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COMPLETENESS OF L^p-SPACES OVER FINITELY ADDITIVE SET FUNCTIONS

BY

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Suppose S is a set, Σ is a field of subsets of S, and μ is a non-negative bounded finitely additive set function on Σ. For $1 \leqslant p < \infty$, the real normed linear space $L^p(S, \Sigma, \mu)$ is not, in general, a complete space. In this paper*, necessary and sufficient conditions for $L^p(S, \Sigma\ \mu)$ to be complete are obtained and then applied to certain Banach limits.

1. Necessary and sufficient conditions. We begin this section by demonstrating the existence of a measure space (S', Σ', μ') and a linear isometry i from $L^p(S, \Sigma, \mu)$ into $L^p(S', \Sigma', \mu')$ such that the image of i is dense in $L^p(S', \Sigma', \mu')$. Then, since $L^p(S', \Sigma', \mu')$ is complete, $L^p(S, \Sigma, \mu)$ is complete if and only if i is an onto function. We conclude Section 1 by showing conditions under which i is onto.

Let I be the ideal of all μ-null sets in Σ, and let Ω be the quotient Boolean algebra Σ/I. By the Stone Representation Theorem, there is a compact Hausdorff space S' such that the field Ω' of clopen (simultaneously open and closed) subsets of S' is isomorphic as a Boolean algebra to Ω. Let E' denote that element of Ω' which corresponds to the equivalence class $[E] \in \Omega$ of $E \in \Sigma$. If we define μ_0' on Ω' by $\mu_0'(E') = \mu(E)$, then μ_0' is a non-negative bounded regular finitely additive set function on Ω'. In addition, μ_0' is countably additive since, given any denumerable collection $\{E_n'\}$ of pairwise disjoint non-empty sets in Ω', the union $\bigcup_{n=1}^{\infty} E_n'$ is *never* in Ω'. Therefore μ_0 can be uniquely extended to a regular Borel measure μ' on S'. Let Σ' denote the σ-field of all Borel subsets of S'. Then (S', Σ', μ') is a measure space and, moreover, if G is a non-void open subset of S', then $\mu'(G) > 0$.

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Let χ_E denote the characteristic function of a set E. If

$$f = \sum_{k=1}^{n} a_k \chi_{E_k}$$

is a μ-simple function on S and $g = f$ in $L^p(S, \Sigma, \mu)$, we define a function i_0 by the formula

$$i_0(g) = \sum_{k=1}^{n} a_k \chi_{E_k'}.$$

It is clear that i_0 is a linear isometry from the simple functions in $L^p(S, \Sigma, \mu)$ into $L^p(S', \Sigma', \mu')$. Since the simple functions are dense in $L^p(S, \Sigma, \mu)$, i_0 can be extended to a linear isometry i from $L^p(S, \Sigma, \mu)$ into $L^p(S', \Sigma', \mu')$. Moreover, since μ' is regular, the image of i is dense in $L^p(S', \Sigma', \mu')$.

Two elements of Σ' are said to be *equivalent* if their symmetric difference has μ'-measure zero.

A topological space is *extremally disconnected* if and only if the closure of every open set is clopen.

THEOREM. *The linear isometry i as defined above is onto if and only if S' is extremally disconnected and every open subset of S' is equivalent to its closure.*

Proof. Suppose i is onto. Then every element of Σ' is equivalent to a clopen set, and thus (recalling that the μ'-measure of every non-empty open set is positive) we conclude that S' is extremally disconnected. (See [6], p. 277.) Moreover, if G is open in S' and is equivalent to H' in Ω', then $G\setminus H'$ is an open set of μ'-measure zero, hence empty. (Let $\bar{B}$ denote the closure, and B^0 denote the interior, of a set B.) Thus $H' \supset \bar{G} \supset G$, which implies $\mu'(\bar{G}\setminus G) = 0$, which in turn implies that G is equivalent to its closure.

Conversely, suppose S' is extremally disconnected and every open subset of S' is equivalent to its closure. By taking complements, it follows that every closed set is equivalent to its interior. We show first that every element of Σ' is equivalent to an element of Ω'.

Let $A \in \Sigma'$ such that $\mu'(A) > 0$. By the regularity of μ', there is a family $\{K_n\}_{n \geqslant M}$ of compact subsets of A such that $\mu'(A) - \mu'(K_n) < 1/n$, where M is the first integer such that $1/M < \mu'(A)$. Since S' is extremally disconnected, $(K_n)^0$ is clopen. By hypothesis, $\mu'(K_n^0) = \mu'(K_n)$, and hence $\mu'(A) - \mu'(K_n^0) < 1/n$. Let

$$K = \bigcup_{n=M}^{\infty} (K_n)^0.$$

Then K is an open subset of A equivalent to A. Hence $\bar{K}$ is an element of Ω' equivalent to A.

Applying Theorem 6 of [6], we conclude that every bounded measurable function on S' is equal a.e. $[\mu']$ to a continuous function. Since every element of Σ' is equivalent to an element of Ω', i takes the set of simple functions in $L^p(S, \Sigma, \mu)$ onto the set of simple functions in $L^p(S', \Sigma', \mu')$.

Let f be a continuous function on S'. Choose a sequence $\{f_n\}$ of continuous simple functions converging uniformly to f. There exists a sequence $\{g_n\}$ of μ-simple functions on S such that $i(g_n) = f_n$ and $\{g_n\}$ is uniformly Cauchy. If $g(x) = \lim_{n\to\infty} g_n(x)$, then $g \in L^p(S, \Sigma, \mu)$ and $i(g) = f$. Hence i is onto the bounded elements of $L^p(S', \Sigma', \mu')$.

If $f \in L^p(S', \Sigma', \mu')$ is non-negative and unbounded, let $f_n(x) = f(x)$ if $f(x) \leqslant n$ and $f_n(x) = 0$ if $f(x) > n$. Choose $E'_n \in \Omega'$ equivalent to $\{x : f(x) > n\}$. By induction, choose a decreasing sequence $\{E_n\}$ of elements of Σ such that $[E_n] \in \Omega$ corresponds to E'_n. Then

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu'(E'_n) = 0.$$

By the preceding section there is a function g_1 on S such that $g_1(x) = 0$ for all $x \in E_1$ and $i(g_1) = f_1$ a.e. $[\mu']$. By induction, there exist functions g_n on S such that $g_n(x) = g_{n-1}(x)$ for all $x \in E_n \cup (S \setminus E_{n-1})$ and such that $i(g_n) = f_n$ a.e. $[\mu']$, $n \geqslant 1$. Let

$$g(x) = \lim_{n\to\infty} g_n(x).$$

Then it is clear that g_n converges to g in μ-measure, and since $\{g_n\}$ is Cauchy in $L^p(S, \Sigma, \mu)$, $g \in L^p(S, \Sigma, \mu)$ and $i(g) = f$ a.e. $[\mu']$.

Since any function in $L^p(S', \Sigma', \mu')$ can be expressed as the difference of two non-negative functions, we conclude that i is onto $L^p(S', \Sigma', \mu')$, and the theorem is proved.

2. Application. We call μ *non-atomic* if for every set $E \in \Sigma$ such that $\mu(E) > 0$, and for every real number a such that $0 < a < \mu(E)$, there is a set $F \subset E$ such that $F \in \Sigma$ and $\mu(F) = a$.

COROLLARY. *If μ is non-atomic, $\mu(S) = 1$, and S' is separable, then $L^p(S, \Sigma, \mu)$ is not complete.*

Proof. We demonstrate the existence of an open subset of S' which is not equivalent to its closure. Let $\{x_n\}_{n=1}^{\infty}$ be dense in S'. Choose $A_1 \epsilon$st such that $\mu(A_1) < 1/2$ and $x_1 \in A'_1$. Let n_1 be 1, and let n_2 be the leaΣ integer such that $x_{n_2} \notin A'_1$. Choose $A_2 \in \Sigma$ such that $A_1 \cap A_2 = \emptyset$, $\mu(A_2) < 1/4$, and $x_{n_2} \in A'_2$. Now let n_3 be the least integer such that $x_{n_3} \notin A'_1 \cup A'_2$, and choose $A_3 \in \Sigma$ such that $A_3 \cap (A_1 \cup A_2) = \emptyset$, $\mu(A_3) < 1/2^3$, and $x_{n_3} \in A'_3$. Continue this process to get a sequence $\{A'_n\}$ of pairwise disjoint clopen subsets of S' such that

$$\{x_n\}_{n=1}^{\infty} \subset \bigcup_{n=1}^{\infty} A'_n.$$

Since $\{x_n\}_{n=1}^{\infty}$ is dense, $\overline{\bigcup_{n=1}^{\infty} A'_n} = S'$. Thus $\mu'\left(\overline{\bigcup A'_n}\right) = 1$, while

$$\mu'\left(\bigcup A'_n\right) = \sum_{n=1}^{\infty} \mu'(A'_n) = \sum_{n=1}^{\infty} \mu(A_n) < 1.$$

Therefore, by the previous theorem, $L^p(S, \Sigma, \mu)$ is not complete.

Example. Let S be the set of positive integers, and let Σ be the field of all subsets of S. Let $t: S \to S$ be defined by $t(n) = n+1$. If $f = (f_1, f_2, \ldots)$ is in l^∞, then $f \circ t = (f_2, f_3, \ldots)$ is in l^∞. Let $B = \{\varphi \in (l^\infty)^* \mid \varphi \geqslant 0, \|\varphi\| = 1$, and $\varphi(f \circ t) = \varphi(f)$ for all $f \in l^\infty\}$, where $(l^\infty)^*$ is the dual space of l^∞. The elements of B are known as *Banach limits*. Since $(l^\infty)^*$ is isometrically isomorphic to $ba(S, \Sigma)$ (the Banach space of all bounded finitely additive set functions on Σ with the total variation norm (see [1], p. 258)), if $\varphi \in B$, then there is an element $\lambda \in ba(S, \Sigma)$ corresponding to φ. We will also call λ a Banach limit. The properties of φ, when transferred to λ, say that λ is a probability element of $ba(S, \Sigma)$ such that $\lambda(A) = \lambda(tA)$ for all $A \in \Sigma$. This last property enables one to show that λ is non-atomic. Let the Stone space S' of Section 1 be denoted by K_λ (to indicate that S' varies as λ varies). K_λ is extremally disconnected since it is the Stone space of a complete Boolean algebra. (For a proof, see [2], p. 1241.) We could apply the Corollary if we knew that K_λ were separable. The map $t: S \to S$ can be extended to a map $T: \beta S \to \beta S$ such that the restriction of T to $\beta S - S$ is a homeomorphism onto $\beta S - S$. (βS is the Stone-Čech compactification of S.) A set $A \subset \beta S$ is called *T-invariant* if $T(A) = A$. We observe ([5], pp. 31f) that K_λ is homeomorphic to a closed subspace of βS. In [4], p. 3, Raimi has shown that the "support set" K_λ is homeomorphic to a (1) non-empty, (2) closed, (3) T-invariant subset of $\beta S - S$. If K_λ is *minimal* with respect to properties (1), (2), and (3), then it is separable. In fact, $\{T^m(x)\}_{n=-\infty}^{\infty}$ is dense in K_λ for any $x \in K_\lambda$. Thus, by the Corollary, $L^p(S, \Sigma, \lambda)$ is not complete if K_λ is minimal. This is sometimes the case. The set B of Banach limits is non-empty, convex, and compact in the weak-star topology on $(l^\infty)^*$. Hence by the Krein-Milman Theorem, the set B is the closed convex hull of its extreme points. Raimi [3] has shown that every minimal non-empty closed T-invariant subset of $\beta S - S$ is the support set of (at least) two extreme Banach limits. Thus there are Banach limits λ such that K_λ is separable. We have not been able to decide if the set K_λ is separable for every Banach limit, nor even for every *extreme* Banach limit (**P 709**).

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