ANNALES
POLONICI MATHEMATICI
XXVII (1973)

A note on a pentomino functional equation

by SHIGERU HARUKI (Waterloo, Canada)

The following square functional equation for $f\colon R^2 \to R$,

$$(1)\quad f(x+\nu, y+\nu)+f(x+\nu, y-\nu)+f(x-\nu, y+\nu)+f(x-\nu, y-\nu) = 4f(x, y)$$

was considered in [7], and generalizations and applications were considered in [1], [2], [8], [12], [13], [15], [16], [17].

It was shown in [1] that the square functional equation (1), or alternatively, the equation

$$(2)\qquad (X^\nu Y^\nu + X^\nu Y^{-\nu} + X^{-\nu} Y^\nu + X^{-\nu} Y^{-\nu})f(x, y) = 4f(x, y)$$

with $X^\nu f(x, y) \overset{\mathrm{df}}{=} f(x+\nu, y)$ and $Y^\nu f(x, y) \overset{\mathrm{df}}{=} f(x, y+\nu)$, has the harmonic polynomials

$$(3)\qquad f(x, y) = \mathrm{Re}(ia_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0)$$

as the only measurably bounded solutions (bounded on a set of positive measure), where ν is real, a_4 is real, and a_j, $j = 0, 1, 2, 3$ are complex constants.

Further, (1) and

$$(4)\qquad (X^\nu + X^{-\nu} + Y^\nu + Y^{-\nu})f(x, y) = 4f(x, y)$$

are equivalent without any regularity assumptions.

We shall consider the following functional equation:

$$(5)\quad (X^{-\nu} Y^{-\nu} + X^{-\nu} Y^{-p\nu} + X^\nu Y^{-p\nu} + X^\nu Y^{-\nu} + X^{p\nu} Y^{-\nu} + X^{p\nu} Y^\nu + X^\nu Y^\nu + \\ + X^\nu Y^{p\nu} + X^{-\nu} Y^{p\nu} + X^{-\nu} Y^\nu + X^{-p\nu} Y^\nu + X^{-p\nu} Y^{-\nu})f(x, y) = 12f(x, y),$$

for some arbitrary real p. Equation (5) is a special case of the mean-value equation for $f\colon R^n \to R$,

$$(6)\qquad \sum_{i=1}^{N} \mu_i f(x + a_i t) = f(x),$$

where $\sum_{i=1}^{N} \mu_i = 1$, $\sum_{i \in I} \mu_i \neq 0$ for any subset $I \subset \{1, 2, \ldots, N\}$, $x \in R^n$, $a_i \in R^n$, a_i, $i = 1, 2, \ldots, N$, span the space R^n. Equations of this form were considered in [3], [4], [5], [6], [14] (among others).

THEOREM. *The only measurably bounded solutions of equation* (5) *for arbitrary fixed* p ($|p| \neq \sqrt{3+\sqrt{10}}$) *are the harmonic polynomials of the form* (3).

Proof. In [14] the following theorem is proved: *if* $f\colon V^n \to R$ *is a solution of* (6) *for all* $x \in V^n$, $t \geqslant 0$, *such that* $x, x+a_1t, \ldots, x+a_nt$ *lie in a domain* $D \subseteq V^n$ *and if*

(a) *the* a_i*'s span* V^n (*hence again* $N \geqslant n$),

(b) $\sum_{i\in J} \mu_i \neq 0$ *for* $J \subset I$ (*assuming* $\sum_{i\in I} \mu_i = 1$),

(c) f *is bounded on some set of positive measure in* D,

then f *is* C^∞ *on* D, *and hence a polynomial of degree at most* $N(N-1)/2$.

We may therefore assume that $f(x, y)$ is of class C^∞. Equation (5) in the plane R^2 yields, by repeated differentiation of both sides with respect to ν for $\nu = 0$,

$$f_{xx}+f_{yy} = 0, \tag{7}$$

$$(2+p^4)f_{xxxx}+(2+p^4)f_{yyyy}+(6+12p^2)f_{xxyy} = 0. \tag{8}$$

From equation (7) follows

$$f_{xxxx}+f_{xxyy} = 0, \quad f_{xxyy}+f_{yyyy} = 0, \tag{9}$$

and substituting (9) into (8) we get $(p^4-6p^2-1)f_{xxyy} = 0$. Since $|p| \neq \sqrt{3+\sqrt{10}}$, we obtain

$$f_{xxyy} = 0, \tag{10}$$

which together with (9) implies

$$f_{xxxx} = 0, \quad f_{yyyy} = 0. \tag{11}$$

Equations (7), (10), (11) yields the form (3). Conversely, by substituting (3) into (5), one verifies that (3) satisfies equation (5). Q.E.D.

COROLLARY. *If* (5) *is satisfied for fixed* p ($|p| \neq \sqrt{3+\sqrt{10}}$), *by a measurably bounded function* $f(x, y)$, *then this function satisfies* (5) *for all* p ($|p| \neq \sqrt{3+\sqrt{10}}$).

Remark. By the corollary, equations (5), for various values of p ($|p| \neq \sqrt{3+\sqrt{10}}$), are for measurably bounded solutions equivalent. This does not seem to be true for the general solutions of (5).

The case $p = 1$ yields the square functional equation (1). In the case $p = 3$, on account of the geometric sense (similar type equations can be found in [9], [10], [11]) of (1), we call equation (5) a *pentomino functional equation.* Without any regularity assumptions, the square functional equation implies the pentomino functional equation; this may readily be verified in view of (1) and (4).

The author would like to sincerely thank Professor M. A. McKiernan and the referee for their helpful suggestions and understanding guidance.

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STUDENT
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA

Reçu par la Rédaction le 21. 5. 1971