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On full derivatives

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In papers [2]–[4] J. Mikusiński introduced the notion of a full derivative and listed certain properties, theorems and comments concerning full derivatives. Functions considered in those papers are defined on subsets of the q-dimensional Euclidean space R^q; their values are real numbers or elements of a Banach space. In [2] and [4] J. Mikusiński stated the following

Theorem. *If the full derivatives $f_i = f^{(e_i)}$ and $f_j = f^{(e_j)}$ exist in a neighbourhood of a point x_0 and if, moreover, the full derivatives $f_i^{(e_j)}(x_0)$ and $f_j^{(e_i)}(x_0)$ exist at x_0, then they are equal.*

This theorem is proved by applying the mean value theorem in a real domain. But this method fails, if the values of functions are in an arbitrary Banach space. We therefore must look for another proof. It can be based on a certain theorem about mappings in Banach spaces.

Let f be a function defined in the q-dimensional Euclidean space and taking values in a fixed Banach space. Let e_i denote the point whose i-th coordinate is 1 and the remaining ones are 0. If a point x has coordinates $\xi_1, \ldots, \xi_q$, then the point $x+e_i\lambda$ differs from x only by the i-th coordinate which is equal to $\xi_i+\lambda$.

We say (see [2]) that u is the *i-th full derivative* of f at x_0, if for each $\varepsilon > 0$ there is a neighbourhood V of x_0 such that

$$\frac{\|f(x)-f(y)-u(x_0)\,e_i(x-y)\|}{|x-y|} < \varepsilon$$

for any points $x, y \in V$ differring from each other by their i-th coordinates, and only by those coordinates. (The inner product $e_i(x-y)$ is equal to the difference of the i-th coordinate of x and y.) In other words, the points x and y belong to the intersection of V with the straight line (not necessarily passing through x_0) parallel to the i-th coordinate axis.

In the proof of the Theorem we shall use the following

Lemma ([1], Theorem 8.6.2). *Let E and F be two Banach spaces, let*

a, $b \in E$ and let f be a differentiable mapping of an open neighbourhood A of the segment S joining a and b into F. Then, for each $x_0 \in A$, we have

$$\|f(b)-f(a)-f'(x_0)(b-a)\| \leqslant \|b-a\| \sup_{x\in S} \|f'(x)-f'(x_0)\|. \tag{1}$$

Proof of Theorem. Since f_i exists in a neighbourhood of x_0, there exists, for any $\varepsilon > 0$, a ball K_r with centre at x_0 and radius $r > 0$, such that

$$\|f(x)-f(y)-f_i(x_0)e_i(x-y)\| < \tfrac{1}{2}\varepsilon|x-y|$$

holds for any two points x, $y \in K_r$ which differ from each other by their i-th coordinates only. Consider the function

$$g(\xi) = f\big(x_0+(\xi e_i+e_j)\lambda\big)-f(x_0+\xi e_i\lambda),$$

where $\xi \in [0, 1]$ and $|\lambda| \leqslant \frac{1}{2}r$, $\lambda \neq 0$. We have

$$g'(\xi) = \big[f_i\big(x_0+(\xi e_i+e_j)\lambda\big)-f_i(x_0+\xi e_i\lambda)\big]\lambda. \tag{2}$$

Since $f_i^{(e_j)}$ exists in a neighbourhood of x_0, there exists, for any $\varepsilon > 0$, a radius $r' \leqslant r$ such that for $|\lambda| \leqslant \frac{1}{2}r'$ the following inequality holds:

$$\big\|f_i\big(x_0+(\xi e_i+e_j)\lambda\big)-f_i(x_0+\xi e_i\lambda)-f_i^{(e_j)}(x_0)\lambda\big\| \leqslant \tfrac{1}{2}\varepsilon|\lambda|,$$

because the points $x_0+(\xi e_i+e_j)\lambda$, $x_0+\xi e_i\lambda$ differ only by their j-th coordinates and belong to the ball $K_{r'}$ with centre at x_0 and radius r'. Hence by (2) we obtain

$$\|g'(\xi)-f_i^{(e_j)}(x_0)\lambda^2\| \leqslant \tfrac{1}{2}\varepsilon|\lambda|^2 \quad \text{for } \xi \in [0, 1],\ |\lambda| \leqslant \tfrac{1}{2}r'. \tag{3}$$

Applying the triangle inequality for norms, Lemma and inequality (3) we have

$$\begin{aligned}\|g(1)-g(0)-f_i^{(e_j)}(x_0)\lambda^2\| &\leqslant \|g(1)-g(0)-g'(\xi_0)\|+\|g'(\xi_0)-\lambda^2 f_i^{(e_j)}(x_0)\|\\ &\leqslant \sup_{\xi\in[0,1]} \|g'(\xi)-g'(\xi_0)\|+\tfrac{1}{2}\varepsilon\lambda^2 \leqslant \tfrac{3}{2}\varepsilon\lambda^2.\end{aligned}$$

Since the difference

$$g(1)-g(0) = f\big(x_0+(e_i+e_j)\lambda\big)-f(x_0+e_i\lambda)-f(x_0+e_j\lambda)+f(x_0)$$

is symmetric with respect to e_i and e_j, we have

$$\|g(1)-g(0)-f_j^{(e_i)}(x_0)\lambda^2\| \leqslant \tfrac{3}{2}\varepsilon\lambda^2.$$

The last two inequalities and the triangle inequality imply

$$\|f_i^{(e_j)}(x_0)\lambda^2-f_j^{(e_i)}(x_0)\lambda^2\| \leqslant 3\lambda^2\varepsilon \quad \text{for } |\lambda| \leqslant \tfrac{1}{2}r'.$$

Hence

$$\|f_i^{(e_j)}(x_0) - f_j^{(e_i)}(x_0)\| \leqslant 3\varepsilon.$$

Since ε can be taken arbitrarily small, it follows that

$$f_i^{(e_j)}(x_0) = f_j^{(e_i)}(x_0).$$

References

[1] J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York and London 1969.

[2] J. Mikusiński, *O pochodnych cząstkowych i pełnych* (*On partial and full derivatives*, in Polish), Wiadom. Mat. 16 (1973), p. 23–35.

[3] —, *On partial derivatives*, Bull. Acad. Polon. Sci., Sér. sci. math., astr., phys. 20 (1972), p. 941–944.

[4] —, *The Bochner integral*, Birkhäuser Verlag Basel und Stuttgart 1978.

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