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On quasi-starlike functions

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Abstract. Let S^* be the usual class of normalized starlike functions $F(z)$ on the unit disk $U = \{z: |z| < 1\}$. If $g(z)$ is regular in U and satisfies the condition $MF[g(z)] = F(z)$, $z \in U$, for some $F \in S^*$ and some positive number $M > 1$, then g is said to be in G^M. In Ann. Polon. Math. 20 (1968), p. 280–282 and ibidem 26 (1972), p. 175–197, I. Dziubiński defined the class G^M and called g in G^M a *quasi-starlike function*. He raised the question of inclusion relations between S^* and G^M and asked if every bounded starlike function is quasi-starlike. We answer the question in the negative by exhibiting a bounded starlike function that is not quasi-starlike. We also show that if F is either a strongly starlike function of order $1/2$ as defined by Brannan and Kirwan in J. London Math. Soc. (2) 1 (1969), p. 431–443, or if F is a circularly symmetric function, then g defined by $MF[g(z)] = F(z)$ is starlike. We also show that the $1/2$ is best possible in the sense that for every ε, $0 < \varepsilon < 1/2$, there exists a strongly starlike function f of order $\varepsilon + 1/2$ such that the g defined by $Mf[g(z)] = f(z)$ is not starlike.

1. Introduction. Let S denote the class of regular univalent functions $f(z) = z + a_2 z^2 + \ldots$ in the unit disk U. Let S^* denote the subclass of S of functions f such that $f(U)$ is starlike with respect to the origin. We use starlike to mean starlike with respect to the origin. In [3] and [4] I. Dziubiński introduced the class of functions $\tilde{S}^*_M$ that he called *quasi-starlike*. He defined for $M > 1$,

$$\tilde{S}^*_M = \{g: \ Mf[g(z)] = f(z), f \in S^*, z \in U\},$$

where g is said to be *generated by* f. Then $Mg(z) = z + \ldots$ is a normalized quasi-starlike function and is in S. In [4] Dziubiński posed the problem as to whether every starlike function bounded in U is a normalized quasi-starlike function. He also discussed the difficulty in obtaining conditions for a quasi-starlike function to be starlike. He stated that the difficulty arises because a quasi-starlike function can be easily constructed from any given starlike function.

In this note we give an example of a bounded starlike function that is not a normalized quasi-starlike function and give some sufficient conditions for a normalized quasi-starlike function to be starlike.

2. An example. Let F be the bounded starlike function such that $F(U)$ is a disk minus two radial slits. The disk is centered at the origin and of radius M, while the slits are non-vertical, non-horizontal and symmetric about the real axis. We will show that F can not be a normalized quasi-starlike function. Assume to the contrary. Then there exists an f in S^* and an $M > 1$ such that $F(z) = Mf^{-1}[f(z)/M]$. For any set X let $f(X) = \{f(x): x \in X\}$. We first show that $f(U)$ must be a slit domain. Let $g(z) = F(z)/M$ with F as defined above. Since $f \in S^*, f[g(U)] = f(U)/M$ is a starlike domain and $f[g(U)] = f(U) - f(l_1) \cup f(l_2)$, where Ml_1 and Ml_2 are the symmetric, radial, linear slits in $F(U)$. Since $f[g(U)]$ is starlike, $f(l_1)$ and $f(l_2)$ must be radial slits. It now follows easily from the equation

$$f(U) - f(l_1) \cup f(l_2) = \frac{f(U)}{M}$$

that $f(U)$ is the plane minus two radial slits.

From this geometric description, f must assume the following form:

$$f(z) = \frac{z}{(1-\sigma_1 z)^{\alpha}(1-\sigma_2 z)^{2-\alpha}} \tag{2}$$

for some α, $0 < \alpha < 2$, and $|\sigma_k| = 1$, $k = 1, 2$. Dziubiński showed in [4], Theorem 3, that the only time a function of the form (2) generates a quasi-starlike function that is starlike is when $\alpha = 1$ and $\sigma_k = \exp i(-1)^{k-1}\theta$, $k = 1, 2$, for any $\theta \in (0, \pi)$. This would imply the two radial slits in $f(U)$ are opposing slits (i.e., their arguments differ by π). But this would force the slits in $F(U)$ to be opposing slits also. This contradicts the definition of F. Therefore F is a bounded starlike function that is not a normalized quasi-starlike function.

3. Conditions for starlikeness. To establish these conditions we need the definitions of two subclasses of S. Jenkins stated in [6] that a domain D is circularly symmetric with respect to the positive reals if every circle centred at the origin intersects D in at most one arc γ such that γ is symmetric with respect to the positive reals. We say a function f is in Y if f is in S and $f(U)$ is circularly symmetric with respect to the positive reals. We will suppress the term "with respect to the positive reals". Also, in [1], Brannan and Kirwan defined the class of strongly starlike functions $S^*(\alpha)$, where, for given α, $0 \leqslant \alpha \leqslant 1$, $f \in S^*(\alpha)$, if and only if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqslant \frac{\alpha\pi}{2}, \quad z \in U. \tag{3}$$

The main theorem is as follows:

THEOREM. *Let F be in $\tilde{S}_M^*$ with F defined by*

$$Mf[F(z)] = f(z), \quad z \in U \tag{4}$$

for $f \in S^*$ *and* $M > 1$. *If either*

(a) $f \in Y$, *or*

(b) $f \in S^*(a)$, $0 \leqslant a \leqslant 1/2$,

then MF *is in* S^*. *The* 1/2 *in* (b) *is sharp*.

Remark. The sharpness result of (b) is in the sense that for every $\varepsilon > 0$ there exists a function f_ε in $S^*(\varepsilon + 1/2)$ that generates a function in S_M^* that is not starlike for some $M = M(\varepsilon)$.

Proof. Take the logarithmic derivative of (4) with respect to z and then multiply by z to obtain:

$$\frac{\left\{\frac{d}{dF} f[F(z)]\right\} z \frac{d}{dz} F(z)}{f[F(z)]} = \frac{z \frac{d}{dz} f(z)}{f(z)}.$$

Let $w = F(z)$, where $|w| < 1$ and let $f'(w) = \frac{df(w)}{dw}$. Then using (4) we have

$$\frac{zF'(z)}{F(z)} = \frac{f(w)}{wf'(w)} \frac{zf'(z)}{f(z)}. \tag{5}$$

Since MF is starlike if and only if

$$\left|\arg \frac{zF'(z)}{F(z)}\right| < \frac{\pi}{2}, \quad z \in U,$$

we need only show that conditions (a) and (b) separately imply that

$$\left|\arg \frac{zf'(z)}{f(z)} \frac{f(w)}{wf'(w)}\right| < \frac{\pi}{2}, \quad z \in U. \tag{6}$$

To prove part (a) of the theorem, consider an f in Y. Let $P(\zeta) = \zeta f'(\zeta)/f(\zeta)$ for any $\zeta \in U$. It follows from a result of Jenkins in [6] that if $f \in Y$, then either f is the identity function or

$$\begin{aligned} &\operatorname{Im}\{z\}\operatorname{Im}\{f(z)\} \geqslant 0, \\ &\operatorname{Im}\{z\}\operatorname{Im}\{P(z)\} \geqslant 0, \end{aligned} \qquad z \in U. \tag{7}$$

The case when f is the identity follows immediately, so assume f is not the identity. Consider the two cases, $\operatorname{Im}\{z\} \geqslant 0$ and $\operatorname{Im}\{z\} \leqslant 0$. When $\operatorname{Im}\{z\} \geqslant 0$, since $F(z) = w$ is defined by (4), we have that $\operatorname{Im}\{w\} \geqslant 0$. Hence, since $f \in Y$, property (7) assures that $\operatorname{Im}\{P(z)\} \geqslant 0$ and $\operatorname{Im}\{P(w)\} \geqslant 0$. Thus, since $\operatorname{Re}\{P(z)\}$ and $\operatorname{Re}\{P(w)\}$ are positive from the starlikeness of f, we have $0 \leqslant \arg P(z) < \pi/2$ and $0 \leqslant \arg P(w) < \pi/2$ for $\operatorname{Im}\{z\} \geqslant 0$. Hence $|\arg[P(z)/P(w)]| = |\arg P(z) - \arg P(w)| = \big||\arg P(z)| - |\arg P(w)|\big| \leqslant \max[\arg P(w), \arg P(z)] < \pi/2$. A corresponding argument will show that if $\operatorname{Im}\{z\} \leqslant 0$, then $|\arg[P(z)/P(w)]| < \pi/2$. Therefore (6) follows.

For part (b) of the theorem, let $f \in S^*(a)$ for $0 \leqslant a \leqslant 1/2$. Then using (3),

$$\left|\arg \frac{zf'(z)}{f(z)} \cdot \frac{f(w)}{wf'(w)}\right| \leqslant \left|\arg \frac{zf'(z)}{f(z)}\right| + \left|\arg \frac{wf'(w)}{f(w)}\right| \leqslant \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Thus (6) follows.

To verify the sharpness result, we show that for every $\varepsilon > 0$ there exists a function in $S^*(\varepsilon + 1/2)$ that generates a function in $\tilde{S}_M^*$ that is not starlike for some $M > 1$. Let $D(a)$ denote the pie shaped convex domain bounded by the left half of the unit circle $\widehat{AC}$ and two line segments $\overline{AB}$ and $\overline{BC}$ having angles of inclination with the positive real axis $\pi \pm (1-a)\pi/2$ $(0 < a < 1)$, respectively. Let g be the corresponding mapping function such that $g(U) = D(a)$ with $g(0) = 0$ and $g'(0) > 0$. It is clear that g extends to a continuous function on the closure of U that is differentiable on U except at the preimages of the three corners of $D(a)$. We denote the extended function as g also. Note the function $g(z) = a_1 z + \ldots$, where a_1 is positive, is such that $\frac{1}{a_1} g$ is in $S^*(a)$. Given an $\varepsilon > 0$, let $a_\varepsilon = 1/2 + \varepsilon/2$. Choose an $M > 1$ such that $Mg(w_0) = g(z_0)$ defines a w_0 with $\arg[w_0 g'(w_0)/g(w_0)] = (1/2 + \varepsilon/8)\pi/2$. M can be chosen in this manner since $w_0 \to z_0$ as $M \to 1$ and $\arg[w_0 g'(w_0)/g(w_0)]$ increases to $\arg[z_0 g'(z_0)/g(z_0)] = (1/2 + \varepsilon/4)\pi/2$. Now we shall construct a sequence of domains which converge to $D(a_\varepsilon)$ and such that their corresponding mapping functions will converge uniformly on compact subsets of U to g. Let n be a large positive integer. Consider the domain bounded by an arc $\widehat{A_n C_n}$ that is the left half of a circle centered at the origin with $\operatorname{Im}\{A_n\} > 0$, the line segment $\overline{C_n B_n}$ parallel to $\overline{CB}$, a line segment $\overline{E_n D_n}$ of length $1/n$, parallel to $\overline{CB}$ and having $g(z_0)$ as its midpoint, and the line segments $\overline{A_n D_n}$ and $\overline{E_n B_n}$ that are parallel to $\overline{AB}$ and that complete the boundary of this simply connected domain. Denote this domain as G_n with corresponding mapping function g_n such that $g_n(U) = G_n$. It is clear that as $n \to \infty$, G_n converges to $D(a_\varepsilon)$ in the sense of Carathéodory. From the Carathéodory convergence theorem [5] g_n converges to g uniformly on compact subsets of U. For each n, let z_n be the point on the unit circle such that $g_n(z_n) = g(z_0)$. From the construction of g_n we have $\arg[z_n g_n'(z_n)/g_n(z_n)] = -(1/2 + 3\varepsilon/4)\pi/2$ for each n. Let w_n be the point in U such that $Mg_n(w_n) = g_n(z_n) = g(z_0)$. From the uniform convergence of g_n to g on compact subsets of U we have that $w_n \to w_0$ as $n \to \infty$, while Weierstrass' Theorem assures that $\arg[w_n g_n'(w_n)/g_n(w_n)]$ approaches $\arg[w_0 g'(w_0)/g(w_0)]$. Thus there exists an integer N such that g_N is in

$S^*(\varepsilon+1/2)$ while

$$\left|\arg\frac{z_N g_N'(z_N)}{g(z_N)}-\arg\frac{w_N g_N'(w_N)}{g_N(w_N)}\right| \geqslant \left|-\left(\frac{1}{2}+\frac{3\varepsilon}{4}\right)\frac{\pi}{2}-\left(\frac{1}{2}+\frac{\varepsilon}{8}\right)\frac{\pi}{2}\right|$$
$$=\left(1+\frac{7\varepsilon}{8}\right)\frac{\pi}{2}>\frac{\pi}{2}.$$

Therefore it follows from (6) that g_N generates a quasi-starlike function that is not starlike. This completes the proof of the theorem.

Let $C(B)$ denote the subclass of S of function f such that $f(U)$ is convex and $|f(z)|\leqslant B$, $z\in U$. The author can show by long, but straightforward, arguments that there exist finite B's for which there are functions in $C(B)$ that generate quasi-starlike functions that are not starlike. Thus there exists a finite B_0 that is the supremum of all B's such that if $f\in C(B)$, then f generates a quasi-starlike function that is starlike for all $M>1$. The following corollary gives a lower bound for B_0.

COROLLARY. *If $f\in C(B)$ with $B\leqslant\sqrt{32/27}$, then f generates a quasi-starlike function that is starlike for all $M>1$.*

Proof. In [2] Brannan and Kirwan proved that if $f\in C(B)$, then $f\in S^*(\alpha)$ with

$$\alpha = 1-\frac{2}{\pi}\arcsin[\delta(B)/B], \tag{8}$$

where $\delta(B)$ denotes the Koebe constant for $C(B)$ (i.e., the radius of the largest open disk centered at the origin and contained in the image of U under every function in $C(B)$ for a fixed B). The value of $\delta(B)$ has been determined by Krzyż in [7] to satisfy

$$\delta(B) = B\sin\theta, \tag{9}$$

where θ is the unique solution of the equation,

$$(\pi+2\theta)\sin\frac{4\pi\theta}{\pi+2\theta} = 2\pi B^{-1}\cos\theta. \tag{10}$$

The result follows by letting $\alpha = 1/2$ in (8) and then solving for B in (9) and (10).

References

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