UNIFORM COMPLETENESS OF TOPOLOGICAL SPACES

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In this note there are proved some theorems which are generalizations of three known theorems, namely of a theorem of Alexandroff [1] and Hausdorff [3] that each $G_{\delta}$-subset of a complete metric space is homeomorphic to a complete metric space, of a theorem of Nagata [7] and Kelley [5] that each paracompact space is topologically complete, and of a theorem of Čech [2] that each metric $G_{\delta}$-subspace of a compact space is homeomorphic to a complete metric space.

Our theorems extend those known theorems in two directions. First, we consider intersections of arbitrary families of open subsets instead of $G_{\delta}$’s and a property induced by the uniformity with a base having prescribed cardinality (not necessarily $\aleph_0$) instead of metrizability. Second, we consider generalized uniformities, namely $f$-uniformities from our paper [5], instead of uniformities. As a consequence, results are concerned with general topological spaces without assuming any separation axioms.

In [5] there was introduced a notion of an $f$-uniformity on a set $X$, being a generalization of a uniformity in the sense of Tukey.

A family $\mathcal{U} \subset 2^X$ is called an $f$-uniformity on the set $X$ if the following conditions are satisfied:

F1. $X = \bigcup \{ \bigcup P : P \in \mathcal{U} \}$.

F2. $Q \in \mathcal{U}$ iff for each $x \in \bigcup Q$ there exists $P(x) \in \mathcal{U}$ such that $x \in \bigcup P(x)$ and $P(x) \triangleright Q$.

F3. If $P_1, P_2 \in \mathcal{U}$, then for each $x \in \bigcup P_1 \cap \bigcup P_2$ there exists $P(x) \in \mathcal{U}$ such that $P(x) \triangleright P_1 \land P_2$.

F4. If $P, Q \in \mathcal{U}$ and $x \in \bigcup Q$, then $\text{st}(x, Q) \cap \bigcup P \neq \emptyset$.

The symbols $P \triangleright Q$ and $P \triangleright Q$ mean that $P$ is a refinement and a star-refinement, respectively, of $Q$.

If we assume that the elements of $\mathcal{U}$ are coverings of $X$, then we obtain axioms of uniformity without the axiom of separation.
A weight of an \( f \)-uniformity \( \mathcal{U} \) is the minimum of \( \text{card} \mathcal{B} \), where \( \mathcal{B} \) is a base for \( \mathcal{U} \).

Each \( f \)-uniformity \( \mathcal{U} \) on a set \( X \) induces a topology \( T_\mathcal{U} \) on \( X \): \( G \in T_\mathcal{U} \) iff for each \( x \in G \) there exists \( P \in \mathcal{U} \) such that \( x \in \bigcup P \) and \( \text{st}(x, P) \subset G \). Conversely, each topology on \( X \) is induced by some \( f \)-uniformity on \( X \) (see [5]).

A filter \( \xi \subset 2^X \) is called a Cauchy filter if, for each \( P \in \mathcal{U} \), \( P \cap \xi \) is non-empty. For each Cauchy filter \( \xi \) there exists a minimal Cauchy filter \( \xi_0 \subset \xi \) such that if \( \eta \subset \xi \) is a Cauchy filter, then \( \xi_0 \subset \eta \). The filter \( \xi_0 \) has a base of the form \( \{ \text{st}(A, P) : A \in \xi, P \in \mathcal{U} \} \). Let \( \xi(x) \) be the filter of neighbourhoods of the point \( x \). The filter \( \xi(x) \) is induced by the base \[ \{ \text{st}(x, P) : x \in \bigcup P, P \in \mathcal{U} \} \] .

If \( \xi \) is a Cauchy filter, then the conditions
(a) \( x \in \lim \xi \),
(b) \( x \in \lim \xi_0 \), where \( \xi_0 \) is the minimal Cauchy filter of \( \xi \),
(c) \( x \in \bigcap \{ \text{cl} A : A \in \xi \} \),
where \( x \in \lim \xi \) means that \( \xi(x) \subset \xi \), are equivalent.

An \( f \)-uniformity \( \mathcal{U} \) on \( X \) is complete if each Cauchy filter converges.

We say that \( f \)-completeness (completeness) of a space \( X \) is not greater than \( m \), and write \( \text{fcp} X \leq m \) (\( \text{cp} X \leq m \)), if there exists a complete \( f \)-uniformity (uniformity) \( \mathcal{U} \) of the weight not greater than \( m \) compatible with the topology on \( X \).

We say that a space \( X \) is well embedded in a space \( Y \) if \( Y = \text{cl}_Y X \) and \( \bigcap \{ \text{cl}_Y V : V \in \xi(x) \} \subset X \) for each \( x \in X \).

A space \( X \) is said to be \( f \)-compact if for each open covering \( P \) there exists a finite subfamily \( Q \subset P \) such that \( X = \text{cl} \bigcup Q \). If a space \( X \) is Hausdorff, then it is usually called \( H \)-closed. Note that there is an equivalence between the \( f \)-compactness and the convergence of each open filter.

**Theorem 1.** Each space \( X \) can be well embedded in an \( f \)-compact space \( Y \). If the topology on \( X \) is induced by a uniformity, then \( X \) can be well embedded in a compact space \( Y \).

**Proof.** In [5] it was proved that each space \( X \) has a totally bounded \( f \)-uniformity compatible with the topology on \( X \). There was constructed an \( f \)-completion (\( \tilde{X}, \tilde{\mathcal{U}} \)) of the \( f \)-uniform space (\( X, \mathcal{U} \)), where \( \tilde{X} = X \cup X_0 \), \( X_0 \) is the set of \( \xi_0 \) such that \( \xi_0 \) is a minimal Cauchy filter in \( \mathcal{U} \) having an empty limit, and the \( f \)-uniformity \( \tilde{\mathcal{U}} \) is induced by the base \( \{ \tilde{P} : P \in \mathcal{U} \} \) with \( \tilde{P} = \{ \tilde{U} : U \in P \} \) and \( \tilde{U} = U \cup \{ \xi_0 \in X_0 : U \in \xi_0 \} \).

It was proved also there that \( X \) is densely embedded in \( \tilde{X} \) and that the topology \( T_{\tilde{\mathcal{U}}} \) is \( f \)-compact. To see that \( X \) is well embedded in \( Y = \tilde{X} \)
notice that if \( x \in X \) and \( \xi_0 \in X_0 \), then there exist \( P \in \mathcal{U} \) and \( U \in \xi_0 \cap P \) such that \( x \notin \text{cl}_X U \). Hence and from the axiom F3 we infer that there exist open (in \( Y \)) neighbourhoods \( V_x \) and \( V_{\xi_0} \) of the points \( x \) and \( \xi_0 \), respectively, such that \( V_x \cap V_{\xi_0} = \emptyset \).

If the space \( X \) is uniformizable, then the topology on \( X \) is also induced by a totally bounded uniformity \( \mathcal{U} \). Since the above-described completion \( \mathcal{U} \) is also a uniformity, the topology \( T_{\mathcal{U}} \) is compact, which completes the proof.

A set \( X \) is a \( G_m \)-subset of a space \( Y \) if \( X \) is an intersection of no more than \( m \) open subsets of \( Y \).

**Theorem 2.** If \( \text{fep} \ X \leq m \) and a space \( X \) is well embedded in \( Y \), then \( X \) is a \( G_m \)-subset of \( Y \).

**Proof.** Let \( \mathcal{B} \subset \mathcal{U} \) with \( \text{card} \mathcal{B} \leq m \) be a base of a complete \( f \)-uniformity \( \mathcal{U} \) compatible with the topology on \( X \). For each \( P \in \mathcal{B} \) and \( x \in X \) let \( U_P(x) \) be an open subset of \( Y \) such that \( U_P(x) \cap X \subset V \in P \). Put

\[ R_P = \bigcup \{ U_P(x) : x \in X \}. \]

It suffices to prove that \( X = \bigcap \{ R_P : P \in \mathcal{B} \} \). Clearly, \( X \subset \bigcap \{ R_P : P \in \mathcal{B} \} \). Now, suppose that there exists \( y \in \bigcap \{ R_P : P \in \mathcal{B} \} \) such that \( y \in Y \setminus X \). For each \( P \in \mathcal{B} \) choose \( x_P \in X \) such that \( y \in U_P(x_P) \). The family \( \{ U_P(x_P) \cap X : P \in \mathcal{B} \} \) is a base for a Cauchy filter \( \xi \). Since \( \mathcal{U} \) is complete, there exists

\[ x \in \bigcap \{ \text{cl}_X U_P(x_P) \cap X : P \in \mathcal{B} \}. \]

Now \( \xi \) is a Cauchy filter, and so, for each \( V \in \xi(x) \), \( V \subset Y \), there exists \( P \in \mathcal{U} \) such that \( U_P(x_P) \cap X \subset V \). Since \( \text{cl}_Y X = Y \), we have

\[ \text{cl}_Y (U_P(x_P) \cap X) \subset \text{cl}_Y U_P(x_P) \cap \text{cl}_Y V. \]

Thus \( y \notin \text{cl}_Y V \) for each neighbourhood \( V \subset Y \) of \( x \). This implies

\[ y \notin \bigcap \{ \text{cl}_Y V : V \in \xi(x) \} \subset X, \]

which contradicts \( y \in Y \setminus X \).

**Theorem 3.** Each topological space has a complete \( f \)-uniformity compatible with the topology on \( X \).

**Proof.** Let \( Y \) be an \( f \)-compact extension of \( X \) such that \( Y = \hat{X} \), where \( \hat{X} \) is a space with the topology induced by the completion of the greatest totally bounded \( f \)-uniformity \( \mathcal{U} \) on \( X \). For each \( y \in Y \setminus X \) let \( P(y) \) be the set of \( V \) such that \( V \) is an open set in \( X \) with \( y \notin \text{cl}_Y V \) (cf. the remark in the proof of Theorem 1: for each \( x \in X \) and \( y \in Y \setminus X \) there exist open neighbourhoods \( V_x \) and \( V_y \) with \( V_x \cap V_y = \emptyset \)). The covering \( P(y) \) of \( X \) belongs to the greatest \( f \)-uniformity \( \mathcal{U}^* \) on the space \( X \) (see [5]). Now, it is easy to see that \( \mathcal{U}^* \) is complete.
Let \( \xi_0 \subset 2^X \) be a minimal Cauchy filter. For each \( A \in \xi_0 \) there is \( \text{int}_X A \neq \emptyset \). We have
\[
\bigcap \{ \text{cl}_X A : A \in \xi_0 \} \supset \bigcap \{ \text{cl}_X \text{int}_X A : A \in \xi_0 \} = \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi_0 \} \cap X,
\]
but, in view of the \( f \)-compactness of \( Y \),
\[
\emptyset \neq \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi_0 \} \subset \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi_0 \cap P(y), y \in Y \setminus X \} \subset X.
\]
Hence \( \bigcap \{ \text{cl}_X A : A \in \xi_0 \} \neq \emptyset \).

Let \( X \) be a uniformizable space and let \( X \subset Y \). An open set \( G \subset Y \) is said to be a uniform neighbourhood of \( X \) if there exists an open covering \( P \) of \( X \) such that \( \bigcup \{ \text{cl}_Y U : U \in P \} \subset G \) and \( P \) belongs to the greatest uniformity compatible with the topology on \( X \).

Note that if \( X \) is paracompact, then each open set \( G \) containing \( X \) is a uniform neighbourhood of \( X \), since each open covering of \( X \) belongs to the greatest uniformity on \( X \).

**Theorem 4.** Let \( X \subset Y \), \( \text{cl}_Y X = Y \), be an intersection of no more than \( m \) uniform neighbourhoods of \( X \). If the topology on \( Y \) is induced by a complete uniformity of the weight not greater than \( m \), then the topology on \( X \) is induced by a complete uniformity of the weight not greater than \( m \cdot n \), i.e., \( \text{cp} Y \leq n \) implies \( \text{cp} X \leq m \cdot n \).

**Proof.** Let \( \mathcal{G} \) be a family of open neighbourhoods of \( X \) with \( \text{card} \mathcal{G} \leq m \) and such that \( X = \bigcap \{ G : G \in \mathcal{G} \} \). For each \( G \in \mathcal{G} \) choose a \( P_G \) belonging to the greatest uniformity \( \mathcal{U}^* \) on the space \( X \) such that
\[
\bigcup \{ \text{cl}_Y U : U \in P_G \} \subset G.
\]

Let \( \mathcal{V} \) be a complete uniformity on \( Y \) with weight \( \mathcal{V} \leq n \). By a countable operation (see [6], p. 246) we can find a uniformity
\[
\mathcal{U}_o \supset \mathcal{V} \cap X \cup \{ P_G : G \in \mathcal{G} \}
\]
compatible with the topology on \( X \) and such that
\[
\text{weight} \mathcal{U}_o = \text{weight} \mathcal{V} \cdot \text{card} \mathcal{G}.
\]

To see that \( \mathcal{U}_o \) is complete notice that each minimal Cauchy filter \( \xi \subset 2^X \) in the sense of \( \mathcal{U}_o \) is a Cauchy filter in the sense of \( \mathcal{V} \) and
\[
\bigcap \{ \text{cl}_X A : A \in \xi \} \supset \bigcap \{ \text{cl}_X \text{int}_X A : A \in \xi \} = \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \} \cap X \neq \emptyset,
\]
since \( \mathcal{V} \) is complete and
\[
\bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \} \subset \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \cap P_G, G \in \mathcal{G} \} \subset X.
\]

Theorem 4 is a generalization of the Alexandroff [1] and Hausdorff [3] Theorem that each \( G \) subset of a complete metric space is metrizable in a complete manner.

Put \( \text{uw}X = \min \{ \text{weight} \mathcal{U} : \mathcal{U} \) is a uniformity compatible with the topology on \( X \)\}. 

THEOREM 5. Suppose that \( \text{uw} X \leq n \), \( X \) is a dense subspace of an \( f \)-compact space \( Y \), and \( X \) is an intersection of no more than \( m \) uniform neighbourhoods of \( X \). Then \( \text{cp} X \leq n \cdot m \).

Proof. The idea of the proof is the same as that of Theorem 4. Let \( \mathcal{G} \) be a family of uniform neighbourhoods of \( X \) with \( \text{card} \mathcal{G} \leq m \). For each \( G \in \mathcal{G} \) choose an open covering \( P_G \in \mathcal{U} \) such that

\[
\bigcup \{ \text{cl}_Y U : U \in P_G \} = G.
\]

Let \( \mathcal{U} \) be a uniformity on \( X \) with weight not greater than \( n \). There exists a uniformity \( \mathcal{V}_0 \subset \mathcal{U}^* \) such that

\[
\mathcal{V}_0 \supset \mathcal{U} \cup \{ P_G : G \in \mathcal{G} \} \quad \text{and} \quad \text{weight} \mathcal{V}_0 \leq m \cdot n.
\]

Take a minimal Cauchy filter \( \xi \in 2^X \) in the sense of \( \mathcal{V}_0 \). We have

\[
\bigcap \{ \text{cl}_Y A : A \in \xi \} \supset \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \} \cap X = \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \} \neq \emptyset,
\]

since \( Y \) is \( f \)-compact and

\[
\bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \} \supset \bigcap \{ \text{cl}_Y \text{int}_X A : A \in \xi \cap P_G, G \in \mathcal{G} \} \subset X.
\]

Theorem 5 is a generalization of the fact that each paracompact space \( X \) is topologically complete ([Nagata [7], Kelley [4]]) and it is also a generalization \((m = n = m_0)\) of the theorem that each metrizable \( G_\delta \) of a compact space is completely metrizable (Čech [2]).

REFERENCES


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