Characterizations of $H_{x_1}$-sufficiency of jets

by Le Tien Tam (Hanoi)

Abstract. The aim of this paper is to investigate germs of maps in manifolds with boundary. The main results are contained in Theorems 1, 2, 3.

1. Definitions and notation. Let $R^n$ denote the Euclidean space of dimension $n$ and $E(n, p)$ the set of all germs of maps $f: R^n \rightarrow R^p$, $f(0) = 0$, $J'(n, p)$ is the space of jets.

Definition 1. A homeomorphism $\sigma: R^n \rightarrow R^n$ is boundary preserving if

$$\sigma(\{0 \times R^{n-1}\}) \subset \{0 \times R^{n-1}\}.$$ 

The set of all homeomorphisms of $R^n$ preserving boundaries will be denoted by $H_{x_1}$.

Definition 2. An $r$-jet $w \in J'(n, p)$ is called $H_{x_1}$-sufficient if for every $f: R^n \rightarrow R^p$, $f(0) = 0$, and $f'(f) = w$ there exists $\sigma \in H_{x_1}$ such that $f_0 \sigma = w$.

Given $f: R^n \rightarrow R^p$, let us put

$$v_i = \left( x_1 \frac{\partial f_i}{\partial x_1}, x_2 \frac{\partial f_i}{\partial x_2}, \ldots, x_n \frac{\partial f_i}{\partial x_n} \right), \quad i = 1, \ldots, p.$$ 

Theorem 1. An $r$-jet $w \in J'(n, p)$ is $H_{x_1}$-sufficient if there exist a positive number $c$ and a neighbourhood $U$ of 0 such that

$$d(v_1^w, \ldots, v_p^w) \geq c |x|^{r-1}$$

for every $x \in U$.

Theorem 2. If $w \in J'(n, p)$ is $H_{x_1}$-sufficient, then there exists a positive number $c$ and a neighbourhood $U$ of 0 such that

$$d(v_1^w, \ldots, v_p^w) \geq c |x|^{r+1}$$

for every $x \in U$.

Theorem 3. Let $w \in J'(n, p)$. If there exist a positive number $c$ and a neighbourhood $U$ of 0 such that

$$d(\Phi_1, \ldots, \Phi_p) \geq c |x|^r$$

for every $x \in U$, 

\[ \begin{array}{c}
\downarrow \\
\cup \quad W
\end{array} \]
then \( w \) is \( H_{x_i} \)-sufficient for every \( i \), where

\[
\Phi_i = \left( x_1 \frac{\partial w_i}{\partial x_1}, x_2 \frac{\partial w_i}{\partial x_2}, \ldots, x_n \frac{\partial w_i}{\partial x_n} \right), \quad i = 1, \ldots, p.
\]

For manifolds without boundary, Bochnak [1] has shown that \( w \) is \( H_{x_1} \)-sufficient if and only if condition (1) is satisfied. The following example shows that for manifolds with boundary condition (1) is only sufficient and not necessary.

**Example.** Let \( w(x, y) = x^2 + y^2 \in J^2(2, 1) \). Then \( w \) is \( H_{x^2} \) and \( H_{y^2} \)-sufficient.

2. Proof of Theorems 1, 2, 3.

2.1. Proof of Theorem 1. For every \( f: (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0) \) and \( f'(f) = w \), let us put

\[
F(x, t) = w(x) + t(f(x) - w(x)) \quad \text{for } (x, t) \in \mathbb{R}^p \times \mathbb{R}
\]

and

\[
Y(x, t) = \sum_{i=1}^{p} \left( (0, 1, f_i', \ldots, f_i') \right) N_i \quad \text{where}
\]

\[
N_i = \sum_{i=1}^{p} \left( f_i - w_i \right) N_i^{(i, i-2)} N_i,
\]

\[
v_i^p = \left( w_1 \frac{\partial F(x, t)}{\partial x_1}, w_2 \frac{\partial F(x, t)}{\partial x_2}, \ldots, w_n \frac{\partial F(x, t)}{\partial x_n} \right)
\]

where \( N_i \) is the projection of \( v_i^p \) onto the space orthogonal to the space spanned by \( v_j^p (x_1, \ldots, x_n, i) \) with \( j \neq i \). \( Y(x, t) \) is the projection of \( (0, \ldots, 0, 1) \) onto the subspace spanned by \( v_1^p, \ldots, v_p^p \).

Then

\[
V(x, t) = (V_1, \ldots, V_{n+1}) = (0, \ldots, 0, 1) - Y(x, t)
\]

is orthogonal to \( v_i^p \) for every \( i \), i.e.,

\[
x_1 \frac{\partial F_i}{\partial x_1} V_1 + x_2 \frac{\partial F_i}{\partial x_2} V_2 + \ldots + \frac{\partial F_i}{\partial t} V_{n+1} = 0.
\]

Write

\[
X(x, t) = (x_1 V_1, V_2, \ldots, V_{n+1}).
\]

Then \( x(x, t) \) is orthogonal to \( \left( \frac{\partial F_i}{\partial x_1}, \ldots, \frac{\partial F_i}{\partial x_n}, \frac{\partial F_i}{\partial t} \right) \), \( i = 1, \ldots, p \).

**Lemma 2.1 [2].** In the notation of Theorem 1, we have

\[
d(v_1^p, \ldots, v_p^p) \geq \frac{1}{4} c \lVert \gamma \rVert^{-1} \quad \text{for every } x \in U.
\]
Let us consider the differential equation

\[
\frac{dx}{dt} = X(x, t),
\]

where \( X(x, t) \) is a map from an open subset of \( \mathbb{R}^n \times \mathbb{R} \) containing \( B_\varepsilon^o \times [0, 1] \) into \( \mathbb{R}^{n+1} \),

\[
B_\varepsilon^o = \{ x \in \mathbb{R}^n : |x| \leq \varepsilon \}.
\]

Let us note that \( X(\cdot, t) \) satisfies the following conditions:

(i) \( X(x, t) \) is continuous for every \( (x, t) \in B_\varepsilon^o \times [0, 2] = A_\varepsilon \),

(ii) \( \lim_{\varepsilon \to 0} \frac{|X(x, t) - X(x, 0)|}{|x|} = 0 \) uniformly with respect to \( t \),

(iii) \( \langle X(x, t), l_{n+1} \rangle > 0 \) for every \( (x, t) \in A_\varepsilon \).

In fact, by Lemma 2.1 we have

\[
|N_i| \geq d(e_1^x, \ldots, e_p^x) \geq \frac{1}{2} c|x|^{r-1}, \quad i = 1, \ldots, p.
\]

This implies that

\[
|Y(x, t)| \leq 2 \sum_{i=1}^p |f_i - w_i| \frac{1}{|N_i|} = o(|x|).
\]

Hence \( V(x, t) \) satisfies conditions (i)-(iii). Thus \( X(x, t) \) does so. By [1], (2.4) has the unique solution \( \eta(x, t; \tau) \) satisfying the condition \( \eta(x, t, 0) = (x, t) \) and

(a) \( \tau \to \eta(x, t; \tau) \) is the unique solution of (2.4) satisfying the condition \( \eta(x, t; 0) = (x, t) \) in a neighbourhood \( W \) of \( B_\varepsilon \times [0, 1] \times \{ 0 \} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \),

(b) \( \eta(0, 0; \tau) = (0, \tau) \),

(c) \( \eta(\{ x \times 0 \} \times \mathbb{R}) \cap W \cap (\mathbb{R}^n \times \{ 1 \}) \) contains the point \( (\sigma(x), 1) \),

(d) \( F \) is constant on \( \eta((x, t) \times \mathbb{R}) \cap W \).

From conditions (a)-(d) it follows that the map \( \sigma \) is a homeomorphism such that \( f \circ \sigma = w \). It remains to show that \( \sigma \in F(x_1) \).

In fact,

\[
X(x, t)|_{x_1 = 0} = (0, V_2, \ldots, V_{n+1}) = X_1(x', t),
\]

where \( x = (0, x_2, \ldots, x_n) \).

We have

\[
\frac{\partial F_1(x', t)}{\partial x_2} \cdot V_2 + \ldots + \frac{\partial F_1(x', t)}{\partial t} \cdot V_{n+1} = 0.
\]

On the other hand, since \( X(x, t) \) satisfies (i)-(iii), \( X_1(x', t) \) does so. This implies that the equation

\[
\frac{dx'}{dt} = X_1(x', t)
\]
has the unique solution \( \eta_*(x', t; \tau) \) satisfying the condition \( \eta_*(x', t; 0) = (x', t) \).

By the continuity of \( \eta(x, t; \tau) \) we infer that \( \eta(x, t; \tau)|_{x_1 = 0} \) is a solution of (2.5).

Thus

\[
\eta(x, t; \tau)|_{x_1 = 0} = \eta_*. 
\]

Since \( \eta_* \) induces a homeomorphism \( \sigma_*: \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \), we have by (2.6)

\[
\sigma|_{x_1 = 0} = \sigma_. \quad \text{Thus} \quad \sigma \text{ preserves the boundary}.
\]

2.2. Proof of Theorem 2. We first prove the following claim, under the hypothesis of Theorem 2:

For every \( f: U \to \mathbb{R}^p, f \in C^r \), and \( f'(f) = w \) and for every sequence \( \{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n \), \( a_i \to 0 \), \( a_i \neq 0 \), \( f^{-1}(f(a_i)) \) is a topological manifold with boundary of codimension \( p \) and the boundary of \( f^{-1}(f(a_i)) \) is the intersection of \( f^{-1}(f(a_i)) \) with the hyperspace \( x_1 = 0 \).

To prove (d_x), we need the following lemma, which is an immediate consequence of Lemma 1 [1].

**Lemma.** Let \( U, V \) be open subsets of \( \mathbb{R}^k \) and \( \mathbb{R}^l \), respectively, \( F: U \times V \to \mathbb{R}^p \) be a \( C^\infty \)-map and \( B \) be a countable set of \( \mathbb{R}^p \) contained in the set of all regular values of \( F \) and \( \partial F|_{x_1 = 0} = 0 \).

*Then there exists a point \( y_0 \in V \) such that \( B \) is contained in the set of regular values of both \( F(x_1, \ldots, x_k, y_0) \) and \( F(0, x_2, \ldots, x_k, y_0) \).*

Applying this lemma we can prove (d_x).

Let \( a_i \to 0, a_i \in \mathbb{R}^n \setminus \{0\} \), and \( f: \mathbb{R}^n \to \mathbb{R}^p \) be a \( C^r \)-map, with \( f'(f) = w \).

Consider the map \( \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \) defined by

\[
F_j(x, y) = w_j(x) + y_j|x|^{2r}, \quad j = 1, 2, \ldots, p;
\]

obviously, \( f(a) \) is a regular value of both \( F(x, y) \) and \( F(x, y)|_{x_1 = 0} \).

We now show that condition (d_x) implies Theorem 2.

Indeed, if it is not the case then there exists a sequence \( \{a_i\}; a_i \neq 0, a_i \to 0 \), such that

\[
d(v_{1}(a_i), \ldots, v_{p}(a_i)) \in o(|a_i|^{r + 1}).
\]

Then by [1] there are linearly independent vectors \( \lambda_1, \ldots, \lambda_p \) satisfying the conditions:

(a) The vectors \( v_1^*(a_i) + \lambda_1, \ldots, v_p^*(a_i) + \lambda_p \) are linearly independent,

(b) \( v_1^*(a_i) \in \text{Span} \{ v_1^*(a_i) + \lambda_1, \ldots, v_p^*(a_i) + \lambda_p \} \),

(c) \( |\lambda_k| \in o(|a_i|^{r + 1}) \), \( k = 2, \ldots, p \).

Since \( w(x) \) is \( H_{x_1} \)-sufficient, \( w(x)|_{x_1 = 0} \) is \( H \)-sufficient in \( \mathbb{R}^{n-1} = 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^n \).
Indeed, let \( f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^p \) be a map with \( J'(f) = w(x)|_{x_1 = 0} \). Write \( w(x_1, x_2, \ldots, x_n) \) in the form
\[
w(x_1, x_2, \ldots, x_n) = w(x_1, x_2, \ldots, x_n)|_{x_1 = 0} + x_1 \varphi(x_1, x_2, \ldots, x_n).
\]
Put \( g = f(x_1, \ldots, x_n) + x_1 \varphi(x_1, x_2, \ldots, x_n) \). Then
\[
J'(g) = f'(x) + f'(x_1 \varphi) = w|_{x_1 = 0} + f'(x_1 \varphi) = w.
\]
Since \( w \) is \( H_{x_1} \)-sufficient, there exists \( \sigma \in H_{x_1} \) such that
\[
(2.7) \quad g \circ \sigma = w, \quad g \circ \sigma = f \circ \sigma + (x_1 \varphi) \circ \sigma.
\]
From (2.6) we get
\[
(2.8) \quad g \circ \sigma|_{x_1 = 0} = w(x_1, \ldots, x_n)|_{x_1 = 0}.
\]
By (2.7)
\[
(2.9) \quad g \circ \sigma|_{x_1 = 0} = f \circ \sigma|_{x_1 = 0} + (x_1 \varphi) \circ \sigma|_{x_1 = 0} = f \circ \sigma|_{x_1 = 0} = f \circ \sigma_*,
\]
where \( \sigma_* = \sigma|_{x_1 = 0} \).

From (2.8) and (2.9) we have
\[
f \circ \sigma_* = w(x_1, \ldots, x_n)|_{x_1 = 0}.
\]
Thus \( w|_{x_1 = 0} \) is \( H \)-sufficient in \( \mathbb{R}^{n-1} \).

Since \( w|_{x_1 = 0} \) is \( H \)-sufficient, by a well-known result, there exist \( c > 0 \) and a neighbourhood \( U \) of \( 0 \) in \( \mathbb{R}^{n-1} \) such that
\[
d(\mathcal{V}_1 w(x), \ldots, \mathcal{V}_p w(x)) \geq c|x'|^{r-1} \geq c|x'|^{r+1}, \quad x' = (0, x_2, \ldots, x_n).
\]
On the other hand,
\[
d(\mathcal{V}_1 w(x), \ldots, \mathcal{V}_p w(x)) = d(v^1_w, \ldots, v^p_w)|_{x_1 = 0}.
\]
Therefore, considering \( d \) in the hyperspace \( x_1 = 0 \), we have
\[
d(v^1_w, \ldots, v^p_w) \geq c|x'|^{r+1} \quad \forall x \in U.
\]
Thus \( a_{i_1} \neq 0 \) for every \( i \).

Write the vectors \( \lambda^i_k, \) \( k = 2, \ldots, p, \) in the form
\[
\lambda^i_k = (\alpha^i_{k,1}, \alpha^i_{k,2}, \ldots, \alpha^i_{k,n}).
\]
Since
\[
v^1_w(a_i) \in \text{Span} \{ v^1_w(a_i) + \lambda^i_2, \ldots, v^p_w(a_i) + \lambda^i_p \},
\]
we have
\[
\mathcal{V}_1 w(a_i) = \left( \frac{\partial w_1}{\partial x_1}(a_i), \ldots, \frac{\partial w_1}{\partial x_n}(a_i) \right)
\]
\[ \epsilon \text{ Span} \left\{ \frac{\partial w_1}{\partial x_1} (a_i) + \frac{\alpha_{i,1}}{a_i}, \frac{\partial w_2}{\partial x_2} (a_i) + \alpha_{i,2}, \ldots, \frac{\partial w_n}{\partial x_n} (a_i) + \alpha_{i,n} \right\}, \ldots \]
\[ \ldots, \left( \frac{\partial w_{p-i}}{\partial x_1} (a_i) + \frac{\alpha_{p-i,1}}{a_i}, \frac{\partial w_{p-i}}{\partial x_2} (a_i) + \alpha_{p-i,2}, \ldots, \frac{\partial w_{p-i}}{\partial x_n} (a_i) + \alpha_{p-i,n} \right) \right\} \]

Let us put
\[ \lambda_i^2 = \left( \frac{\alpha_{i,1}}{a_i}, \alpha_{i,2}, \ldots, \alpha_{i,n} \right), \]
\[ \lambda_i^p = \left( \frac{\alpha_{p-i,1}}{a_i}, \alpha_{p-i,2}, \ldots, \alpha_{p-i,n} \right). \]

Then we have \(|\lambda_i^k| = o(|a_i|^{-1})\), \(k = 2, \ldots, p\).

In fact, since \(|\lambda_i^k| = o(|a_i|^{r+1})\), \(k = 2, \ldots, p\) and
\[ \lambda_i^k = (\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,n}), \]
we get \(|\alpha_{k,j}| = o(|a_i|^{r+1})\), \(j = 1, 2, \ldots, n\). On the other hand,
\[ |\alpha_{k,1}| = |a_i| \left| \frac{\alpha_{k,1}}{a_i} \right| \quad \text{and} \quad |\alpha_{k,1}| = o(|a_i|^{r+1}); \]
we get
\[ (2.10) \quad |a_i| \left| \frac{\alpha_{k,1}}{a_i} \right| = o(|a_i|^{r+1}). \]

Since \(|a_{i,1}|/|a_i| \leq |a_{i,1}|/|a_i|^2 = 1/|a_{i,1}| \to \infty\), we have
\[ \lim_{i \to x} \frac{|a_{i,1}|}{|a_i|^2} = \infty. \]

Thus from (2.10) we obtain
\[ |\frac{\alpha_{k,1}}{a_i}| = o(|a_i|^{r-1}). \]

Since \(|\alpha_{k,j}| = o(|a_i|^{r-1})\), we have \(|\lambda_i^k| = o(|a_i|^{r-1})\), \(k = 2, \ldots, p\). Finally, we get
\[ \text{grad} \ w_1 (a_i) = \sum_{k=2}^{p} \alpha_k (\text{grad} \ w_k (a_i) + \lambda_i^k), \quad |\lambda_i^k| = o(|a_i|^{r-1}). \]

Repeating the proof of Theorem in [1], we conclude the proof of Theorem 2.

2.3. Proof of Theorem 3. Assume that \(f: \mathbb{R}^n \to \mathbb{R}^p, f(0) = 0\) and \(f'(f) = w\). Let us show that there exists a homeomorphism
\[ \sigma: \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma \{ x \in \mathbb{R}^n: x_1 = 0 \} \subset \{ x \in \mathbb{R}^n: x_1 = 0 \} \]
such that \(f \circ \sigma = w\).
Put
\begin{equation}
F(x, t) = w(x) + t(f(x) - w(x)).
\end{equation}

We first show that the inequality \(d(\Phi_1(w), \ldots, \Phi_p(w)) \geq c|x|^r\) for every \(x \in U\) implies
\begin{equation}
d(\Phi_1(F), \ldots, \Phi_p(F)) \geq \frac{1}{2}c|x|^r, \quad \forall x \in U \quad \text{and} \quad \forall t,
\end{equation}
where
\[
\Phi_1(F) = \left( x_1 \frac{\partial F}{\partial x_1}, \ldots, x_n \frac{\partial F}{\partial x_n}, -t \frac{\partial F}{\partial t} \right),
\]
\[
\Phi_p(F) = \left( x_1 \frac{\partial F_p}{\partial x_1}, \ldots, x_n \frac{\partial F_p}{\partial x_n}, -t \frac{\partial F_p}{\partial t} \right).
\]

In fact, consider \(\Phi_k(w)\) as a vector of the \((x, t)\)-space. Since the last components of the vectors are zero, we get
\begin{equation}
|\Phi_k(F) - \Phi_k(w)|
\end{equation}
\[
= \left| \left( t x_1 \left( \frac{\partial f_k}{\partial x_1} - \frac{\partial w_k}{\partial x_1} \right), \ldots, t x_n \left( \frac{\partial f_k}{\partial x_n} - \frac{\partial w_k}{\partial x_n} \right), f_k - w_k \right) \right| \leq c_1 |x|^{r+1},
\]
(3.3)
\[
\left| \sum_{i=1}^{p} \lambda_i \Phi_i(F) \right| \geq \left| \sum_{i=1}^{p} \lambda_i \Phi_i(w) \right| - \left| \sum_{i=1}^{p} \lambda_i (\Phi_i(F) - \Phi_i(w)) \right|.
\]
\begin{equation}
\sum_{i=1}^{p} \lambda_i \Phi_i(F)
\end{equation}
\[
= \frac{|\Phi_k(F) - \Phi_k(w)|}{|\Phi_k(w) + \sum_{i \neq k} \lambda_i^{-1} \Phi_i(w)|}
\]
\[
\leq \frac{c_1 |x|^{r+1}}{d(\Phi_1(w), \ldots, \Phi_p(w))} \leq \frac{c_1 |x|^{r+1}}{c |x|^r} = \frac{c_1}{c} |x|.
\]
\begin{equation}
\text{For } \lambda_k(x, t) = 0 \text{ the inequality is obvious. From (3.3) and (3.4) we infer that}
\end{equation}
\[
\lim_{t \to 0} \left| \sum_{i} \lambda_i \Phi_i(w) \right|^{-1} \left| \sum_{i} \lambda_i (\Phi_i(F) - \Phi_i(w)) \right| = 0
\]
uniformly with respect to \(t\).
Thus by (3.4) we have
\[
\left| \sum_{i} \lambda_i \Phi_i(F) \right| \geq \frac{1}{2} \left| \sum_{i} \lambda_i \Phi_i(w) \right|
\]
when \(x\) is sufficiently small. Thus (3.2) is proved.
To continue the proof of the theorem, we put
\begin{equation}
Y(x, t) = \sum_{i=1}^{p} \left( (0, 1) \cdot \Phi_i(F) \right) |N_i|^{-2} N_i.
\end{equation}
where $N_i$ is the projection of the vector $\Phi_j(F)$ onto the space spanned by $\Phi_j(F)$, $j \neq i$.

Note that $Y(x, t)$ is the projection of $(0, \ldots, 0, 1)$ onto the subspace spanned by $\Phi_1(F), \ldots, \Phi_p(F)$.

Thus

$$X_1(x, t) = (0, \ldots, 0, 1) - Y(x, t)$$

is orthogonal to $\Phi_1(F), \ldots, \Phi_p(F)$. Note that

$$\Phi_k(F) = \left( x_1 \frac{\partial F_k}{\partial x_1}, \ldots, x_n \frac{\partial F_k}{\partial x_n}, \frac{\partial F_k}{\partial t} \right).$$

Thus if we put $X_1(x, t) = (X_{1,1}, \ldots, X_{1,n}, X_{1,n+1})$, we have

$$x_1 \frac{\partial F_k}{\partial x_1} \cdot X_{1,1} + \ldots + x_n \frac{\partial F_k}{\partial x_n} \cdot X_{1,n} + \frac{\partial F_k}{\partial t} \cdot X_{1,n+1} = 0$$

for $k = 1, \ldots, p$. Let

$$X(x, t) = (x_1 X_{1,1}, x_2 X_{1,2}, \ldots, x_n X_{1,n}, X_{1,n+1}).$$

Then $X(x, t)$ is orthogonal to $\nabla F_k$, $k = 1, 2, \ldots, p$.

From (3.6) and (3.2) it follows that

$$\lim_{x \to 0} |x|^{-1} Y(x, t) = 0.$$

Thus the vector $X_1(x, t)$ has properties (i), (ii), (iii) occurring in the proof of Theorem 1. It is easy to see that $X(x, t)$ has properties (i), (ii), (iii).

Repeating the last part of the proof of Theorem 1, we obtain the proof of Theorem 3.

References


DEPARTMENT OF MATHEMATICS  
THE UNIVERSITY OF HANOI  
HANOI, VIETNAM

Reçu par la Rédaction le 1980.12.18