ANNALES
POLONICI MATHEMATICI
XLII (1983)

# Approximate iterative method and the existence of solutions of non-linear parabolic differential-functional equations

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*Dedicated to the memory of Jacek Szarski*

**Abstract.** Let us consider a system of non-linear second order parabolic differential equations which include functionals

(1) $$F^i[z^i] = f^i(t, x, z, z(t, \cdot)) \quad (i = 1, \ldots, m),$$

where

$$F^i = \frac{\partial}{\partial t} - \sum_{\alpha,\beta=1}^{n} a^i_{\alpha\beta}(t, x) \frac{\partial^2}{\partial x_\alpha \partial x_\beta},$$

$x = (x_1, \ldots, x_n)$, $z = (z^1, \ldots, z^m)$, $G \subset \mathbf{R}^n$, $(t, x) \in [0, T) \times G$, $T < +\infty$.

For fixed $t \in [0, T)$ we denote by $z(t, \cdot) = (z^1(t, \cdot), \ldots, z^m(t, \cdot))$ the function

$$z(t, \cdot): \bar{G} \ni x \to z(t, x) \in \mathbf{R}^n,$$

which is an element of the space of continuous functions $C_m(\bar{G})$.

In particular, equations (1) may be considered as differential-integral or charge equations which are applied especially to the Markov stochastic processes and to the description of flow of underground water.

For system (1) we consider the first Fourier's boundary problem and to the solutions of this problem we apply the iterative method, so we obtain the existence theorem. We shall show a construction of two sequences which will approximate uniformly and monotonically the solution of our equation. Moreover, it will be the exponential type of approximation. In these constructions we shall use results due to J. Szarski [3]-[5].

**1. Notation, definitions and assumptions.** Let $G$ be open and bounded set in the space $(x_1, \ldots, x_m)$ such that boundary $\partial G$ is a finite union of $C^1$ surfaces.

We denote

$$D = (0, T) \times G, \quad \sigma = (0, T) \times \partial G,$$

$$S_0 = \{(t, x): t = 0, x \in \bar{G}\}, \quad \Sigma = S_0 + \sigma, \quad \bar{D} = D + \Sigma.$$

Let us assume that the quadratic forms

$$\sum_{\alpha,\beta=1}^{n} a^i_{\alpha\beta}(t,x)\lambda_\alpha\lambda_\beta \qquad (i=1,\ldots,m)$$

are positive definite in $D$.

A function $z(t,x)$ is called *regular in* $\bar{D}$ if $z$ is continuous in $\bar{D}$ and $\partial z/\partial t$, $\partial z/\partial x_\alpha$, $\partial^2 z/\partial x_\alpha \partial x_\beta$ are continuous in $D$.

For system (1) we consider the following first Fourier's boundary problem:

Find the regular solution $z(t,x)$ in $\bar{D}$ fulfilling the boundary condition

$$(2) \qquad z(t,x)=0 \quad \text{for } (t,x)\in\Sigma.$$

We denote by $C_m(\bar{G})$ the space of continuous functions

$$z(\cdot)\colon\ \bar{G}\ni x\to z(x)\in \boldsymbol{R}^n,$$

and for the subspace of those $z$ which are bounded we introduce the norm

$$\|z\| = \max_i \max\{|z^i(x)|\colon\ x\in\bar{G}\}.$$

In the space $C_m(\bar{G})$ the following order is introduced: for

$$z=\big(z^1(\cdot),\ldots,z^m(\cdot)\big)\in C_m(\bar{G}), \qquad \tilde{z}=\big(\tilde{z}^1(\cdot),\ldots,\tilde{z}^m(\cdot)\big)\in C_m(\bar{G})$$

the inequality $z\leqslant\tilde{z}$ means that

$$z^j(x)\leqslant\tilde{z}^j(x) \quad \text{for arbitrary } x\in\bar{G},\ j=1,\ldots,m.$$

Let $f^i(t,x,z,s)$ $(i=1,\ldots,m)$, where $z=(z^1,\ldots,z^m)$ and $s=(s^1,\ldots,s^m)$, be defined for $(t,x)\in D$, $z$ arbitrary and $s\in C_m(\bar{G})$.

We introduce the following assumptions:

($\mathrm{H}_f$) the functions $f^i(t,x,z,s)$ $(i=1,\ldots,m)$ are continuous in $D$ and fulfil locally the Hölder condition with respect to $x$,

(L) the functions $f^i(t,x,z,s)$ $(i=1,\ldots,m)$ satisfy the Lipschitz condition with respect to $z$ and $s$: for arbitrary $z,\tilde{z},s,\tilde{s}$ we have the inequality

$$|f^i(t,x,z,s)-f^i(t,x,\tilde{z},\tilde{s})|\leqslant L_1\sum_{j=1}^{m}|z^j-\tilde{z}^j|+L_2\sum_{j=1}^{m}\|s^j-\tilde{s}^j\|,$$

where $L_1, L_2$ are positive constants,

(W) the functions $f^i(t,x,z,s)$ $(i=1,\ldots,m)$ are non-decreasing with respect to $z$ and $s$,

($\mathrm{H}_a$) the coefficients $a^i_{\alpha\beta}(t,x)$ $(i=1,\ldots,m;\ \alpha,\beta=1,\ldots,m)$ are continuous with respect to $t,x$ and are bounded in $\bar{D}$, moreover, they fulfil locally the Hölder condition with respect to $x$.

The functions $u(t,x) = \big(u^1(t,x), \ldots, u^m(t,x)\big)$ and $v(t,x) = \big(v^1(t,x), \ldots, v^m(t,x)\big)$, regular in $\bar{D}$, satisfying the differential inequalities

$$F^i[u^i] \leqslant f^i\big(t, x, u(t,x), u(t,\cdot)\big), \tag{3}$$

$$F^i[v^i] \geqslant f^i\big(t, x, v(t,x), v(t,\cdot)\big) \quad (i = 1, \ldots, m) \tag{4}$$

in $D$ and the boundary condition (2), will be called the *lower* and *upper functions* for problem (1), (2) in $\bar{D}$, respectively.

Assumption A. Let us assume that there exists at least one pair $u_0(t,x)$, $v_0(t,x)$ of lower and upper functions for problem (1), (2) in $D$, respectively.

**2. Iterative method. Existence theorem.** If $u(t,x)$ and $v(t,x)$ are the lower and upper functions for problem (1), (2) in $D$ and if $z(t,x)$ is the regular solution for this problem (in our case we shall assume that there exists a solution) and conditions (W) and (L) hold, then from Szarski's theorem on the differential-functional inequalities [3], [5] we obtain

$$u(t,x) \leqslant z(t,x) \leqslant v(t,x) \quad \text{for } (t,x) \in \bar{D}. \tag{5}$$

In particular, if Assumption A holds, then

$$u_0(t,x) \leqslant z(t,x) \leqslant v_0(t,x) \quad \text{for } (t,x) \in \bar{D}. \tag{6}$$

We shall start from arbitrary functions $u_0(t,x)$ and $v_0(t,x)$ (of course, the functions $u_0$ and $v_0$ are the lower and upper functions for problem (1), (2) in $\bar{D}$) and we shall construct two sequences of the lower $\{u_n(t,x)\}$ and upper $\{v_n(t,x)\}$ functions using the iterative method. In details, this construction will be explain in following theorems.

Theorem. *Let the functions $f^i(t,x,z,s)$ $(i = 1, \ldots, m)$ fulfil conditions $(\mathrm{H}_f)$, (L), (W), let the coefficients $a^i_{\alpha\beta}(t,x)$ $(i = 1, \ldots, m;\ \alpha, \beta = 1, \ldots, n)$ fulfil conditions $(\mathrm{H}_a)$ and let Assumption A be satisfied. Assume that the successive terms of sequences $\{u_n(t,x)\}$ and $\{v_n(t,x)\}$ are defined as regular solutions in $\bar{D}$ of the linear parabolic equations*

$$F^i[u^i_n] = f^i\big(t, x, u_{n-1}(t,x), u_{n-1}(t,\cdot)\big), \tag{7}$$

$$F^i[v^i_n] = f^i\big(t, x, v_{n-1}(t,x), v_{n-1}(t,\cdot)\big) \quad (i = 1, \ldots, m) \tag{8}$$

*satisfying the boundary condition* (2).

*Then*

(i) *there exist unique regular solutions $u_n(t,x)$ and $v_n(t,x)$ for $n = 1, 2, \ldots$ of system* (7), (8) *with condition* (2) *in $\bar{D}$,*

(ii) *the inequalities*

$$u_{n-1}(t,x) \leqslant u_n(t,x), \quad v_n(t,x) \leqslant v_{n-1}(t,x) \quad (n = 1, \ldots, m)$$

*hold for $(t,x) \in \bar{D}$,*

(iii) *the functions* $u_n(t, x)$ *and* $v_n(t, x)$ $(n = 1, 2, \ldots)$ *are the lower and upper functions for problem* (1), (2) *in* $\bar{D}$, *respectively,*

(iv) $\lim_{n\to\infty} [v_n(t, x) - u_n(t, x)] = 0$ *uniformly in* $\bar{D}$,

(v) *the function*

$$z(t, x) = \lim_{n\to\infty} u_n(t, x)$$

*is the regular unique solution of problem* (1), (2) *in* $\bar{D}$.

Proof. Ad (i). Using assumptions $(\mathrm{H}_f)$, (L) about the functions $f^i$ and using the regularity of $u_0(t, x)$, we conclude that $f^i\big(t, x, u_0(t, x), u_0(t, \cdot)\big)$ are continuous in $D$ and fulfil locally the Hölder condition. Moreover, since the coefficients $a^i_{\alpha\beta}$ fulfil conditions $(\mathrm{H}_a)$, there exists a unique regular solution $u_1(t, x)$ of system

$$F^i[u^i_1] = f^i\big(t, x, u_0(t, x), u_0(t, \cdot)\big)$$

with condition (2) in $D$. Using the same method, we obtain (i) for every $n$.

Ad (ii). It follows from (3) for $n = 0$ and from (7) for $n = 1$ that

$$F^i[u^i_0 - u^i_1] \leqslant f^i\big(t, x, u_0(t, x), u_0(t, \cdot)\big) - f^i\big(t, x, u_0(t, x), u_0(t, \cdot)\big) = 0 \qquad (i = 1, \ldots, m)$$

in $D$ and $u_0 - u_1 = 0$ on $\Sigma$.

Hence by the theorem on differential inequalities [2] we have

$$u_0(t, x) \leqslant u_1(t, x) \quad \text{in } \bar{D}.$$

By assumi ng a sufficient inequality for $u_{n-1}(t, x)$ and using monotonicity, we get for every $n$

$$(9) \qquad u_0(t, x) \leqslant u_1(t, x) \leqslant \ldots \leqslant u_{n-1}(t, x) \leqslant u_n(t, x) \leqslant \ldots \quad \text{in } \bar{D}.$$

In the same way we obtain the inequality

$$(10) \qquad \ldots \leqslant v_n(t, x) \leqslant v_{n-1}(t, x) \leqslant \ldots \leqslant v_1(t, x) \leqslant v_0(t, x) \quad \text{in } \bar{D}.$$

Ad (iii). By the definition of the function $u_1(t, x)$ and by condition (W) for $f^i$ using (9) we obtain the following inequality

$$F^i[u^i_1] - f^i\big(t, x, u_1(t, x), u_1(t, \cdot)\big) = f^i\big(t, x, u_0(t, x), u_0(t, \cdot)\big) - f^i\big(t, x, u_1(t, x), u_1(t, \cdot)\big) \leqslant 0 \quad \text{in } D,$$

so, by (3), the function $u_1(t, x)$ is the lower function for problem (1), (2) in $\bar{D}$. The proof of the induction step is the same as above, since $\{u_n(t, x)\}$ is the sequence of the lower functions for problem (1), (2) in $\bar{D}$.

Using a similar method, we obtain that $\{v_n(t, x)\}$ is the sequence of the upper functions for problem (1), (2) in $\bar{D}$.

From the last two results and by (9), (10) and (5) it follows that

$$(11)\qquad u_0(t,x) \leqslant u_1(t,x) \leqslant \ldots \leqslant u_n(t,x) \leqslant \ldots \leqslant z(t,x) \leqslant \ldots$$
$$\ldots \leqslant v_n(t,x) \leqslant \ldots \leqslant v_1(t,x) \leqslant v_0(t,x) \quad \text{for } (t,x) \in \bar{D}.$$

Ad (iv). Using mathematical induction, we shall first show the following inequality

$$(12)\qquad w_n^i(t,x) \leqslant N\frac{[m(L_1+L_2)t]^n}{n!} \quad (n=1,\ldots,m) \text{ for } (t,x) \in \bar{D},$$

where

$$w_n(t,x) = v_n(t,x) - u_n(t,x),$$
$$(13)\qquad N = \max_i \max_{\bar{D}} [v_0^i(t,x) - u_0^i(t,x)]$$

but owing to the regularity of the functions $u_0$ and $v_0$ we have $N = \text{const} < +\infty$.

By (13) we obtain inequality (12) for $n = 0$. Let inequality (12) hold for $w_n$. Since the functions $f^i$ fulfil assumption (L), we get by (7), (8) and (11)

$$F^i[w_{n+1}^i] = f^i\big(t,x,v_n(t,x),v_n(t,\cdot)\big) - f^i\big(t,x,u_n(t,x),u_n(t,\cdot)\big)$$
$$\leqslant L_1\sum_{j=1}^{m}(v_n^j - u_n^j) + L_2\sum_{j=1}^{m}\|v_n^j(t,\cdot) - u_n^j(t,\cdot)\|$$
$$\leqslant L_1\sum_{j=1}^{m} w_n^j + mL_2\|w_n(t,\cdot)\|.$$

By the definition of norm in the space $G_n(\bar{G})$ and by inequality (12) we get

$$\|w_n(t,\cdot)\| \leqslant N\frac{[m(L_1+L_2)t]^n}{n!},$$

so, we finally obtain

$$(14)\qquad F^i[w_{n+1}^i] \leqslant N\frac{[m(L_1+L_2)]^{n+1}t^n}{n!} \quad (i=1,\ldots,m) \text{ for } (t,x) \in \bar{D},$$

where $w_{n+1}^i(t,x) = 0$ for $(t,x) \in \Sigma$.

Let us consider the comparison system of equations

$$(15)\qquad F^i[M_{n+1}^i] = N\frac{[m(L_1+L_2)]^{n+1}t^n}{n!} \quad (i=1,\ldots,m) \text{ in } D$$

with boundary condition

$$M_{n+1}^i(t,x) \geqslant 0 \quad \text{on } \Sigma.$$

The functions

$$M^i_{n+1}(t,x) = N\frac{[m(L_1+L_2)t]^{n+1}}{(n+1)!} \qquad (i=1,\ldots,m)$$

are the regular solutions of the comparison system in $\bar{D}$.

Applying the theorem on differential inequalities [2] to systems (14) and (15), we find that

$$w^i_{n+1}(t,x) \leqslant M^i_{n+1}(t,x) = N\frac{[m(L_1+L_2)t]^{n+1}}{(n+1)!} \qquad (i=1,\ldots,m)$$

$$\text{for } (t,x)\in\bar{D},$$

so, the induction step is proved and inequality (12) is proved.

As a direct conclusion from formula (12) we get

$$(16)\qquad \lim_{n\to\infty}[v_n(t,x)-u_n(t,x)] = 0 \quad \text{uniformly in } \bar{D}.$$

Ad (v). The functional sequences $\{u_n\}$ and $\{v_n\}$ are monotonous and (16) holds, so there exists a continuous function $U(t,x)$ such that

$$(17)\qquad u_n(t,x)\to U(t,x) \quad \text{and} \quad v_n(t,x)\to U(t,x) \quad (n\to\infty)$$

uniformly in $\bar{D}$.

Since the functions $f^i$ fulfil condition (W), we conclude by (11), (17) that the right-hand sides of system (7) are bounded in $D$. Then by Pogorzelski's theorem [1] about the properties of weak singular integrals (the solution of system (7) can be expressed by these integrals) we get $u_n(t,x)$ satisfying locally (uniformly with respect to $n$) the Lipschitz condition with respect to $x$.

Let us now consider the system of equations

$$(18)\qquad F^i[z^i] = f^i\big(t,x,U(t,x),U(t,\cdot)\big) \quad (i=1,\ldots,m)$$

with condition (2) in $D$. By the fact that the functions $f^i$ fulfil conditions ($H_f$), (L) and by the above conclusions we get that the right-hand sides of system (18) are continuous and fulfil locally the Hölder condition. By Assumption ($H_f$) about the coefficients $a^i_{\alpha\beta}$ there exists a regular solution $z(t,x)$ of problem (18), (2) in $\bar{D}$.

On the other hand, by the fact that the functions $f^i$ fulfil conditions (L), ($H_f$) and by (17) we get

$$(19)\qquad \lim_{n\to\infty} f^i\big(t,x,u_n(t,x),u_n(t,\cdot)\big) = f^i\big(t,x,U(t,x),U(t,\cdot)\big)$$

uniformly in $\bar{D}$.

Applying the comparison theorem on the differential inequalities and using (19), we get

$$(20)\qquad \lim_{n\to\infty} u_n(t,x) = z(t,x) \quad \text{uniformly in } \bar{D}.$$

Moreover, with (17) we obtain

$$z(t, x) = U(t, x) \quad \text{for } (t, x) \in \bar{D},$$

so, the function $z(t, x)$ is a regular solution of problem (1), (2) in $\bar{D}$.

The uniqueness of the solution of this problem follows directly from J. Szarski's results [4].

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*Reçu par la Rédaction le 10.03.1981*