ON EXTENDING OF PARTIAL BOOLEAN ALGEBRAS
TO PARTIAL *-ALGEBRAS

BY

JANUSZ CZELAKOWSKI (WROCLAW)

Kochen and Specker introduced and examined in [3] and [4] the concepts of a partial Boolean algebra (PBA) and of a partial algebra over an arbitrary field $K$. As to the second name, perhaps the term "partial (commutative) linear algebra" would be more suitable. A very general concept of "partial algebra" has been presented for years in algebra. In [4] it is noticed that the set of all idempotents of a partial algebra (in the sense of Kochen and Specker) forms a PBA.

In this note the concept of a partial *-algebra is examined. This notion is a slight modification of that of a partial algebra; namely, by a partial *-algebra we mean any partial algebra equipped with an operation of involution. We prove that any PBA can be extended to a partial *-algebra, which is equivalent to the proposition that the converse of the result by Kochen and Specker holds.

Definition 1. A system $\mathcal{A} = \langle A; \cdot; +, \cdot, *; 1 \rangle$ is said to be a partial *-algebra over the field of complex numbers $C$ if the following conditions are satisfied:

1. $\cdot \subseteq A \times A$ is a non-empty symmetric and reflexive relation. $\cdot$ is called the relation of commeasurability.
2. $+$ and $\cdot$ are partial binary operations whose domains and ranges satisfy the connections $\text{Dom}(+) = \text{Dom}(\cdot) = \cdot$, $\text{Rg}(+) = \text{Rg}(\cdot) = A$.
3. $*: A \mapsto A$.
4. $1$ is a distinguished element in $A$. $1$ is the unit of the partial *-algebra $\mathcal{A}$.
5. $a \cdot 1$ for every $a \in A$.
6. If $a \cdot b$, then $a \cdot b^*$ and $\lambda a \cdot b$ for any complex number $\lambda$.
7. If $a$, $b$, $c$ are pairwise in the relation $\cdot$, then $a + b \cdot c$ and $a \cdot b \cdot c$. 
(8) If \( a, b, c \) are pairwise in the relation \( \circ \), then the set \( \{a, b, c\} \) generates in \( \mathcal{A} \) a commutative linear algebra with the involution \( * \) and the unit \( 1 \).

Condition (8) may be replaced by the following system of axioms:

(8') If \( a, b, c \) are mutually in the relation \( \circ \), then:

\[(L1)\ a + b = b + a,
\[(L2)\ (a + b) + c = a + (b + c),
\[(L3)\ \text{if } a + c = b + c, \text{ then } a = b,
\[(L4)\ \lambda(a + b) = \lambda a + \lambda b,
\[(L5)\ (\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a,
\[(L6)\ (\lambda_1 \lambda_2)a = \lambda_1 (\lambda_2 a),
\[(L7)\ 1a = a, \text{ where } 1 \in C;\]

\[(A1)\ a \cdot b = b \cdot a,
\[(A2)\ (a \cdot b) \cdot c = a \cdot (b \cdot c),
\[(A3)\ (a + b) \cdot c = a \cdot c + b \cdot c,
\[(A4)\ (\lambda a) \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b) \text{ for any complex number } \lambda,
\[(A5)\ 1 \cdot a = a = a \cdot 1, \text{ where } 1 \text{ is the unit of } \mathcal{A};\]

\[(A^*1)\ (a + b)^* = a^* + b^*,
\[(A^*2)\ (\lambda a)^* = \overline{\lambda} a^*,
\[(A^*3)\ (a \cdot b)^* = b^* \cdot a^*,
\[(A^*4)\ (a^*)^* = a.\]

Recall that (L1)-(L7), (A2)-(A5), and (A^*1)-(A^*4) are axioms of linear algebras with involution.

Notice that every commutative linear algebra \( \mathcal{A} = \langle A; +, \cdot, \ast; 1 \rangle \) with involution becomes a partial \( \ast \)-algebra provided \( \circ = A \times A \).

Here we write some simple properties of partial \( \ast \)-algebras:

1.1. \( 0a = 01 \) for any \( a \in A \) (\( 0 \in C \)).

The element \( 0a \) (\( a \) arbitrary) is the zero of a partial \( \ast \)-algebra and is denoted by \( 0 \).

1.2. For any \( a \in A \) and \( \lambda \in C \),

\[a \circ 0, \quad a + 0 = a, \quad 0a = 0, \quad \lambda 0 = 0.\]

Let \( -a = (-1)a \) and \( a - b = a + (-1)b \), where \( a \circ b \).

1.3. Let \( a \circ b \). Then the equation \( a + x = b \) has exactly one solution \( x \) such that \( x \circ b \), namely \( x = b - a \).

A partial Boolean algebra \( \mathcal{B} = \langle B; \circ, \lor, \land, 1 \rangle \) is given by a non-empty set \( B \), a binary relation \( \circ \subseteq B \times B \), a unary function \( \neg \) from \( B \) into \( B \), a partial binary function \( \lor \) from \( B \times B \) into \( B \), and an element \( 1 \in B \) called the unit of \( \mathcal{B} \). The domain of \( \lor \) is a subset of \( B \times B \). The properties of \( \mathcal{B} \) are the following:
1. The relation $\triangleleft$ (called also the relation of commeasurability) is symmetric and reflexive.
2. For all $a \in B$, $a \triangleleft 1$.
3. The partial function $\lor$ is defined exactly on $\triangleleft$.
4. If $a \triangleleft b$, then $a \triangleleft \neg b$.
5. If $a, b, c$ are mutually commeasurable, then $a \lor b \triangleleft c$.
6. If $a, b, c$ are mutually commeasurable, then the Boolean polynomials in $a, b, c$ form a Boolean algebra.

Let $\mathbf{PBA}$ denote the class of all partial Boolean algebras.

Basic properties of partial Boolean algebras can be found in [1], [3], and [4].

Let $\mathcal{A} = \langle A; \triangleleft, +, \cdot, \ast; 1 \rangle$ be a partial $\ast$-algebra. An element $a \in A$ is said to be a Hermitian idempotent if $a = a \cdot a = a \ast$. Let $B$ be the set of all Hermitian idempotents in $\mathcal{A}$. Notice that $1 \in B$. Let $a, b \in B$. We put

$$\neg a = 1 - a,$$

$$a \lor b = (a + b) - (a \cdot b) \quad \text{if} \quad a \triangleleft b.$$

The system $\mathcal{B} = \langle B; \triangleleft, \lor, \neg; 1 \rangle$ is a partial Boolean algebra (here $\triangleleft$ is a restriction of the relation of commeasurability to the set $B \times B$).

In what follows we admit the following convention: algebraic objects are denoted by capital script letters (eventually, with subscripts) and their underlying sets are denoted by the same italic letters (with the same subscripts).

Let $\mathcal{A} \in \mathbf{PBA}$ and let $\{\mathcal{A}_k\}_{k \in A}$ be the indexed family of all maximal Boolean subalgebras contained in $\mathcal{A}$. Let $X_A$ denote the Stone space of all maximal filters in $\mathcal{A}_A$ and let $\mathcal{F}_A$ be the Stone field of all simultaneously open and closed subsets of $X_A$. Let $i_A$ be the Stone isomorphism of $\mathcal{A}_A$ onto $\mathcal{F}_A$.

Let

$$X = \prod_{k \in A} X_A.$$

For every $A \subseteq X_A$ let $A^\ast$ be the set of all functions $\varphi \in X$ such that $\varphi(\lambda) \in A$, and let $\mathcal{F}_A^\ast$ be the field of subsets of $X$ formed from all sets $A^\ast$, where $A \in F_A$.

Let $\mathcal{F}_A^\ast$ be the least field of subsets of $X$ containing all algebras $\mathcal{F}_A^\ast$. The fields $\{\mathcal{F}_A^\ast\}_{k \in A}$ are independent in $\mathcal{F}_A^\ast$.

A function $h_A: \mathcal{A} \mapsto \mathcal{F}_A^\ast$, where $h_A a = (i_A a)^\ast$, maps $\mathcal{A}$ isomorphically onto $\mathcal{F}_A^\ast$.

It is easy to show that the set

$$B^\ast = \bigcup_{k \in A} F_k^\ast \subseteq 2^X,$$
equipped with the relation of commeasurability $\triangleleft$, where $A_1^* \triangleleft A_2^*$ iff there exists a $\lambda \in A$ such that $A_1^*, A_2^* \in B^*_\lambda$ ($A_1^*, A_2^* \in B^*$), and with the usual set-theoretical operations of join restricted to $\triangleleft$ and complementation, is a partial Boolean algebra. This partial Boolean algebra is denoted by $B^*$.

We define the following equivalence relation $\sim$ in $B^*$:

$$A_1^* \sim A_2^* \iff A_1^* = h_{\lambda_1} a \text{ and } A_2^* = h_{\lambda_2} a$$

for a certain (and unique) $a \in B$.

Let $|A_1^*|, |A_2^*| \in B^*/\sim$. Then we define

$$|A_1^*| \uparrow |A_2^*| \iff A_1^* = h_{\lambda_1} a_1, A_2^* = h_{\lambda_2} a_2, \text{ and } a_1 \triangleleft a_2.$$  

The function $\neg$ is defined by

$$\neg|A^*| = |X - A^*|.$$  

Notice that if $|A_1^*| \uparrow |A_2^*|$, then there exists $B_{\lambda_0}$ such that $a_1, a_2 \in B_{\lambda_0}$, where $A_1^* = h_{\lambda_1} a_1, A_2^* = h_{\lambda_2} a_2$ ($B_{\lambda_0}$ is the carrier of $B_{\lambda_0}$). Let $A_3^* = h_{\lambda_0}(a_1 \lor a_2)$. Then we define the function $\lor$ as follows:

$$|A_1^*| \lor |A_2^*| = |A_3^*|.$$  

**THEOREM 1** (see [3] and [4]). Let $B \in PBA$. Then

(i) $B^*/\sim \in PBA$.

(ii) A mapping $\varphi(A^*) = |A^*|$ maps $B^*$ homomorphically onto $B^*/\sim$.

(iii) $B^*/\sim$ and $B$ are isomorphic.

Let $X$ be a non-empty set and let $F$ be any fixed field of subsets of $X$. Then $f : X \rightarrow C$ is a simple function (over $F$) if $f$ equals a finite linear combination of characteristic functions of sets from $F$. Let $W_F$ be a commutative linear algebra of all simple functions over $F$ with the usual addition and multiplication of complex functions and with conjugation as an involution.

**LEMMA 1.** Let $F$ and $W_F$ be as above. Then each element $w \in W_F (w \neq 0)$ has a unique representation of the form

$$(*) \quad w = \sum_{i=1}^{n} a_i \chi_{A_i} \quad (A_i \in F, \ n < \omega)$$

up to a permutation of the numbers $\{1, 2, \ldots, n\}$, where $a_i \neq 0$, $a_i \neq a_j$ for $i \neq j$, and $A_i \neq \emptyset$, $A_i \cap A_j = \emptyset$ for $i \neq j$.

The proof is straightforward. $(*)$ is called the canonical representation of $w$.

Our aim is to prove the following theorem:

**THEOREM 2.** Every partial Boolean algebra is isomorphic to the partial Boolean algebra of all Hermitian idempotents of a certain partial $*$-algebra.
In a shortened form this theorem can be expressed as follows:

Every partial Boolean algebra is embeddable into a certain partial *-algebra.

Proof. Let $\mathcal{B} = \langle B; \cup, \cap, 1 \rangle \in \text{PBA}$. Let $X_A, \mathcal{F}_i, \mathcal{U}, X, \mathcal{F}_i^*, h_i, \mathcal{F}^*, \mathcal{B}^*, \mathcal{B}^*/\sim$ be defined as above. Let $\mathcal{W}_{i \lambda}$ denote a commutative *-algebra of simple functions, spanned over characteristic functions of sets from $\mathcal{F}_i^*$. Let

$$W^* = \bigcup_{A^* \lambda} W_{i \lambda}^*.$$ 

Let $w_1, w_2$ be in $W^*$ and let

$$w_1 = \sum_{i=1}^m a_i \chi_{A_i^*} \quad \text{and} \quad w_2 = \sum_{j=1}^n \beta_j \chi_{B_j^*}$$

be canonical representations of $w_1$ and $w_2$.

Let $\approx$ be defined in $W^*$ as follows:

$w_1 \approx w_2$ iff $m = n$ and there exists a permutation $\sigma$ of the numbers $\{1, 2, \ldots, m\}$ such that $A_i^* \sim B_{\sigma(i)}$ for every $1 \leq i \leq m$ and $a_i = \beta_{\sigma(i)}$ for every $1 \leq i \leq m$.

$\approx$ is an equivalence relation in $W^*$. Let $\langle w \rangle, \langle u \rangle \in W^*/\approx$. Define $\langle w \rangle \hat{\ast} \langle u \rangle$ iff for the canonical representations of the elements $w$ and $u$,

$$w = \sum_{i=1}^m a_i \chi_{A_i^*} \quad \text{and} \quad u = \sum_{j=1}^n \beta_j \chi_{B_j^*},$$

we have $|A_i^*| \hat{\ast} |B_j^*|$ for each $i (1 \leq i \leq m)$ and for each $j (1 \leq j \leq n)$.

The definition of $\hat{\ast}$ does not depend on the choice of representatives of $\langle w \rangle$ and $\langle u \rangle$. Notice that $\langle w \rangle \hat{\ast} \langle u \rangle$ iff there exist a $\lambda_0 \in A$ and elements $w_1 \in \langle w \rangle$, $u_1 \in \langle u \rangle$ such that $w_1, u_1 \in W_{\lambda_0}^*$.

Let $\langle w \rangle \hat{\ast} \langle u \rangle$. Then there are a $\lambda_0 \in A$ and elements $w_1 \in \langle w \rangle$, $u_1 \in \langle u \rangle$ such that $w_1, u_1 \in W_{\lambda_0}^*$. Then $w_1 + u_1 \in W_{\lambda_0}^*$ and $w_1 \cdot u_1 \in W_{\lambda_0}^*$. We put

$$\langle w \rangle + \langle u \rangle = \langle w_1 + u_1 \rangle \quad \text{and} \quad \langle w \rangle \cdot \langle u \rangle = \langle w_1 \cdot u_1 \rangle.$$ 

The definitions of $+$ and $\cdot$ are correct. Notice that classes $\langle u \rangle, \langle w \rangle, \langle v \rangle$ are mutually in the relation $\hat{\ast}$ iff there are a $\lambda_0 \in A$ and elements $u_1 \in \langle u \rangle$, $w_1 \in \langle w \rangle$, $v_1 \in \langle v \rangle$ such that $u_1, w_1, v_1 \in W_{\lambda_0}^*$.

An involution $\ast$ is defined in $W^*/\approx$ by

$$\langle w \rangle^* = \langle \overline{w} \rangle,$$

where $\overline{w}$ is conjugate to $w$.

The unit element in $W^*/\approx$ is an equivalence class determined by the characteristic function of the whole set

$$X = \prod_{i \in A} X_i.$$
It is easy to check that the system
\[ W^*_{/\omega} = \langle W^*_{/\omega}; \text{\; \textcircled{\,\textdagger}\, \; +, \cdot, \; \text{\; \textasteriskcentered}\, \; 1 \rangle \]
satisfies all the axioms of partial \(*\)-algebras.

Let
\[ A_1^* \subseteq B^* = \bigcup_{\lambda \in A} F^*_1. \]

Notice that
\[ A_1^* \sim A_2^* \text{ \; iff \; } \chi_{A_1^*} \approx \chi_{A_2^*}. \]

Each Hermitian idempotent in \( W^*_{/\omega} \) is of the form \( \langle \chi_A \rangle \), where \( A^* \subseteq B^* \). Let \( \mathcal{L} \) be a partial Boolean algebra of all Hermitian idempotents in \( W^*_{/\omega} \). A function \( \psi: B^*_{/\omega} \rightarrow \mathcal{L} \) defined by \( \psi(|A^*|) = \langle \chi_A \rangle \) maps \( B^*_{/\omega} \) isomorphically onto \( \mathcal{L} \). By Theorem 1, the partial Boolean algebra \( \mathcal{B} \) is isomorphic to \( B^*_{/\omega} \). Hence \( \mathcal{B} \) and \( \mathcal{L} \) are isomorphic. Thus the proof is completed.

Thus, with any partial Boolean algebra \( \mathcal{B} \) we can associate, in a unique way, a partial \(*\)-algebra \( W^*_{/\omega} \) constructed as above. This particular partial \(*\)-algebra will be denoted by \( W_{\mathcal{B}} \).

Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be partial \(*\)-algebras. A mapping \( h: \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) is said to be a homomorphism if the following conditions are fulfilled:

(i) if \( a \cdot b \), then \( ha \cdot hb \) \( \text{\; \textcircled{\,a\,\textdagger\, \; (a, b \in A_1)} \}
(ii) if \( a \cdot b \), then \( h(a + b) = ha + hb \) and \( h(a \cdot b) = ha \cdot hb \);
(iii) \( h(\lambda a) = \lambda ha \);
(iv) \( h(a^*) = (ha)^* \);
(v) \( h1 = 1 \).

A one-to-one homomorphism is called a monomorphism.

**Theorem 3.** Let \( \mathcal{A}_1, \mathcal{A}_2 \in PBA \). Let \( h_0: \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) be a homomorphism (see [3]). Then \( h_0 \) can be extended to a homomorphism \( h: W_{\mathcal{A}_1} \rightarrow W_{\mathcal{A}_2} \).

If \( h_0 \) is an epimorphism (a monomorphism), then \( h \) is an epimorphism (a monomorphism).

**Proof.** It suffices to notice that any element \( w \in W_{\mathcal{A}} \) \( (w \neq 0) \), where \( \mathcal{A} \in PBA \), has a unique representation of the form

\[ w = \sum_{i=1}^{n} a_i p_i \]

up to a permutation of the set \( \{1, 2, ..., n\} \), where \( p_i \) is a Hermitian idempotent, \( p_i \neq 0 \) \( (i = 1, 2, ..., n) \), \( p_i \cdot p_j = 0 \) for \( i \neq j \), and \( a_i \neq 0 \) \( (i = 1, 2, ..., n) \), \( a_i \neq a_j \) for \( i \neq j \).

\((***) \) is called the canonical representation of \( w \).
\( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) can be identified with partial Boolean algebras of Hermitian idempotents in \( \mathcal{W}_{\mathcal{A}_1} \) and \( \mathcal{W}_{\mathcal{A}_2} \), respectively. We put

\[
\mathcal{h} \left( \sum_{i=1}^{n} a_i p_i \right) = \sum_{i=1}^{n} a_i h_0 p_i.
\]

Thus we obtain a function \( \mathcal{h} \) which maps \( \mathcal{W}_{\mathcal{A}_1} \) into \( \mathcal{W}_{\mathcal{A}_2} \). The function \( \mathcal{h} \) is well defined, since representation (**) is unique. \( \mathcal{h} \) is also an extension of \( h_0 \). Simple computations show that \( \mathcal{h} \) is a homomorphism.

In case where \( h_0 \) is a monomorphism we need some comments. If (**) is the canonical representation of \( \mathcal{w} \), then the sum

\[
\sum_{i=1}^{n} a_i h_0 p_i
\]

is the canonical representation of \( \mathcal{h}\mathcal{w} \). It follows that if \( \mathcal{w}_1 \neq \mathcal{w}_2 \), then \( \mathcal{h}\mathcal{w}_1 \neq \mathcal{h}\mathcal{w}_2 \).

**Corollary 1.** A partial Boolean algebra \( \mathcal{A} \) is embeddable into a Boolean algebra iff \( \mathcal{W}_{\mathcal{A}} \) is embeddable into a commutative linear algebra with involution.

Indeed, \( \mathcal{W}_{\mathcal{A}} \) is a commutative algebra with involution iff \( \mathcal{A}' \) is a Boolean algebra.

There exist partial Boolean algebras which cannot be even homomorphically mapped into a Boolean algebra (see [2] and [3]). Hence there exist partial \( \ast \)-algebras which cannot be extended to commutative linear algebras with involution.

A partial Boolean algebra \( \mathcal{A} \) is transitive if the relation \( \leq \) defined in \( \mathcal{A} \) by

\[
a \leq b \quad \text{iff} \quad a \ast b \quad \text{and} \quad a \lor b = b
\]

is a partial order in \( \mathcal{A} \).

If \( \mathcal{A} = \langle A; +, \cdot, \ast; 1 \rangle \) is a linear algebra (not necessarily commutative) with the involution *, then the set \( L_{\mathcal{A}} \) of all Hermitian idempotents in \( \mathcal{A} \) forms a transitive partial Boolean algebra \( L_{\mathcal{A}} \), where

\[
a \ast b \quad \text{iff} \quad a \cdot b = b \cdot a \quad (a, b \in L_{\mathcal{A}}).
\]

The remaining operations in \( L_{\mathcal{A}} \) are defined as in case of partial \( \ast \)-algebras.

Let \( \mathcal{A} \) be a linear algebra with involution and with a unit 1. Let \( \{\mathcal{A}_i\}_{i \in I} \) be the family of all maximal, with respect to the inclusion, commutative algebras with involution, contained in \( \mathcal{A} \). Let

\[
\mathcal{A}_0 = \bigcup_{i \in I} \mathcal{A}_i.
\]
Let
\[ a \leftrightarrow b \text{ iff there is a } \lambda \in A \text{ such that } a, b \in A_\lambda (a, b \in A_\mu). \]

It is easy to check that the system
\[ A_\mu = (A_\mu; \leftrightarrow; +, \cdot, *, 1), \]
with operations inherited from \( A \), is a partial *-algebra.

**Corollary 2.** A partial Boolean algebra \( B \) is embeddable into a partial Boolean algebra of Hermitian idempotents of a certain linear algebra with involution iff \( W_B \) can be extended to a linear algebra with involution.

Indeed, let \( A \) be a linear algebra with involution and let \( h_0 \) be an embedding of \( B \) into \( L_A \). Let
\[ w = \sum_{i=1}^{m} a_i p_i \quad \text{and} \quad u = \sum_{j=1}^{n} \beta_j q_j \quad (w \neq u) \]
be elements of \( W_B \) in their canonical representations. Easy computations show that
\[ \sum_{i=1}^{m} a_i h_0 p_i \neq \sum_{j=1}^{n} \beta_j h_0 q_j. \]

A mapping \( h \),
\[ h w = \sum_{i=1}^{m} a_i h_0 p_i, \]
is well defined. Moreover, \( h \) is an embedding of \( W_B \) into \( A \).

There exist partial Boolean algebras not embeddable into transitive ones (see [3]). Hence there exist partial *-algebras which cannot be extended to linear algebras with involution.

Let \( B \in \text{PBA} \). We define a **finitely additive spectral measure** as a homomorphism of the algebra \( B(C) \) of Borel sets of complex numbers into \( B \). A spectral measure \( E: B(C) \mapsto B \) has a **finite carrier** iff there exists a finite set \( \Delta = \{a_1, a_2, \ldots, a_n\} (\Delta \subset C) \) such that \( E(\Delta) = 1 \).

**Theorem 4.** Let \( B \in \text{PBA} \). Then there is a one-to-one correspondence between elements of a partial *-algebra \( W_B \) and finitely additive spectral measures with finite carriers and values in \( B \).

**Proof.** Let
\[ w = \sum_{i=1}^{n} a_i p_i \]
be the canonical representation of \( w \) (\( w \in W_B \)). We have to consider two cases.

(1) \[ \sum_{i=1}^{n} p_i \neq 1. \]
Let
\[ a_{n+1} = 0 \in C \text{ and } p_{n+1} = 1 - \sum_{i=1}^{n} p_i. \]

We define a spectral measure \( E_w \) corresponding to \( w \) as follows. Let \( A \) be a Borel set (\( A \subseteq C \)). Put
\[ \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} = A \cap \{a_1, a_2, \ldots, a_n, a_{n+1}\}. \]
Then
\[ E_w(A) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}. \]
(II)
\[ \sum_{i=1}^{n} p_i = 1. \]

Let \( A \) be a Borel set and put
\[ \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} = A \cap \{a_1, a_2, \ldots, a_n\}. \]
Then
\[ E_w(A) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}. \]

It is clear that \( w_1 \neq w_2 \) implies \( E_{w_1} \neq E_{w_2} \) (\( w_1, w_2 \in W_\mathcal{A} \)).

Now, let \( E \) be any finitely additive spectral measure with a finite carrier \( \Delta_0 \) (\( E: \mathcal{A}(C) \to \mathcal{A} \)). Let \( \Delta_0 = \{a_1, a_2, \ldots, a_n\} \). We put
\[ w = \sum_{i=1}^{n} a_i E(\{a_i\}). \]
Then \( w \in W_\mathcal{A} \) and the formula for \( w \) is the canonical representation of \( w \) iff \( 0 \notin \Delta_0 \). Moreover, \( E = E_w \).

REFERENCES


Reçu par la Rédaction le 20. 4. 1976; en version modifiée le 26. 2. 1977