $t_n \to t_0$ as $n \to \infty$.*

Proof. By Theorem 8 and the mean value theorem, the possibility that $\psi'(t) > y_0'(t)$ on $(t_0, t_0+\delta]$ for some $\delta > 0$ can be ruled out immediately. The only possibilities which remain are cases (i) and (ii).

If $f(t, y, y')$ is non-decreasing in y, then Kamke [6] and others have proved that the maximal solution and its derivative dominate the given upper solution and its derivative on their common interval of existence. Of course, non-decreasing in y is only a sufficient condition. For example, $y'' = -y$, $y(0) = 0$, $y'(0) = 1$ has $y_0(t) = \sin t$ as its maximal solution on $[0, \pi]$ and $\psi(t) = -\frac{1}{2}(t-1)^2 + \frac{1}{2}$ as an upper solution with $y(t) \geqslant \psi(t)$, and $y'(t) \geqslant \psi'(t)$ on $[0, \pi]$.

We conclude this paper by giving an example of a differential equation (1) with continuous right-hand side such that:

(A) there exists a $C^{(1)}$-upper solution $\psi(t)$ such that no solution of (13) dominates $\psi(t)$ on $[t_0, t_0+\delta]$ for any $\delta > 0$;

(B) there exists a $C^{(1)}$-upper solution $\psi(t)$ such that $y_0(t) - \psi(t)$ and $y_0'(t) - \psi_0'(t)$ change sign infinitely often as $t \to t_0^+$, where $y_0(t)$ is the maximal solution of (13).

Consider

$$y'' = f(t, y, y'), \quad y(0) = 0, \quad y'(0) = 0, \tag{16}$$

where

$$f(t, y, y') = \begin{cases} 0, & t \geqslant 0, y \geqslant 0, \\ 5\left(\dfrac{20y}{t^2} - \dfrac{y}{t^4}\right), & 0 < t \leqslant 1, -t^5 \leqslant y \leqslant 0, \\ 5(-20t^3 + t), & 0 < t \leqslant 1, y \leqslant -t^5. \end{cases}$$

Then $f(t, y, y')$ is continuous on $[0, 1/2\sqrt{5}] \times R \times R$, and $y_0(t) = 0$ is a solution of (16). In fact, $y_0(t)$ is the maximal solution of (16) on $[0, 1/2\sqrt{5}]$. By the definition of f, there exists no solution of (16) which is non-negative on any right neighborhood of 0 except $y_0(t)$ and there also exists no solution $y(t) \leqslant -t^5$ on any right neighborhood of 0 since $y(t) \leqslant -t^5$ implies $y''(t) > 0$. Assume $y_0(t)$ is not the maximal solution to (16). Then there exists a solution $y(t)$ of (16) such that $y(t) < 0$ on $(0, \delta)$, $y(\delta) = 0$, and $y(t) > 0$ on $(\delta, \delta+\varepsilon)$ for some $\varepsilon > 0$. For any $\eta \in (0, \delta)$,

$$y(\eta) - y_0(\eta) = \big(y(0) - y_0(0)\big) + \big(y'(0) - y_0'(0)\big)t + \big(y''(\mu) - y''(\mu)\big)t^2/2$$

for some $\mu \in (0, \eta)$ implies that if $y(\eta) < 0$, then $y''(\mu) < 0$. But

$$y''(\mu) = \frac{5y(\mu)}{\mu^4}\,[20\mu^2 - 1] > 0$$

or

$$y''(\mu) = 5[-20\mu^3 + \mu] > 0 \quad \text{if} \quad \mu < \frac{1}{2\sqrt{5}}.$$

Hence, if $\delta < 1/2\sqrt{5}$, $y''(t) \geqslant 0$ whenever $y(t) \leqslant 0$ implies that $y(t) \equiv 0$ on $[0, 1/2\sqrt{5}]$. Assume then that there exists $t_1 > 1/2\sqrt{5}$ such that $y(t_1) > 0$. Since the only solution $z(t)$ of $y'' = f(t, y, y')$, $y(1/2\sqrt{5}) = 0$, $y'(1/2\sqrt{5}) = 0$ that is non-negative is $z(t) = 0$, $y(t)$ must be negative at some points $t > 1/2\sqrt{5}$. But if $y(t) < 0$, $t > 1/2\sqrt{5}$, then $y''(t) < 0$ which implies $y(t_1) < 0$. We conclude $y_0(t)$ is the maximal solution on $[0, 1]$.

Define $\bar{\psi}(t) = t^5 \sin t^{-1}$. $\bar{\psi}(t)$ is not a $C^{(1)}$-upper solution. However, it can be modified in such a way that the modified function is. We will only sketch the procedure. One can verify that $\bar{\psi}(t)$ is a $C^{(2)}$-upper solution on any interval $\left[\dfrac{2}{(4n+1)\pi}, \dfrac{2}{(4n-1)\pi}\right]$, $n = 1, 2, \ldots$

Let $\{\sigma_n\}$ and $\{\mu_n\}$ be sequences of relative maxima and minima of $\bar{\psi}(t)$, respectively, such that $0 < \sigma_{n+1} < \sigma_n < 1/2\sqrt{5}$, $0 < \mu_{n+1} < \mu_n < 1/2\sqrt{5}$ and $\sigma_n \to 0$, $\mu_n \to 0$ as $n \to \infty$. Then we have

$$\frac{2}{(4n+3)\pi} < \mu_n < \frac{1}{(2n+1)\pi} < \frac{2}{(4n+1)\pi} < \sigma_n < \frac{1}{2n\pi} < \frac{2}{(4n-1)\pi} < \mu_{n-1}.$$

For n sufficiently large, $n \geq N_1$, one can verify that $\bar{\psi}'' \leq f(t, \bar{\psi}, \bar{\psi}')$ on $[\sigma_n, \mu_{n-1}]$.

On the intervals $[\mu_n, \sigma_n]$, $\bar{\psi}(t)$ must be modified in such a way that the resulting function is a $C^{(1)}$-upper solution on $[0, \mu_{N_1-1}]$. To do this, define

$$y_n(t) = \bar{\psi}(\mu_n)\sin c_1(t-a_1) \tag{17}$$

and

$$z_n(t) = \bar{\psi}(\sigma_n)\sin c_2(t-a_2). \tag{18}$$

Let $t_n \in (\mu_n, \sigma_n)$ and choose c_1, a_1, c_1, a_2 such that

$$\begin{gathered} \sin c_1(\mu_n - a_1) = 1, \quad \sin c_2(\sigma_n - a_2) = 1, \\ \sin c_1(t_n - a_1) = 0, \quad \sin c_2(t_n - a_2) = 0, \\ -c_1\bar{\psi}(\mu_n) = c_2\bar{\psi}(\sigma_n). \end{gathered} \tag{19}$$

From (19), we find that:

$$c_1 = \frac{\pi}{2}\frac{1}{(t_n - \mu_n)}, \quad a_1 = 2\mu_n - t_n,$$
$$c_2 = \frac{\pi}{2}\frac{1}{(\sigma_n - t_n)}, \quad a_2 = t_n,$$

where

$$t_n = \frac{\mu_n\bar{\psi}(\sigma_n) - \sigma_n\bar{\psi}(\mu_n)}{\bar{\psi}(\sigma_n) - \bar{\psi}(\mu_n)}.$$

$z_n(t) \leq 0 = f(t, z_n, z_n')$ on $[t_n, \sigma_n]$ and for n sufficiently large, $n \geq N_2 \geq N_1$, $y_n''(t) \leq f(t, y_n(t), y_n'(t))$ on $[\mu_n, t_n]$. Define

$$\psi(t) = \begin{cases} \bar{\psi}(t), & t \in [\sigma_n, \mu_{n-1}], \\ z_n(t), & t \in [t_n, \sigma_n], \\ y_n(t), & t \in [\mu_n, t_n] \end{cases}$$

for $n \geq N_2$. Then $\psi(t)$ is of class $C^{(1)}$ on $[0, \mu_{N_2-1}]$, and is $C^{(2)}$ with $\psi''(t) \leq f(t, \psi(t), \psi'(t))$ on $\bigcup_{n=N_2}^{\infty} [(\mu_n, t_n) \cup (t_n, \sigma_n) \cup (\sigma_n, \mu_{n-1})]$. For $\eta = t_n$, σ_n, or μ_n,

$$\bar{D}\psi'(\eta) \leq f(\eta, \psi(\eta), \psi'(\eta))$$

for all $n \geq N_2$. Hence, $\psi(t)$ is a $C^{(1)}$-upper solution on $[0, \mu_{N_2-1}]$.

As a final remark, we note that the above example shows that a maximal solution to (1) may exist even if the conditions of Theorems 2, 3, or 4 are not satisfied.

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