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Fredholm eigenvalues and Grunsky matrices

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Dedicated to the memory of my friend Stefan Bergman

Abstract. The close relation between the Hilbert transform and the Fredholm integral equation is shown and is used to derive the basic properties of the Fredholm eigenfunctions and their eigenvalues. Applications are given to the theory of conformal mapping and the Grunsky inequalities. The sharp bounds for the inequalities are shown to be expressible in terms of the lowest Fredholm eigenvalue of the image domain. An analogous theory is developed for the case of univalent functions which map onto mutually disjoint domains.

Introduction. In this paper we shall show the close connection between coefficient inequalities of the Grunsky type and the theory of the Hilbert transform and the Fredholm eigenvalues of a domain. We arrive at some illuminating insights into the character of such inequalities. Since the change of the Fredholm eigenvalues under quasi-conformal mapping has been well explored [14], [17], our results may be useful in the theory of univalent functions with k-quasi-conformal extensions. In order to make the paper self-contained, I retrace briefly the theory of the Fredholm eigenvalues and of the Hilbert transform, which are developed from a new point of view. I started this theory almost thirty years ago with my friend Stefan Bergman [3], and I dedicate this paper to his memory.

1. Single and double-layer potentials in the plane. Let Γ be a closed curve in the complex z-plane which encloses a bounded domain Δ and let $\tilde{\Delta}$ be its unbounded complement. To avoid unnecessary arguments, we assume that Γ is three times differentiable with respect to its arc length s. We define two continuous functions of $\zeta(s) \in \Gamma$, say $\mu(\zeta)$ and $\nu(\zeta)$ and introduce the integrals

$$(1) \qquad S(z) = \int_{\Gamma} \mu(\zeta) \log \frac{1}{|\zeta - z|} ds, \qquad D(z) = \int_{\Gamma} \nu(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta - z|} ds,$$

with $\int_{\mu} \mu ds = 0$ and where $\vec{n}_{\zeta}$ denotes the normal to Γ at ζ pointing into Δ.

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One calls $S(z)$ a single-layer potential with density μ on Γ and $D(z)$ a double-layer potential with density ν on Γ. Both integrals represent harmonic functions in Δ and $\tilde{\Delta}$. It is well known that $S(z)$ is continuous in the entire plane, while $D(z)$ jumps when z crosses Γ. However, $\partial D/\partial n$ has the same value when evaluated in Δ or in $\tilde{\Delta}$.

We define the scalar product typical for potential theory

$$(2) \qquad (\varphi, \psi) = \int_E \nabla\varphi \cdot \nabla\psi \, dxdy, \qquad E = \Delta + \tilde{\Delta}$$

for any two functions $\varphi(z)$, $\psi(z)$ defined in $\Delta+\tilde{\Delta}$. Now, we use the above continuity behavior of $S(z)$ and $D(z)$ to find

$$(3) \qquad (S, D) = \int_\Delta \nabla S \cdot \nabla D \, dxdy + \int_{\tilde{\Delta}} \nabla S \cdot \nabla D \, dxdy = -\int_\Gamma S \frac{\partial D}{\partial n} ds + \int_\Gamma S \frac{\partial D}{\partial n} ds = 0.$$

Thus, every single-layer potential is orthogonal in the Dirichlet metric to every double-layer potential.

Let next $h(z)$ be harmonic in Δ and continuously differentiable in the closure $\Delta+\Gamma$. We have the fundamental identity

$$(4) \qquad \frac{1}{2\pi}\int_\Gamma h(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|\zeta-z|}\,ds - \frac{1}{2\pi}\int_\Gamma \frac{\partial h}{\partial n_\zeta}\log\frac{1}{|\zeta-z|}\,ds = h(z)\chi_\Delta(z),$$

where $\chi_\Delta(z)$ is the characteristic function of Δ. Thus, every such harmonic function can be expressed as the sum of a single-layer and a double-layer potential since $\int_\Gamma \frac{\partial h}{\partial n} ds = 0$.

If $F(z)$ is any real-valued function which is continuously differentiable in $\Delta+\Gamma$, we can find a harmonic function $h(z)$ which is continuously differentiable in $\Delta+\Gamma$ and has the same boundary values as $F(z)$. We can write

$$(5) \qquad F(z) = N(z)+h(z) = N(z)+S(z)+D(z),$$

where $N(z)$ is continuously differentiable in $\Delta+\Gamma$ and vanishes on Γ. For each such null-function $N(z)$, we have the identity, valid for harmonic h,

$$(6) \qquad \iint_\Delta \nabla N \cdot \nabla h \, dxdy = -\int_\Gamma N \frac{\partial h}{\partial n} ds = 0.$$

If we consider the class of single-layer potentials, double-layer potentials and functions $N(z)$ which vanish in $\tilde{\Delta}+\Gamma$, we have three linear function spaces which are orthogonal in the Dirichlet sense and every continuously differentiable function $F(z)$ can be decomposed in a unique way into three elements of these spaces.

To use complex notation define

$$(7)\qquad s(z) = \frac{\partial S}{\partial z}, \quad d(z) = \frac{\partial D}{\partial z}, \quad n(z) = \frac{\partial N}{\partial z}.$$

Then every complex-valued function $f(z)$ which is square-integrable over Δ can be split into

$$(8)\qquad f(z) = s(z)+d(z)+n(z)$$

and the three components are orthogonal in the sense of the hermitian metric

$$(9)\qquad \int \varphi\bar{\psi}\,dxdy = |\varphi, \psi|.$$

Since every $F(z)$ which is continuously differentiable in the entire plane can be written in the form

$$(10)\qquad F(z) = F(z)\chi_\Delta(z)+F(z)\chi_{\tilde{\Delta}}(z),$$

we find that the decomposition (8) is also valid and unique for every $f(z)$ which is square-integrable over the entire complex plane.

2. The Hilbert transformation. For every $f(z)$ of the above type we introduce the integral transformation

$$(11)\qquad \mathrm{T}f = \frac{1}{\pi}\int_E \frac{\overline{f(\zeta)}}{(\zeta-z)^2}\,d\xi d\eta$$

understood in the Cauchy improper integral sense. It is called the *Hilbert transform of f* [3], [13].

To understand the significance of this transformation we will discuss its effect on a function $n(z)$, $s(z)$ and $d(z)$ separately.

We start with a function $n(\zeta) = \partial N/\partial\zeta$, where $N(z)$ vanishes on Γ. Let Δ_r be the domain Δ minus a disk of radius r around z. Then

$$(12)\qquad \mathrm{T}n(z) = \lim_{r\to 0}\frac{1}{\pi}\int_{\Delta_r}\frac{\partial N}{\partial\bar{\zeta}}\frac{1}{(\zeta-z)^2}\,d\xi d\eta.$$

Integration by parts leads to

$$(13)\qquad \frac{1}{\pi}\int_{\Delta_r}\frac{\partial N}{\partial\bar{\zeta}}\frac{1}{(\zeta-z)^2}\,d\xi d\eta = \frac{1}{2\pi i}\int_{|\zeta-z|=r} N(\zeta)\frac{d\zeta}{(\zeta-z)^2}.$$

Observe that

$$(14)\qquad \frac{1}{2\pi i}\int_{|\zeta-z|=r} N(\zeta)\frac{d\zeta}{(\zeta-z)^2} = \frac{\partial N}{\partial z}+O(r) = n(z)+O(r).$$

Hence, combining (12), (13) and (14) we find

$$(15)\qquad \mathrm{T}n = n.$$

The Hilbert transformation on all n-functions is the identity.

Next, let us start with Green's identity for all single-layer potentials

$$\frac{1}{2\pi}\int_E \nabla_\zeta \log\frac{1}{|\zeta-z|}\cdot\nabla_\zeta S(\zeta)\,d\xi d\eta = S(z). \tag{16}$$

We use complex notation and write

$$\nabla A\cdot\nabla B = 4\,\mathrm{Re}\left\{\frac{\partial A}{\partial\zeta}\frac{\partial B}{\partial\bar\zeta}\right\} \tag{17}$$

and hence (16) becomes

$$-\frac{1}{2\pi}\int_E \frac{1}{\zeta-z}\left(\overline{\frac{\partial S}{\partial\zeta}}\right)d\xi d\eta-\frac{1}{2\pi}\int_E \frac{1}{(\overline{\zeta-z})}\frac{\partial S}{\partial\zeta}\,d\xi d\eta = S(z). \tag{18}$$

We differentiate in z and use the Laplace–Poisson identity to find

$$-\frac{1}{2\pi}\int_E \frac{1}{(\zeta-z)^2}\left(\overline{\frac{\partial S}{\partial\zeta}}\right)d\xi d\eta+\frac{1}{2}\frac{\partial S}{\partial z} = \frac{\partial S}{\partial z}. \tag{19}$$

Hence we proved for every $s(z) = \partial S/\partial s$ the transformation law

$$\mathrm{T}s = -s. \tag{20}$$

Finally, if $D(z)$ is a double-layer potential, we have

$$\frac{1}{2\pi}\int_E \nabla_\zeta \log\frac{1}{|\zeta-z|}\,\nabla_\zeta D(\zeta)\,d\xi d\eta = 0, \tag{21}$$

as follows easily from the continuity of $\partial D/\partial n$ on Γ. Hence the above calculation yields

$$-\frac{1}{2\pi}\int_E \frac{1}{(\zeta-z)^2}\left(\overline{\frac{\partial D}{\partial\zeta}}\right)d\xi d\eta+\frac{1}{2}\frac{\partial D}{\partial z} = 0. \tag{22}$$

For all $d(z) = \partial D/\partial z$ follows the transformation law

$$\mathrm{T}d = d. \tag{23}$$

Formulas (15), (20) and (23) give a clear understanding for the significance of the Hilbert transform. The general $f(z)$ can be decomposed uniquely into

$$f = n+d+s \tag{24}$$

and its norm is

$$\|f\|^2 = \|n\|^2+\|d\|^2+\|s\|^2. \tag{25}$$

Its Hilbert transform is

(26) $$\mathrm{T}f = n+d-s$$

with the norm

(27) $$\|\mathrm{T}f\| = \|n\|^2+\|d\|^2+\|s\|^2 = \|f\|^2.$$

Also, we have obviously

(28) $$\mathrm{TT}f = f.$$

This shows that the Hilbert transform is a norm-preserving involution.

This interpretation of the Hilbert transform in the plane suggests also easy generalizations in potential theory for higher dimensions.

3. Fredholm eigenfunctions. Our preceding analysis makes it quite clear that the Hilbert transformation is only of interest within the class of analytic functions since for their othogonal complement of n-functions it is nothing but the identity operation. Now, the remarkable fact appears that in the case of analytic functions the improper kernel of the transformation can be replaced by an analytic regular kernel.

Indeed, let $g(z, \zeta)$ be the Green's function of Δ. We form the two complex-valued kernels [2], [3]

(29) $$K(z, \bar{\zeta}) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z \partial \bar{\zeta}}, \quad L(z, \zeta) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}.$$

The first kernel is the well-known Bergman kernel. It is regular analytic for all z in Δ and anti-analytic for all ζ in Δ. It has hermitian symmetry $\overline{K(z, \bar{\zeta})} = K(\zeta, \bar{z})$ and for all $\varphi(z)$ analytic in Δ it has the reproducing property

(30) $$\int_{\Delta} K(z, \bar{\zeta})\varphi(\zeta)\, d\xi d\eta = \varphi(z).$$

The kernel $L(z, \zeta)$ is symmetric in both arguments, but it has a double pole for $z = \zeta$. It can be written in the form

(31) $$L(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} - l(z, \zeta),$$

where $l(z, \zeta)$ is regular analytic in both arguments. For any analytic function $\varphi(z)$ in Δ, we have the identity

(32) $$\int_{\Delta} L(z, \zeta)\overline{\varphi(\zeta)}\, d\xi d\eta \equiv 0.$$

This implies the identity

(33) $$\frac{1}{\pi}\int_{\Delta}\frac{\overline{\varphi(\zeta)}}{(\zeta-z)^2}\, d\xi d\eta = \int_{\Delta} l(z, \zeta)\overline{\varphi(\zeta)}\, d\xi d\eta.$$

This shows that we may replace the singular kernel in the Hilbert transformation of analytic functions in Δ and $\tilde{\Delta}$ by the corresponding kernels $l(z,\zeta)$ and $\tilde{l}(z,\zeta)$.

Now we find at our disposal the classical methods of integral equation theory. We may ask for analytic functions $w(z)$ in Δ which satisfy the integral equation

$$w(z) = \frac{\lambda}{\pi}\int_{\Delta}\frac{\overline{w(\zeta)}}{(\zeta-z)^2}\,d\xi d\eta, \qquad z\in\Delta. \tag{34}$$

Since $w(\zeta)e^{i\alpha}$ satisfies the same integral equation with $\lambda^* = \lambda e^{-2i\alpha}$, we may be more specific and demand λ to be positive. We may also normalize the eigenfunction by the demand

$$\int_{\Delta}|w|^2\,d\xi d\eta = 1. \tag{35}$$

The significance of the eigenfunctions $w_\nu(z)$ and the eigenvalues λ_ν for potential theory is obvious. Observe that the eigenfunctions satisfy also the integral equation with analytic kernel

$$w_\nu(z) = \lambda_\nu\int_{\Delta} l(z,\zeta)\overline{w_\nu(\zeta)}\,d\xi d\eta. \tag{36}$$

Eigenfunctions to different eigenvalues are obviously orthogonal to each other, and we may assume them to form an orthonormal set. It can be shown that the system is complete and we have the spectral decomposition of the two derivatives of the Green's function

$$K(z,\bar{\zeta}) = \sum_{\nu=1}^{\infty} w_\nu(z)\overline{w_\nu(\zeta)}, \qquad l(z,\zeta) = \sum_{\nu=1}^{\infty}\frac{w_\nu(z)w_\nu(\zeta)}{\lambda_\nu}. \tag{37}$$

The eigenvalues λ_ν occurred early in potential theory. As is well known, Poincaré attacked the first boundary value problem of two-dimensional potential theory by setting up the sought harmonic function as a double-layer potential

$$h(z) = \frac{1}{\pi}\int_{\Gamma} v(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|\zeta-z|}\,ds_\zeta, \qquad z\in\Delta \tag{38}$$

and was led to the boundary condition

$$h(z) = v(z)+\frac{1}{\pi}\int_{\Gamma} v(\zeta)\frac{\partial}{\partial n_\zeta}\log\frac{1}{|\zeta-z|}\,ds_\zeta, \qquad z\in\Gamma. \tag{39}$$

This is an integral equation for the sought density $v(z)$ in terms of the known boundary values $h(z)$ on Γ.

This problem gave rise to the modern theory of integral equations and the fundamental Fredholm alternative. According to it, one has to consider the eigenvalue problem

$$\varphi(z) = \frac{\lambda}{\pi} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta, \quad z \in \Gamma \tag{40}$$

and to show that $\lambda = -1$ is not an eigenvalue. Now it is easy to see that up to one trivial eigenvalue $\lambda = 1$ all eigenvalues of the Poincaré–Fredholm theory coincide with our eigenvalues λ_ν. They are therefore called the *Fredholm eigenvalues* of the curve Γ [10], [11].

Let us return now to the integral equation (34). The eigenfunction $w_\nu(z)$ in Δ defines now also an analytic function

$$\hat{w}_\nu(z) = \frac{\lambda_\nu}{\pi} \int_{\Delta} \frac{\overline{w_\nu(\zeta)}}{(\zeta - z)^2} d\xi d\eta, \quad z \in \tilde{\Delta}. \tag{34'}$$

We may say that the pair $w_\nu(\zeta)$ in Δ and 0 in $\tilde{\Delta}$ has the Hilbert transform consisting of $\frac{1}{\lambda_\nu} w_\nu(z)$ in Δ and $\frac{1}{\lambda_\nu} \hat{w}_\nu(z)$ in $\tilde{\Delta}$. From the norm preservation under the Hilbert transform, we deduce

$$\|w_\nu\|^2 = 1 = \frac{1}{\lambda_\nu^2} \|w_\nu\|^2 + \frac{1}{\lambda_\nu^2} \|\hat{w}_\nu\|^2 = \frac{1}{\lambda_\nu^2} + \frac{1}{\lambda_\nu^2} \|\hat{w}_\nu\|^2. \tag{41}$$

We conclude that for all ν

$$\lambda_\nu \geqslant 1 \tag{42}$$

and $\lambda_\nu = 1$ is only possible if $\hat{w}_\nu \equiv 0$. It is easy to see that this cannot happen if Γ is a smooth curve. We also find

$$\|\hat{w}_\nu\| = \sqrt{\lambda_\nu^2 - 1}. \tag{43}$$

We define the normalized analytic function in $\tilde{\Delta}$

$$\tilde{w}_\nu = \frac{i}{\sqrt{\lambda_\nu^2 - 1}} \hat{w}_\nu, \quad \|\tilde{w}_\nu\| = 1 \tag{44}$$

and rewrite (34′) into

$$\tilde{w}_\nu(z) = \frac{\lambda_\nu i}{\pi \sqrt{\lambda_\nu^2 - 1}} \int_{\Delta} \frac{\overline{w_\nu(\zeta)}}{(\zeta - z)^2} d\xi d\eta, \quad z \in \tilde{\Delta}. \tag{34''}$$

Next we apply the Hilbert transformation to the functions $\frac{1}{\lambda_\nu} w_\nu(z)$ in Δ and $\frac{i}{\lambda_\nu} \sqrt{\lambda_\nu^2 - 1}\, \tilde{w}_\nu(z)$ in $\tilde{\Delta}$. By the involutory character of the T-transformation, we must end up with the function pair $w_\nu(z)$ in Δ and 0 in $\tilde{\Delta}$.

Thus

$$(45)\quad \frac{1}{\lambda_\nu}\frac{1}{\pi}\int_{\Delta}\overline{\frac{w_\nu(\zeta)}{(\zeta-z)^2}}\,d\xi d\eta+\frac{i\sqrt{\lambda_\nu^2-1}}{\lambda_\nu}\frac{1}{\pi}\int_{\tilde\Delta}\overline{\frac{\tilde w_\nu(\zeta)}{(\zeta-z)^2}}\,d\xi d\eta = w_\nu(z)\chi_\Delta(z).$$

Observe that $w_\nu(z)$ satisfies equations (34) in Δ and (34″) in $\tilde\Delta$. Hence for $z\in\tilde\Delta$ we find

$$(46)\qquad \tilde w_\nu(z) = +\frac{\lambda_\nu}{\pi}\int_{\tilde\Delta}\overline{\frac{\tilde w_\nu(\zeta)}{(\zeta-z)^2}}\,d\xi d\eta,\qquad z\in\tilde\Delta$$

and for $z\in\Delta$

$$(47)\qquad w_\nu(z) = \frac{i\lambda_\nu}{\pi\sqrt{\lambda_\nu^2-1}}\int_{\tilde\Delta}\overline{\frac{\tilde w_\nu(\zeta)}{(\zeta-z)^2}}\,d\xi d\eta,\qquad z\in\Delta.$$

We see that in view of (46) $\tilde w_\nu(z)$ is the eigenfunction in $\tilde\Delta$ to the same eigenvalue λ_ν and (47) and (34″) exhibit the complete symmetry between the domains Δ, $\tilde\Delta$ and their eigenfunctions.

Much of this theory can be extended to multiply-connected domains [12], [16], and we will make some application in this respect in Section 7.

It should be observed that in the unbounded domain $\tilde\Delta$ there exists always an eigenfunction $\tilde w_0(z)$ to the eigenvalue $\lambda = -1$. Indeed, let $\tilde g(z,\infty)$ be the Green's function of $\tilde\Delta$ with the logarithmic pole at infinity. Then $\tilde w_0(z) = \frac{\partial}{\partial z}\tilde g(z,\infty)$ will satisfy the integral equation (46), as can be easily verified by integration by parts. However, this eigenfunction does not have a finite norm over $\tilde\Delta$. This observation shows the importance of restricting our considerations to functions with finite norms and the corresponding Hilbert spaces when we pass to orthogonal developments of the eigenfunctions.

4. Fredholm eigenfunctions and conformal mapping. Let $z = f(t)$ be univalent and regular analytic in $|t|<1$ and map this unit disk onto Δ. Also, let $z = g(t)$ be univalent in $|t|>1$ and map that region onto $\tilde\Delta$ such that the point at infinity goes into itself. We then transplant the integral equations for $w_\nu(z)$ and $\tilde w_\nu(z)$ into the t-plane. We have

$$(48)\qquad w_\nu[f(t)]f'(t) = \frac{\lambda_\nu}{\pi}\int_{|\tau|<1}\overline{w_\nu[f(\tau)]f'(\tau)\frac{f'(t)f'(\tau)}{[f(t)-f(\tau)]^2}}\,d\alpha d\beta$$

if $\tau = \alpha+i\beta$. We define the analytic function in the unit disk

$$(49)\qquad m_\nu(t) = w_\nu[f(t)]f'(t).$$

The L-kernel for the unit disk is $1/\pi(z-\zeta)^2$. Hence from (32) and (48) follows the integral equation

(50) $$m_\nu(t) = \frac{\lambda_\nu}{\pi} \int_{|\tau|<1} \overline{m_\nu(\tau)} \left\{ \frac{f'(t)f'(\tau)}{[f(t)-f(\tau)]^2} - \frac{1}{(t-\tau)^2} \right\} d\alpha d\beta.$$

Observe that

(51) $$A(t, \tau) = \frac{f'(t)f'(\tau)}{[f(t)-f(\tau)]^2} - \frac{1}{(t-\tau)^2} = \frac{\partial^2}{\partial t \partial \tau} \log \frac{f(t)-f(\tau)}{t-\tau}.$$

The combination $\log \frac{f(t)-f(\tau)}{t-\tau}$ is well known in the theory of univalent functions [6], [9]. A necessary and sufficient condition for $f(z)$ to be regular analytic and univalent in $|t| < 1$ is the regularity of the expression

(52) $$\log \frac{f(t)-f(\tau)}{t-\tau} = \sum_{m,n=0}^{\infty} c_{mn} t^m \tau^n.$$

The infinite matrix $C = ((c_{mn}))$ was introduced by Grunsky [6] and is called the *Grunsky matrix*. Many necessary conditions for univalence can be expressed by means of it.

Next, we define the analytic function in $\tilde{\Delta}$

(53) $$n_\nu(t) = \tilde{w}_\nu [g(t)] g'(t)$$

and the kernel

(54) $$B(t, \tau) = \frac{g'(t)g'(\tau)}{[g(t)-g(\tau)]^2} - \frac{1}{(t-\tau)^2} = \frac{\partial^2}{\partial t \partial \tau} \log \frac{g(t)-g(\tau)}{t-\tau}.$$

The integral equations (34) and (46) become

(55) $$m_\nu(t) = \frac{\lambda_\nu}{\pi} \int_{|\tau|<1} A(t, \tau) \overline{m_\nu(\tau)} d\alpha d\beta,$$

(56) $$n_\nu(t) = \frac{\lambda_\nu}{\pi} \int_{|\tau|>1} B(t, \tau) \overline{n_\nu(\tau)} d\alpha d\beta.$$

We also introduce the kernel

(57) $$C(t, \tau) = \frac{g'(t)f'(\tau)}{[g(t)-f(\tau)]^2}.$$

We can then express (34″) and (47) in the simple form

(58) $$n_\nu(t) = \frac{i\lambda_\nu}{\pi\sqrt{\lambda_\nu^2 - 1}} \int_{|\tau|<1} \overline{m_\nu(\tau)} C(t, \tau) d\alpha d\beta$$

and

(59) $$m_\nu(t) = \frac{i\lambda_\nu}{\pi\sqrt{\lambda_\nu^2 - 1}} \int_{|\tau|>1} \overline{n_\nu(\tau)} C(\tau, t) d\alpha d\beta.$$

We choose a complete orthonormal set $\{\varphi_\alpha(t)\}$ in $|t| < 1$ and an analogous set $\{\psi_\alpha(t)\}$ in $|t| > 1$. We can develop

$$(60) \qquad m_\nu(t) = \sum x_{\nu\alpha}\varphi_\alpha(t), \quad n_\nu(t) = \sum y_{\nu\alpha}\psi_\alpha(t).$$

Define the matrices $((A_{\alpha\beta}))$, $((B_{\alpha\beta}))$ and $((C_{\alpha\beta}))$ by the developments

$$(61) \quad A(t,\tau) = \sum_1^\infty A_{\alpha\beta}\varphi_\alpha(t)\varphi_\beta(\tau), \quad B(t,\tau) = \sum_1^\infty B_{\alpha\beta}\psi_\alpha(t)\psi_\beta(\tau),$$

$$C(t,\tau) = \sum_1^\infty C_{\alpha\beta}\psi_\alpha(t)\varphi_\beta(\tau).$$

The matrices $((A_{\alpha\beta}))$ and $((B_{\alpha\beta}))$ are obviously symmetric, the matrix $((C_{\alpha\beta}))$ is not. Using the orthogonality properties, we can transform the four integral equations (55), (56), (58) and (59) into the following discrete equations:

$$(62) \quad x_{\nu\alpha} = \frac{\lambda_\nu}{\pi}\sum_{\beta=1}^\infty A_{\alpha\beta}\bar{x}_{\nu\beta}, \qquad y_{\nu\alpha} = \frac{\lambda_\nu}{\pi}\sum_{\beta=1}^\infty B_{\alpha\beta}\bar{y}_{\nu\beta},$$

$$(63) \quad y_{\nu\alpha} = \frac{i\lambda_\nu}{\pi\sqrt{\lambda_\nu^2-1}}\sum_{\beta=1}^\infty C_{\alpha\beta}\bar{x}_{\nu\beta}, \quad x_{\nu\alpha} = \frac{i\lambda_\nu}{\pi\sqrt{\lambda_\nu^2-1}}\sum_{\beta=1}^\infty C_{\beta\alpha}\bar{y}_{\nu\beta}.$$

5. Grunsky matrices and Fredholm eigenvalues. The most obvious choice of an orthonormal function set in the two circular regions would be the set of powers

$$(64) \qquad \varphi_\alpha(t) = \sqrt{\alpha/\pi}\, t^{\alpha-1}, \quad \psi_\alpha(t) = \sqrt{\alpha/\pi}\, t^{-\alpha-1}.$$

In view of definitions (51) and (52), we find

$$(65) \qquad A_{\alpha\beta} = \pi\sqrt{\alpha\beta}\, c_{\alpha\beta}, \quad B_{\alpha\beta} = \pi\sqrt{\alpha\beta}\, d_{\alpha\beta}$$

if we write

$$(66) \qquad \log\frac{g(t)-g(\tau)}{t-\tau} = \sum_{\alpha,\beta=0}^\infty d_{\alpha\beta}t^{-\alpha}\tau^{-\beta}.$$

Finally let

$$(67) \qquad \log[g(t)-f(\tau)] = \log t + \sum e_{\alpha\beta}t^{-\alpha}\tau^\beta.$$

This implies

$$(68) \qquad C_{\alpha\beta} = \pi\sqrt{\alpha\beta}\, e_{\alpha\beta}.$$

We recognize the relation between the Grunsky matrices of the mapping functions $f(t)$ and $g(t)$ and the Fredholm eigenvalues. Let us define the matrices

$$(69) \quad P = ((\sqrt{\alpha\beta}\, c_{\alpha\beta})), \quad Q = ((\sqrt{\alpha\beta}\, d_{\alpha\beta})), \quad R = ((\sqrt{\alpha\beta}\, e_{\alpha\beta})).$$

Equations (62) and (63) can then be expressed as follows. There are associated vector pairs $\mathfrak{x}_\nu$ and $\mathfrak{y}_\nu$ such that

$$(70)\qquad P\bar{\mathfrak{x}}_\nu = \frac{1}{\lambda_\nu}\mathfrak{x}_\nu, \qquad Q\bar{\mathfrak{y}}_\nu = \frac{1}{\lambda_\nu}\mathfrak{y}_\nu,$$
$$R\bar{\mathfrak{x}}_\nu = \frac{1}{i}\frac{\sqrt{\lambda_\nu^2-1}}{\lambda_\nu}\mathfrak{y}_\nu, \qquad R^T\bar{\mathfrak{y}}_\nu = \frac{1}{i}\frac{\sqrt{\lambda_\nu^2-1}}{\lambda_\nu}\mathfrak{x}_\nu.$$

Each set $\{\mathfrak{x}_\nu\}$ and $\{\mathfrak{y}_\nu\}$ is a complete orthonormal set in its corresponding Hilbert space. An arbitrary vector $\mathfrak{v}$ in it can be written as

$$(71)\qquad \mathfrak{v} = \sum_{\nu=1}^{\infty} k_\nu \mathfrak{x}_\nu$$

and

$$(72)\qquad P\bar{\mathfrak{v}} = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu}\bar{k}_\nu \mathfrak{x}_\nu.$$

Hence, because of the orthonormality of the $\mathfrak{x}_\nu$

$$(73)\qquad (P\bar{\mathfrak{v}}, \bar{\mathfrak{v}}) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu}\bar{k}^2, \qquad \|\mathfrak{v}\|^2 = \sum |k_\nu|^2.$$

If λ_1 is the least Fredholm eigenvalue of Δ, we have the inequality

$$(74)\qquad \Big|\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}\sqrt{\alpha\beta}\, x_\alpha x_\beta\Big| \leqslant \frac{1}{\lambda_1}\sum_{\alpha=1}^{\infty} |x_\alpha|^2.$$

Since we know that all $\lambda_\nu \geqslant 1$, we can state

$$(75)\qquad \Big|\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}\sqrt{\alpha\beta}\, x_\alpha x_\beta\Big| \leqslant \sum_{\alpha=1}^{\infty} |x_\alpha|^2.$$

This is the well-known Grunsky inequality which plays such a role in the coefficient problem for univalent functions. Inequality (74) is a considerable improvement on it.

To show an important application of (74), we quote a theorem due to G. Springer [17]. Let $f(z)$ be a univalent mapping of the complex plane and let $f(z)$ be analytic in a domain D and k-quasi-conformal in its complement $\tilde{D}$. If λ_1 is the lowest Fredholm eigenvalue of D and Λ_1 the lowest Fredholm eigenvalue of its image $f(D)$, we have the inequality

$$(76)\qquad \frac{\lambda_1+1}{\lambda_1-1}\,\frac{1-k}{1+k} \leqslant \frac{\Lambda_1+1}{\Lambda_1-1} \leqslant \frac{\lambda_1+1}{\lambda_1-1}\,\frac{1+k}{1-k}.$$

We choose D to be the unit disk and $f(t)$ to be a continuous univalent function in the whole plane, analytic in $|t| < 1$ and k-quasi-conformal in $|t| > 1$. It will map the unit disk onto a domain Δ with the lowest Fredholm eigenvalue $\Lambda_1 \geq 1/k$. Indeed, the eigenvalues of the unit disk are all infinite since $l(z, \zeta) = 0$ for it. Hence we can state the following theorem first proved by R. Kühnau [7], [15]:

If the univalent continuous mapping $f(t)$ is analytic in $|t| < 1$ and k-quasi-conformal in $|t| > 1$, its Grunsky matrix satisfies the inequality

$$\left| \sum_{\alpha,\beta=1}^{\infty} \sqrt{\alpha\beta}\, c_{\alpha\beta} x_\alpha x_\beta \right| \leq k \sum_{\alpha=1}^{\infty} |x_\alpha|^2. \tag{77}$$

These inequalities are sharp as can be shown by variational methods.

6. Identities between complementary Grunsky matrices. Let Δ and $\tilde{\Delta}$ be complementary domains in the complex plane and let $f(t)$ and $g(t)$ map onto them as before. We can ask for relations between such functions which are so closely connected by geometry. Our present results lead indeed to interesting identities for the coefficients of such pairs.

We form the matrix

$$M = \left(\left(\begin{matrix} P & R^T \\ R & Q \end{matrix}\right)\right) \tag{78}$$

which is composed of the three infinite matrices defined by the formulas (69). Let

$$\mathfrak{v}_\nu^{(1)} = \left(\left(\begin{matrix} \mathfrak{x}_\nu \\ 0 \end{matrix}\right)\right), \quad \mathfrak{v}_\nu^{(2)} = \left(\left(\begin{matrix} 0 \\ \mathfrak{y}_\nu \end{matrix}\right)\right) \tag{79}$$

be corresponding doubly-infinite vectors constructed from the eigenvectors of the above problems. By virtue of equations (70) we find that

$$M\overline{\mathfrak{v}_\nu^{(1)}} = \left(\left(\begin{matrix} \dfrac{1}{\lambda_\nu} \mathfrak{x}_\nu \\ \dfrac{1}{i}\sqrt{1-\dfrac{1}{\lambda_\nu^2}}\, \mathfrak{y}_\nu \end{matrix}\right)\right), \quad M\bar{M}\mathfrak{v}_\nu^{(1)} = \mathfrak{v}_\nu^{(1)}. \tag{80}$$

Similarly

$$M\overline{\mathfrak{v}_\nu^{(2)}} = \left(\left(\begin{matrix} \dfrac{1}{i}\sqrt{1-\dfrac{1}{\lambda_\nu^2}}\, \mathfrak{x}_\nu \\ \dfrac{1}{\lambda_\nu} \mathfrak{y}_\nu \end{matrix}\right)\right), \quad M\bar{M}\mathfrak{v}_\nu^{(2)} = \mathfrak{v}_\nu^{(2)}. \tag{81}$$

The $\{\mathfrak{x}_\nu\}$ and $\{\mathfrak{y}_\nu\}$ are complete orthonormal systems in their Hilbert spaces and every doubly-infinite vector $\mathfrak{v}$ in the same space can be written as a linear combination of vectors $\mathfrak{v}_\nu^{(1)}$ and $\mathfrak{v}_\nu^{(2)}$. Thus we have the identity

(82) $$M\bar{M}\mathfrak{v} = \mathfrak{v}$$

valid for all vectors $\mathfrak{v}$. This implies the identity

(83) $$M\bar{M} = I.$$

By its definition (78), M is obviously a symmetric matrix. Now (82) shows that M is also unitary. We find the remarkable identities

(84) $$P\bar{P}+R^T\bar{R} = R\bar{R}^T+Q\bar{Q} = I, \quad R\bar{P}+Q\bar{R} = 0.$$

An immediate consequence of (83) is the fact that the vector transformation

(85) $$\hat{\mathfrak{v}} = M\mathfrak{v}$$

is norm-preserving since

(86) $$(\hat{\mathfrak{v}}_1, \hat{\mathfrak{v}}_2) = (M\mathfrak{v}_1, M\mathfrak{v}_2) = (M\bar{M}\mathfrak{v}_1, \mathfrak{v}_2) = (\mathfrak{v}_1, \mathfrak{v}_2).$$

Hence, applying the Schwarz inequality to the hermitian scalar product we find

(87) $$|(M\mathfrak{v}_1, \mathfrak{v}_2)|^2 \leqslant \|M\mathfrak{v}_1\|^2\cdot\|\mathfrak{v}_2\|^2 = \|\mathfrak{v}_1\|^2\cdot\|\mathfrak{v}_2\|^2.$$

This is equivalent to the generalized Grunsky inequality

(88) $$\Big|\sum_{\alpha,\beta=1}^{\infty} \sqrt{\alpha\beta}\,[c_{\alpha\beta}\,\xi_\alpha\, x_\beta + e_{\alpha\beta}\,(x_\beta\,\eta_\alpha + \xi_\beta\, y_\alpha) + d_{\alpha\beta}\,\eta_\alpha\, y_\beta]\Big|^2 \leqslant \sum_{\alpha=1}^{\infty}(|x_\alpha|^2+|y_\alpha|^2)\sum_{\alpha=1}^{\infty}(|\xi_\alpha|^2+|\eta_\alpha|^2).$$

If we use only vectors whose components with index larger than k is zero, we obtain estimates for the $k\times k$ submatrices of $((c_{\alpha\beta}))$, $((e_{\alpha\beta}))$ and $((d_{\alpha\beta}))$.

7. Fredholm eigenvalues of multiply-connected domains. Inequalities (88) are rarely applicable since one knows very few curves Γ for which the mapping functions $f(t)$ and $g(t)$ onto their interior and exterior are explicitly given. We can now use the Fredholm eigenvalues of multiply-connected domains, generalize these inequalities to the same matrices $((c_{\alpha\beta}))$, $((e_{\alpha\beta}))$, $((d_{\alpha\beta}))$ constructed as before but with functions $f(t)$ and $g(t)$ which map on disjoint domains Δ, $\bar{\Delta}$, not necessarily complementary.

We denote the domain set $\Delta+\tilde{\Delta}$ by D and its complement by $\tilde{D}$. We frame the same eigenvalue problem as before; to find solutions $w_\nu(z)$ and eigenvalues l_ν such that

(89) $$w_\nu(z) = \frac{l_\nu}{\pi}\int_D \frac{\overline{w_\nu(\zeta)}}{(\zeta-z)^2}\,d\xi d\eta, \qquad z\in D.$$

Since the kernel $\dfrac{1}{\pi}(\zeta-z)^{-2}$ may be replaced by $l(\zeta, z)$ and $\tilde{l}(\zeta, z)$ respectively, if z and ζ lie in the same component of D, we are again dealing with an integral equation with regular kernel. We obtain a complete set of eigenfunctions $w_\nu(z)$ which can be orthonormalized by the condition

$$\int_D w_\nu(\zeta)\overline{w_\mu(\zeta)}\,d\xi d\eta = \delta_{\nu\mu} \tag{90}$$

and we may specify

$$l_\nu > 0. \tag{91}$$

The same argument regarding the Hilbert transform leads at once to the result

$$l_\nu \geqslant 1. \tag{92}$$

We can now repeat the formalism of Section 5. We assume that the component $\tilde{\Delta}$ contains the point at infinity and that $f(t)$ maps $|t| < 1$ onto Δ, while $g(t)$ maps $|t| > 1$ onto $\tilde{\Delta}$ such that $g(\infty) = \infty$. Denote the restrictions of $w_\nu(z)$ to Δ and $\tilde{\Delta}$ by $w_\nu^{(1)}(z)$ and $w_\nu^{(2)}(z)$ and define

$$m_\nu(t) = w_\nu^{(1)}[f(t)]\,f'(t), \qquad n_\nu(t) = w_\nu^{(2)}[g(t)]\,g'(t) \tag{93}$$

and using definitions (51), (54) and (57) we can express the integral equation (89) as follows:

$$\begin{aligned} m_\nu(t) &= \frac{l_\nu}{\pi}\Big\{\int_{|\tau|<1} A(t,\tau)\overline{m_\nu(\tau)}\,d\alpha d\beta + \int_{|\tau|>1} C(\tau,t)\overline{n_\nu(\tau)}\,d\alpha d\beta\Big\},\\ n_\nu(t) &= \frac{l_\nu}{\pi}\Big\{\int_{|\tau|<1} C(t,\tau)\overline{m_\nu(\tau)}\,d\alpha d\beta + \int_{|\tau|>1} B(t,\tau)\overline{n_\nu(\tau)}\,d\alpha d\beta\Big\}. \end{aligned} \tag{94}$$

The same calculations as before lead to the system of equations

$$\begin{aligned} x_{\nu\alpha} &= l_\nu\Big\{\sum_{\beta=1}^{\infty}\sqrt{\alpha\beta}\,(c_{\alpha\beta}\,\bar{x}_{\nu\beta} + e_{\beta\alpha}\,\bar{y}_{\nu\beta})\Big\},\\ y_{\nu\alpha} &= l_\nu\Big\{\sum_{\beta=1}^{\infty}\sqrt{\alpha\beta}\,(e_{\alpha\beta}\,\bar{x}_{\nu\beta} + d_{\alpha\beta}\,\bar{y}_{\nu\beta})\Big\}. \end{aligned} \tag{95}$$

In notations (69) and (78) we can combine these equations into

$$M\begin{pmatrix}\bar{\mathfrak{x}}_\nu\\ \bar{\mathfrak{y}}_\nu\end{pmatrix} = \frac{1}{l_\nu}\left(\begin{pmatrix}\mathfrak{x}_\nu\\ \mathfrak{y}_\nu\end{pmatrix}\right). \tag{96}$$

The vectors $\mathfrak{v}_\nu = \left(\begin{pmatrix}\mathfrak{x}_\nu\\ \mathfrak{y}_\nu\end{pmatrix}\right)$ form a complete orthonormal set and every vector $\mathfrak{v}$ can be written as a linear combination of them:

$$\mathfrak{v} = \sum_{\nu=1}^{\infty} k_\nu\,\mathfrak{v}_\nu. \tag{97}$$

We have by iteration of (96)

(98)
$$M\bar{M}\mathfrak{v}_\nu = \frac{1}{l_\nu^2}\mathfrak{v}_\nu .$$

Hence

(99)
$$\|M\mathfrak{v}\|^2 = (M\bar{M}\mathfrak{v}, \mathfrak{v}) = \sum_{\nu=1}^{\infty} |k_\nu|^2 \frac{1}{l_\nu^2} \leqslant \frac{1}{l_1^2}\|\mathfrak{v}\|^2$$

if l_1 is the least eigenvalue of our problem. Again, we find by the Schwarz inequality

(100)
$$|(M\mathfrak{v}, \mathfrak{s})|^2 \leqslant \|M\mathfrak{v}\|^2 \cdot \|\mathfrak{s}\|^2 \leqslant \frac{1}{l_1^2}\|\mathfrak{v}\|^2 \cdot \|\mathfrak{s}\|^2 .$$

This is analogous to (87) and leads to the same consequence (88) because of (92). However, we may now improve this estimate by inserting the factor $1/l_1^2$ before the right-hand term.

Inequalities for the coefficients of pairs of univalent functions which map on disjoint domains are well known [1], [4], [5], [8]. What is new in our derivation is the connection with the theory of the Hilbert transform and the Fredholm eigenvalues. One may generalize further and consider in an analogue manner the Fredholm eigenvalues for sets of k disjoint domains. One obtains so coefficient inequalities for sets of univalent functions which have no common values.

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