ON EMBEDDINGS AND TRACES IN SOBOLEV SPACES
WITH WEIGHTS OF POWER TYPE

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1. Introduction

Let $\mathbf{R}^N$ be the $N$-dimensional Euclidean space, let $\Omega \subset \mathbf{R}^N$ be a bounded (open) domain and let $M$ be a non-empty closed subset of $\mathbf{R}^N - \Omega$. For $x \in \mathbf{R}^N$ set

$$d_M(x) = \text{dist}(x, M).$$

Let $\varepsilon \in \mathbf{R}$, $k \in \mathbf{N}$, $1 < p < \infty$. The weighted Sobolev space $W^{k,p}(\Omega; d_M, \varepsilon)$ is the set of all measurable functions $u$ on $\Omega$ such that

$$\|u\|_W = \left( \sum_{|\alpha| \leq k} \int \sum_{j=1}^{N} (|D^j u(x)|^p d_M^q(x) d\alpha)^{1/p} \right) < \infty,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$, $|\alpha| = \alpha_1 + \ldots + \alpha_N$ and the $D^j u$ are distributional derivatives of $u$. The expression (1) defines a norm in $W^{k,p}(\Omega; d_M, \varepsilon)$ which provided with this norm is a Banach space.

In [5] and [1] it was proved that under certain assumptions on $\varepsilon, k, p, \Omega$ and $M$ the space $W^{k,p}(\Omega; d_M, \varepsilon)$ is continuously embedded in another weighted space of Sobolev type $H^{k,p}(\Omega; d_M, \varepsilon)$ which consists of all functions $u$ such that

$$\|u\|_H = \left( \sum_{|\alpha| \leq k} \int \sum_{j=1}^{N} (|D^j u(x)|^p d_M^{-|\alpha|} d\alpha(x) d\alpha)^{1/p} \right) < \infty.$$

($H^{k,p}(\Omega; d_M, \varepsilon)$ is also a Banach space when equipped with the norm (2).)

The spaces $H^{k,p}(\Omega; d_M, \varepsilon)$ are worth studying for several reasons: the exponents of the weight $d_M$ in (2) are less than the ones in (1), so that the norm (2) reflects more finely the behaviour of functions; in (2) there may be exponents of different signs, i.e. the norm (2) admits simultaneous appearance of weights with both degeneracy and singularity; the spaces $H^{k,p}(\Omega; d_M, \varepsilon)$ occur in applications to boundary value problems.
Let $W^p_0(\Omega; d_M, \varepsilon)$ and $H^p_0(\Omega; d_M, \varepsilon)$ be the closures of the set

$$C^\infty_M(\bar{\Omega}) = \{ v \in C^\infty(\bar{\Omega}) ; \text{supp} \, v \cap M = \emptyset \}$$

in the spaces $W^{k,p}(\Omega; d_M, \varepsilon)$ and $H^{k,p}(\Omega; d_M, \varepsilon)$ respectively. In [1] it is claimed that for all $\varepsilon \in \mathbb{R}$,

$$H^{k,p}(\Omega; d_M, \varepsilon) \subset W^p_0(\Omega; d_M, \varepsilon)$$

(the symbol $\subset$ denotes continuous embedding). Unfortunately, that assertion (Theorem 1.2 in [1]) does not hold, because the assumptions on $\Omega$ and $M$ are too weak and in the proof the estimate preceding (1.13) contains a mistake. We shall give here a correct version of the theorem proving at the same time a bit more: the embedding into $H^p_0(\Omega; d_M, \varepsilon)$.

In Section 3 we shall discuss the existence and value of traces of functions from weighted Sobolev spaces.

2. Embeddings

Throughout this section we shall suppose that the domain $\Omega$ has the segment property outside the set $M$ and satisfies the inner cone condition in a neighborhood of the boundary of the set $\partial \Omega - M$; more precisely: There exists an open covering $\{ U_j \}_{j=1}^\infty$ of $\bar{\Omega}$ with the following properties:

(a) If

(3)

$$\bar{U}_j \cap M = \emptyset,$$

then there exists a vector $\xi_j \in \mathbb{R}^N - \{ 0 \}$ such that $x + t \xi_j \in \Omega$ for all $x \in U_j \cap \bar{\Omega}$ and $0 < t < 1$.

(b) If

(4)

$$U_j \cap M \cap \partial \Omega - M \neq \emptyset,$$

then there exists an open cone $C_j$ with vertex at the origin, congruent to a given cone $C$, and such that $(x + C_j) \subset \Omega$ for all $x \in U_j \cap \bar{\Omega}$.

(c) If neither (3) nor (4) holds then $U_j \cap (\partial \Omega - M) = \emptyset$.

**Theorem 1.** Let $1 < p < \infty$, $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Then

$$H^{k,p}(\Omega; d_M, \varepsilon) = H^p_0(\Omega; d_M, \varepsilon).$$

**Proof.** Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$. Evidently, it suffices to find a sequence of functions $w_n \in C^\infty_0(\bar{\Omega})$ converging to $u$ in $H^{k,p}(\Omega; d_M, \varepsilon)$. For $h > 0$, $x \in \mathbb{R}^N$, denote by $B_h(x)$ the ball in $\mathbb{R}^N$ of radius $h$ with center at $x$. Let $\{|\psi_x|\}_{x=1}^\infty$ be a partition of unity on $\bar{\Omega}$ subordinate to the covering $\{ U_j \}_{j=1}^\infty$. Put $v_j = u \psi_j$ and extend it by zero outside $\Omega$. Let $\varphi \in C^\infty_0(\mathbb{R}^N)$ be a non-negative function such that $\text{supp} \, \varphi \subset B_1(0)$, $\int_{\mathbb{R}^N} \varphi(x) \, dx = 1$ and put $\varphi_h(x) = h^{-N} \varphi(x/h)$ for
$h > 0$. Write $\delta = \min \text{dist}(U_j, M)$, where the minimum is taken over all $j$ satisfying (3). For a function $v$ on $\Omega$ and $t > 0$ define $v^{(t)}(x) = v(x)$ if $d_M(x) \geq t$ and $x \in \Omega$, $v^{(t)}(x) = 0$ otherwise.

Take $j = 1, \ldots, s$.

First suppose that (3) holds. Then $\delta \leq d_M(x) \leq \text{diam} \Omega + \text{dist}(\Omega, M) < \infty$ for $x \in U_j \cap \Omega$, so that $H^{k,p}(U_j \cap \Omega; d_M, \epsilon)$ coincides with the non-weighted Sobolev space $W^{k,p}(U_j \cap \Omega)$ and we can construct in a usual way, by the use of translation and mollification arguments, a sequence of functions $v_{j,h} \in C^\infty(\mathbb{R}^N)$ such that

(5) \quad \text{supp}\, v_{j,h} \cap M = \emptyset,

(6) \quad v_{j,h} \to u_j \text{ in } H^{k,p}(\Omega; d_M, \epsilon) \quad \text{as} \quad h \to 0.

Next, suppose (4). The cone $C_j$ from condition (b) can be expressed in the form

(7) \quad C_j = \bigcup_{0 < r < 1} tB_r(\xi_j),

where $\xi_j \in \mathbb{R}^N - \{0\}$, $0 < r < |\xi_j|$. Set $\sigma = r^{-1} |\xi_j| > 1$ and define functions $v_{j,h} \in C^\infty(\mathbb{R}^N)$, $h > 0$, by

(8) \quad v_{j,h}(x) = \int_{\mathbb{R}^N} \varphi_h(x - y) u_j^{(5\sigma h)}(y + hr^{-1} \xi_j) dy.

We shall prove that the $v_{j,h}$ satisfy (5) and (6). The index $j$ will be omitted.

(i) If $x \in U \cap \Omega$ is such that $d_M(x) \leq 3\sigma h$, then for $y \in B_h(x)$ we have $d_M(y + hr^{-1} \xi) \leq d_M(x) + |y - x| + hr^{-1} |\xi| < 5\sigma h$. Hence,

(9) \quad v_h(x) = 0

and (5) holds.

(ii) Let $x \in U \cap \Omega$ be such that $3\sigma h < d_M(x) \leq 7\sigma h$. Then for $y \in B_{1 + \sigma h}(x)$ we get $\sigma h < d_M(y) < 9\sigma h$, i.e. $d_M(x) \sim d_M(y) \sim h$ ($a \sim b$ means that the ratio $a/b$ is bounded from above and from below by positive constants). Hence, if $|x| \leq k$, we can write

$$D^a v_h(x) = h^{-N - |a|} \int_{\mathbb{R}^N} (D^a \varphi) \left(\frac{x - y}{h}\right) u^{(5\sigma h)}(y + hr^{-1} \xi) dy$$

and

$$|D^a v_h(x)| \leq \sup_z |D^a \varphi(z)| \ h^{-N - |a|} \int_{B_h(x)} |u(y + hr^{-1} \xi)| dy \leq c_1 h^{-N - |a|} \int_{B(1 + \sigma h)} |u(y)| dy.$$
It follows that

\[ |D^\alpha v_h(x)|_d^{|l/p - k + |\alpha|}(x) \leq c_2 |B_{(1+\sigma)h}(x)|^{-1} \int_{B_{(1+\sigma)h}(x)} |u(y)| d_d^{|l/p - k}(y) dy \]
\[ \leq c_2 M (u d_d^{|l/p - k})(x), \]

where the \( c_i \) are positive constants, \(|B|\) denotes the Lebesgue measure of \( B \) and \( M \) is the Hardy–Littlewood maximal operator defined by

\[ Mf(x) = \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| dy. \]

(iii) Suppose that \( x \in U \cap \Omega \) is such that for some \( l > 7, l \sigma h < d_M(x) \leq (l+1) \sigma h \). Then \( y \in B_{(l+\sigma)h}(x) \) implies \( 5 \sigma h < (l-2) \sigma h \leq d_M(y) \leq (l+3) \sigma h \), i.e. \( d_M(x) \sim d_M(y) \sim \sigma h \), and \( u^{(5 \sigma h)}(y) = u(y) \). Moreover, \( x + C \subset \Omega \) by (b), which together with (7) yields \( B = B_h(x + h r^{-1} \xi) \subset \Omega \). Thus for \( |\alpha| \leq k \) we have

\[ \begin{align*}
D^\alpha v_h(x) & = D^\alpha \int_B \varphi_h \left( \frac{x-y}{h} + r^{-1} \xi \right) D^\alpha u(y) dy \\
& = h^{-N} \int_B \varphi_h \left( \frac{x-y}{h} + r^{-1} \xi \right) D^\alpha u(y) dy,
\end{align*} \]

and

\[ |D^\alpha v_h(x)| d_d^{|l/p - k + |\alpha|}(x) \leq \sup_z |\varphi(z)| h^{-N} \int_{B_{(1+\sigma)h}(x)} |D^\alpha u(y)| d_d^{|l/p - k + |\alpha|}(y) dy \]
\[ \leq c_3 M(D^\alpha u d_d^{|l/p - k + |\alpha|})(x). \]

Put

\[ G(x) = \max_{|\alpha| \leq k} M(D^\alpha u d_d^{|l/p - k + |\alpha|})(x). \]

Since \( u \in H^{k,p}(\Omega; d_M, \sigma) \), the functions \( D^\alpha u d_d^{|l/p - k + |\alpha|} \) belong to \( L^p(\Omega) \) and to \( L^p(R^N) \) as well. The boundedness of the maximal operator \( M \) in \( L^p(R^N) \) now implies that \( G \in L^p(R^N) \) and by (9), (10), (11) we have

\[ |D^\alpha v_h(x)| d_d^{|l/p - k + |\alpha|}(x) \leq cG(x), \quad |\alpha| \leq k. \]

It suffices to select a subsequence \( \{v_{h_n}\}_{n=1}^{\infty} \) of \( \{v_h\}_{h > 0} \) such that

\[ D^\alpha v_{h_n}(x) \to D^\alpha u(x) \quad \text{for} \quad |\alpha| \leq k \quad \text{and for a.e.} \quad x \in \Omega. \]

Then by the Lebesgue Dominated Convergence Theorem, (6) holds with \( h_n \) instead of \( h \). The construction of \( \{v_{h_n}\} \) relies on the properties of the mollifier and can be done in the same way as in the proof of Theorem 1.2 in [1].

Finally, if neither (3) nor (4) holds, then \((\partial \Omega - M) \cap U_j = \emptyset \). We define

\[ v_{j,h}(x) = \int_{R^N} \varphi_h(x-y) u_{j}^{(3h)}(y) dy \]
and proceed like in the case of (4): if $d_M(x) \leq 2h$, $2h < d_M(x) \leq 4h$ or $lh < d_M(x) \leq (l+1)h$, $l \geq 4$, then (9), (10) or (11) holds respectively, and we again construct a subsequence of $\{v_{n}\}$ converging to $u$ in $H^{k,p}(\Omega; d_M, \varepsilon)$.

The functions $w_n = \sum_{j=1}^{n} v_{jh_n}$ form the desired sequence.

**Corollary.** Under the assumptions of Theorem 1 the embedding
\[ H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_{M}^{k,p}(\Omega; d_M, \varepsilon) \]
holds.

**Proof.** Since $\Omega$ is bounded, $d_M(x) < \text{diam} \Omega + \text{dist}(\Omega, M) < \infty$ and $\|u\|_W \leq c \|u\|_H$.

**Remark 1.** In other words, Theorem 1 and the corollary establish the density of the set $C^\infty_c(\overline{\Omega})$ in $H^{k,p}(\Omega; d_M, \varepsilon)$ (with respect to both norms (1) and (2)).

We got the approximation by functions smooth up to the boundary $\partial \Omega$ at the cost of relatively strong assumptions. In [7] H. Triebel proved (without any assumptions on $M$ and $\Omega$) that in the weighted (fractional order) Sobolev space $W^{k,p}(\Omega; d_M, \varepsilon)$ the set $\{f; f \in W^{k,p}(\Omega; d_M, \varepsilon), \text{supp} f \cap M = \emptyset\}$ is dense.

Let us recall the inverse embedding proved in [1].

**Proposition ([1], Theorem 2.3).** Let $1 < p < \infty$, $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Let $m \in \{0, 1, \ldots, N-1\}$, $M \subset \partial \Omega$. Suppose that there exists an open covering $\{U_i\}_{i=0}^{\omega}$ (\(\omega \leq \infty\)) of $\overline{\Omega}$ with the properties:

(i) $\bigcup_{i=1}^{\omega} U_i \supset M$, and there exists $s \in \mathbb{N}$ such that every system of $s+1$ sets $U_i$ is disjoint;

(ii) $U_0 \cap M = \emptyset$;

(iii) there are numbers $c_1$, $c_2 > 0$ and a system of one-to-one mappings $T_i: \overline{\Omega} \to \overline{\Omega_i}$, $Q = (0, 1)^N$, such that
\[ T_i(\{x \in \overline{\Omega}; x_{m+1} = \ldots = x_N = 0\}) = M \cap \overline{U_i} \]
and
\[ c_1 |x-y| \leq |T_i(x) - T_i(y)| \leq c_2 |x-y| \quad \text{for all } x, y \in \overline{\Omega}, i = 1, 2, \ldots, \omega. \]

Then
\[ V \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon), \]
where
\[ V = W^{k,p}(\Omega; d_M, \varepsilon) \quad \text{if } \varepsilon > kp + m - N \quad \text{or} \quad \varepsilon \leq m - N, \]
and
\[ V = W_{M}^{k,p}(\Omega; d_M, \varepsilon) \quad \text{if } \varepsilon \neq jp + m - N, \quad j = 1, \ldots, k. \]
The absence of the values $\varepsilon = jp + m - N$, $j = 1, \ldots, k$ in Proposition is essential. J. Kadlec and A. Kučer [2] proved for $\Omega$ with Lipschitzian boundary and for $M = \partial \Omega$ that if $\varepsilon = jp - 1$ with some $j = 1, \ldots, k$, then $W^{k,p}_0(\Omega; d_M, \varepsilon)$ ( = $W^{k,p}_0(\Omega; d_M, \varepsilon)$) is equivalent to the space $H^{k,p}_0(\Omega; d_M, \varepsilon)$ of functions with the norm
\[
\|u\|_{H^k_0} = \left( \sum_{|\alpha| \leq k - j} \int_{\Omega} \left\| D^\alpha u(x) \right\|^p d_M^{-k-|\alpha|} \left( \log \frac{R}{d_M(x)} \right)^{-p} dx + \sum_{|\alpha| \leq k - j} \int_{\Omega} \| D^\alpha u(x) \|^p d_M^{-k-|\alpha|} \left( \log \frac{R}{d_M(x)} \right)^{-p} dx \right)^{1/p} < \infty
\]
($R > 0$ is a sufficiently large number). This result can be extended to more general domains $\Omega$ and sets $M$. Define $H^{k,p}_{0,M}(\Omega; d_M, \varepsilon)$ as the closure of $C^\infty_0(\Omega)$ in $H^{k,p}_{0,M}(\Omega; d_M, \varepsilon)$.

**Theorem 2.** Let $p, k, \varepsilon, \Omega$ and $M$ satisfy the assumptions of Theorem 1 and let $j = 1, \ldots, k$. Then

$H^{k,p}_{0,M}(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}_{0,M}(\Omega; d_M, \varepsilon) \hookrightarrow W^{k,p}_0(\Omega; d_M, \varepsilon)$.

**Theorem 3.** Let $p, k, m, \Omega$ and $M$ satisfy the assumptions of Proposition. Let $\varepsilon = jp + m - N$ for some $j = 1, \ldots, k$. Then

$W^{k,p}_M(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}_{0,M}(\Omega; d_M, \varepsilon)$.

Proof of Theorem 2 or 3 can be done step by step as the proofs of Theorem 1 or of Proposition (Theorem 2.3 in [1]) and the corresponding theorem in [2].

3. Traces

In this section we suppose that the domain $\Omega$ has a Lipschitzian boundary, i.e. there exist a finite number $m$ of coordinate systems $(y'_i, y_{i,N})$. $y'_i = (y_{i,1}, \ldots, y_{i,N})$ and of functions $a_i = a_i(y'_i)$ Lipschitzian on the closures of the $(N - 1)$-dimensional cubes $2\Delta_i = \{y'; |y_j| < 2\delta \text{ for } j = 1, \ldots, N - 1\}$ $(i = 1, \ldots, m)$ and such that:

(i) for each $x \in \partial \Omega$ there is at least one $i \in \{1, \ldots, m\}$ such that $x = (y'_i, y_{i,N})$ and $y_{i,N} = a_i(y'_i)$, $y'_i \in \Delta_i = \{y'; |y_j| < \delta \text{ for } j = 1, \ldots, N - 1\}$,

(ii) there exists $\beta > 0$ such that the sets $B_i = \{y'_i, y_{i,N}; y'_i \in 2\Delta_i, a_i(y'_i) - 2\beta < y_{i,N} < a_i(y'_i) + 2\beta\}$ satisfy

$U_i = B_i \cap \Omega = \{(y'_i, y_{i,N}); y'_i \in 2\Delta_i, a_i(y'_i) - 2\beta < y_{i,N} < a_i(y'_i)\}$

and

$\Gamma_i = B_i \cap \partial \Omega = \{(y'_i, y_{i,N}); y'_i \in 2\Delta_i, y_{i,N} = a_i(y'_i)\}$

$(i = 1, \ldots, m)$.

Further, let us suppose that the set $M$ is a subset of the boundary $\partial \Omega$. It is easily seen that conditions (a), (b), (c) on $\Omega$ and $M$ from Section 2 are
satisfied. Set $M_i = M \cap B_i$. By $L^p(\partial \Omega ; d_M, \varepsilon)$ we denote the space of functions $u$ defined a.e. on $\partial \Omega$ and such that the surface integral

\begin{equation}
(\int_{\partial \Omega} |u(x)|^p d_M^*(x) dS(x))^{1/p}
\end{equation}

is finite.

It can be proved that

\begin{equation}
\left[ \sum_{i=1}^{m} \int_{\partial \Omega} |u(x_i, a_i(x_i))|^p d_M^*(x_i, a_i(x_i)) \right]^{1/p},
\end{equation}

where we put $d_0(x) = 1$, is a norm in $L^p(\partial \Omega ; d_M, \varepsilon)$ which is equivalent to the norm (15).

We shall study the existence of traces on $\Gamma_i$ of functions from $H^{k,p}(\Omega ; d_M, \varepsilon)$ for some index $i = 1, \ldots, m$. If $M_i = \emptyset$, then

\[ 0 < \min(\beta, \delta) \leq d_M(x) < \text{diam} \Omega \quad \text{for} \ x \in B_i \]

and the problem can be reduced to the non-weighted case which is well known.

Thus, suppose $M_i \neq \emptyset$. We shall omit the index $i$. Take $x' \in \Delta$ and suppose first that $(x', a(x')) \notin M$, i.e. $d_M(x', a(x')) > 0$. Following the considerations in the proof of Theorem 2.6 in [3] we can write for $u \in C^\infty(\overline{\Omega})$ and for

\begin{equation}
a(x') - \min(\beta, d(x', a(x'))) < s < a(x')
\end{equation}

that

\begin{equation}
|u(x', a(x'))|^p \leq 2^{p-1} \left\{ |u(x', s)|^p + \left( \int_s^{a(x')} |D_N u(x', t)| dt \right)^p \right\}
\end{equation}

\[ \leq 2^{p-1} \left\{ |u(x', s)|^p + d_M(x', a(x'))^{p-1} \right\}
\end{equation}

\[ \times \left\{ \int_{a(x') - d_M(x', a(x'))}^{a(x')} |D_N u(x', t)|^p dt \right\}. \]

The triangle inequality and the Lipschitz property of the function $a$ yield

\begin{equation}
\frac{1}{2} \leq \frac{d_M(x', a(x'))}{d_M(x', s)} \leq c_1
\end{equation}

for $s$ satisfying (17).

Integrating (18) with respect to $s$ from $a(x') - d_M(x', a(x'))$ to $a(x')$, using estimates (19) and integrating over $\Delta^x = \{x' \in \Delta; (x', a(x')) \notin M\}$ we obtain

\begin{equation}
\int_{\Delta^x} |u(x', a(x'))|^p d_M^{*^{-p+1}}(x', a(x')) dx' \leq c_2 \left\{ \int_{\partial \Omega} |u(x)|^p d_M^*(x) dS(x) + \int_{\partial \Omega} |D_N u(x)|^p d_M^*(x) dS(x) \right\}.
\end{equation}

The last estimate together with Theorem 1 implies
Theorem 4. Let $1 < p < r$, $q \in \mathbb{R}$. Then there exists a unique bounded linear operator $Z : H^{1,p}(\Omega; dM, v) \to L^p(\partial\Omega - M; dM, v - p + 1)$ such that $Zu = u|_{\partial\Omega - M}$ for all $u \in C^\infty(\overline{\Omega})$.

By the same method one can prove

Theorem 5. Let $1 < p < \infty$, $v \in \mathbb{R}$. Then there exists a unique bounded linear operator $Z$ from $H^{1,p}_{\Omega}(\Omega; dM, v)$ into the Lebesgue space $L^p$ on $\partial\Omega - M$ with the weight $dM^{p-1}(x) \left( \log \frac{R}{dM(x)} \right)^p$.

Now, we turn our attention to the case $(\alpha', a(\alpha')) \in M$. Simple examples show that functions from $H^{1,p}(\Omega; dM, v)$ for $v \sim p - 1$ may have singularities on $M$ — although there is a dense set of functions vanishing near $M$. On the other hand, if $v \sim p - 1$ and $u \in H^{1,p}(\Omega; dM, v)$, then

$$(21) \int_{u(\alpha') - \beta}^{u(\alpha')} |u(x', s)|^p dM(x', s) ds + \int_{u(\alpha') - \beta}^{u(\alpha')} |D_N u(x', s)|^p dM(x', s) ds < \infty$$

for a.e. $x' \in A$. By the Hölder inequality we have for $a(\alpha') - \beta \leq s \leq s + h$

$$|u(x', s + h) - u(x', s)|$$

$$\leq \left( \int_s^{s+h} |D_N u(x', t)|^p dM(x', t) dt \right)^{1/p} \left( \int_s^{s+h} dM^{(p-1)}(x', t) dt \right)^{(p-1)/p}$$

where the first term on the right-hand side is bounded for a.e. $x'$ and the second is $o(1)$ as $h \to 0$. Hence, the function $u$ is uniformly continuous on almost all lines $x' = \text{const}$ and there exists a finite limit

$$(22) \lim_{t \to u(x')} u(x', t) = g(x')$$

which must be zero because of the convergence of the first integral in (21). Unfortunately, such considerations do not work if $v = p - 1$. Nevertheless, we have

**Lemma.** Let $0 < a < b < 1$, $0 < \alpha < \beta < \infty$ and $1 < p < \infty$. Then for each function $u \in H^{1,p}((0, 1); d\gamma; p - 1)$ such that $u(a) = \alpha$, $u(b) = \beta$,

$$(23) \int_a^b |u(x)|^p \frac{dx}{x} + \int_a^b |u'(x)|^p x^{p-1}dx > 2^{1-\varepsilon}(p-1)^{1/p} \frac{u(b) - u(a)}{p}.$$
has a general solution \( u_0(x) = Ax^\lambda + Bx^{-\lambda} \), where \( \lambda = (p-1)^{-1/p} \). If we insert \( u_0 \) in \( J \) taking into account the values \( u_0(a) = \alpha \), \( u_0(b) = \beta \), we can estimate \( J(u_n) = \min J(u) \) from below by the right-hand side of (23).

Now, suppose that the limit (22) does not exist. Then we can choose an oscillating sequence of values \( u(u_n) \) such that \( u_n \to u(x') \) and applying the lemma on the intervals \( (a_{2n-1}, a_{2n}) \) we get a contradiction with (21). Hence, the limit (22) exists and must be finite for a.e. \( x' \) because of (21).

In this way we have proved

**Theorem 6.** If \( \varepsilon \leq p-1 \), then functions from \( H^{k,p}(\Omega; d_M, \varepsilon) \) have zero traces on \( M \).

**Remark 2.** The results of this section can be easily reformulated for the spaces \( W^{k,p}(\Omega; d_M, \varepsilon) \) and \( W^{k,p}_M(\Omega; d_M, \varepsilon) \), if we use Theorems 1, 2, 3 and Proposition 6.

**Remark 3.** We treated the question of existence of traces only. The problem of full characterization of traces by direct and inverse theorems is still open. For certain results with \( M = \partial \Omega \) we refer e.g. to [4], [6].

References


Presented to the Semester

**Approximation and Function Spaces**

**February 27–May 27, 1986**